Counting and symmetries

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Sometimes, we want to count the number of certain objects up to symmetries; e.g., we can ask how many non-isomorphic graphs are there on n vertices (counting the graphs that differ only by a permutation of vertices as one graph), or in how many ways can we color the faces of the cube by k colors up to rotations (counting all colorings that differ only by a rotation of the cube as one coloring). To deal with these kinds of problems, we first need to introduce a bit of group theory.

1 Groups

A *group* G is a set with a binary associative operation $\circ$ on G and and a identity element $1 \in G$ such that

- $1 \circ x = x \circ 1 = x$ for every $x \in G$, and
- for every $x \in G$, there exists $y \in G$ such that $x \circ y = y \circ x = 1$.

We say that the element y is *inverse* to x and denote it by x^{-1}. Let us remark that it is easy to see that both the identity element and the inverse are unique. E.g., if $x \circ y_1 = y_1 \circ x = 1$ and $x \circ y_2 = y_2 \circ x = 1$, then $y_1 = (y_2 \circ x) \circ y_1 = y_2 \circ (x \circ y_1) = y_2$.

Example 1. *The set of all permutations of* $\{1, \ldots, n\}$*, with the operation being the composition of permutations and the identity element being the identity permutation, forms a group. This group is called the* symmetric group *and denoted by* Sym_n*. This group has size* $n!$.

The set of rotations of $\mathbb{R}^3$ *that map an unit cube centered at* $(0, 0, 0)$ *to itself, together with the operation of composition (the rotations are just functions mapping* $\mathbb{R}^3$ *to* $\mathbb{R}^3$*) and with the identity element being the identity function, form a group, which we will denote by* R_{cube}*. Note that* R_{cube} *consists of the identity,* 9 *rotations along an axis passing through the center of* *opposite faces (three choices of an axis, rotation by* 90*,* 180*, or* 270 *degrees),* 6 *rotations along an axis passing through the center of opposite edges (six choices of an axis, rotation by* 180 *degrees), and* 8 *rotations along an axis passing through opposite vertices (four choices of an axis, rotation by* 120 *or* 240 *degrees), and thus* $|R_{cube}| = 24$.

Note that the elements of the group often correspond to operations on some objects. More precisely, for a group G and a set T, an *action* of G on T is a function $a : G \to T^T$ that to each element of G assign a function $a_G : T \to T$, with the following properties:

- a_1 is the identity function. I.e., $a_1(x) = x$ for all $x \in G$.
- $a_{g \circ h} = a_g \cdot a_h$, with $\cdot$ denoting the function composition. I.e., $a_{g \circ h}(x) = a_h(a_g(x))$ for every $x \in G$.

Note this implies that $a_{g^{-1}}$ is the inverse function to a_g, since $a_{g^{-1}} \cdot a_g = a_{g^{-1} \circ g} = a_1 = \mathrm{id}$.

Example 2. *For a graph H with vertex set $\{1, \ldots, n\}$ and a permutation π of this set, we can define $a_\pi(H)$ to be the graph obtained from H by permuting the vertices according to π, i.e., $a_\pi(H)$ is the graph with vertex set $\{1, \ldots, n\}$ and with $uv \in E(a_\pi(H))$ if and only if $\pi^{-1}(u)\pi^{-1}(v) \in E(H)$. Then a is an action of Sym_n on the set $\mathcal{H}_n$ of all graphs with vertex set $\{1, \ldots, n\}$.*

Let B_k denote the set of all colorings of faces of the cube by colors $1, \ldots, k$. Then R_{cube} naturally acts on B_k: Each coloring is mapped by a rotation $r \in R_{cube}$ to the appropriately rotated coloring.

We are interested in the number of distinct objects up to some such action. More precisely, let a be an action of a group G on a set T. For $x, y \in T$, we define $x \sim_a y$ if and only if there exists $g \in G$ such that $a_g(x) = y$. It is easy to see that $\sim_a$ is an equivalence, and the classes of the equivalence are *orbits*.

Example 3. *Consider the actions defined in Example 2. Two graphs belong to the same orbit iff they are isomorphic. Two colorings belong th the same orbit iff they only differ by a rotation.*

We would now like to find an easy way how to count the orbits, in the typical situation that the size of the group is much smaller than the number of objects it acts on. To this end, we demonstrate a relationship to the number of objects fixed by each element of the group under the action. For $g \in G$, the set of *fixed points* of g is $\mathrm{Fix}_a(g) = \{x \in T : a_g(x) = x\}$.

Example 4. *Consider the permutation* (12) *that swaps vertices* 1 *and* 2*. In the action* a *defined in Example 2, we have* $H \in \mathrm{Fix}_a((12))$ *iff exchanging the vertices* 1 *and* 2 *does not change the graph* H*, i.e, if the vertices* 1 *and* 2 *have the same sets of neighbors in* $V(H) \setminus \{1, 2\}$*.*

It turns out that the number of orbits is equal to the average number of fixed points over the elements of the group. Before we prove this, we need to introduce another object. For an action a of a group G on a set T and elements $x, y \in T$, let $\mathrm{Map}_a(x, y) = \{g \in G : a_g(x) = y\}$. Obviously, $\mathrm{Map}_a(x, y)$ is non-empty if and only if x and y belong to the same orbit. Suppose z is another element belonging to the same orbit, and thus $a_h(y) = z$ for some h. Then for each $g \in \mathrm{Map}_a(x, y)$, we have $a_{g\circ h}(x) = a_h(a_g(x)) = a_h(y) = z$, and thus $g \circ h \in \mathrm{Map}_a(x, z)$. Conversely, for each $f \in \mathrm{Map}_a(x, z)$, we have $f \circ h^{-1} \in \mathrm{Map}_a(x, y)$. This establishes a bijection between $\mathrm{Map}_a(x, y)$ and $\mathrm{Map}_a(x, z)$, and thus $|\mathrm{Map}_a(x, y)| = |\mathrm{Map}_a(x, z)|$ for any elements y and z belonging to the same orbit. Moreover, note that for any $g \in G$, there exists precisely one y in the orbit of x (namely $y = a_g(x)$)) such that $g \in \mathrm{Map}_a(x, y)$. Therefore, denoting by O_x the orbit containing x, we have

$$|G| = \sum_{y \in O_x} |\mathrm{Map}_a(x, y)| = |O_x| \cdot |\mathrm{Map}_a(x, x)|.$$

Theorem 5 (Burnside's lemma)**.** *For any group finite* G *and an action* a *of* G *on a finite set* T*, the number of orbits of this action is equal to*

$$\frac{1}{|G|} \cdot \sum_{g \in G} |\mathrm{Fix}_a(g)|.$$

Proof. Let O_1, ..., O_n be the orbits of the action a. We will count the number m of pairs (x, g) such that $x \in T$, $g \in G$, and $a_g(x) = x$ in two ways. On one hand, for each g all the elements x with this property belong to $\mathrm{Fix}_a(g)$, and thus

$$m = \sum_{g \in G} |\mathrm{Fix}_a(g)|.$$

On the other hand, for each element $x \in T$ all $g \in G$ with this property belong to $\mathrm{Map}_a(x, x)$, and thus

$$m = \sum_{x \in T} |\mathrm{Map}_a(x, x)| = \sum_{x \in T} \frac{|G|}{|O_x|} = |G| \sum_{x \in T} \frac{1}{|O_x|} = |G| \sum_{i=1}^{n} |O_i| \cdot \frac{1}{|O_i|} = |G|n.$$

Comparing the two expressions, we have

$$n = \frac{1}{|G|} \cdot \sum_{g \in G} |\mathrm{Fix}_a(g)|,$$

as required. □

2 Applications of Burnside's lemma

Example 6. *Let us count the number of colorings of the faces of a cube by k colors up to rotations. Recall R_{cube} is group of rotations that preserve the cube, B_k is the set of all colorings of the faces by k colors, and let a be the natural action of R_{cube} on B_k as defined in Example 2. We want to determine the number of orbits of a, and by Burnside's lemma, it suffices to compute the number of fixed points for each element r of R_{cube}:*

- *If r is the identity, then all colorings are fixed by r, and thus $|\mathrm{Fix}_a(r)| = |B_k| = k^6$.*
- *If r is one of the six rotations by ± 90 degrees along an axis passing through centers of opposite faces, then r fixes exactly the colorings for which the faces not pierced by the axis all have the same color. Hence, we can choose one color for these four faces and one color for each of the two faces pierced by the axis, and $|\mathrm{Fix}_a(r)| = k^3$.*
- *If r is one of the three rotations by 180 degrees along an axis passing through centers of opposite faces, then r fixes exactly the colorings for which the opposite faces not pierced by the axis (of which there are two pairs) have the same color. Hence $|\mathrm{Fix}_a(r)| = k^4$.*
- *If r is one of the six rotations by 180 degrees along an axis passing through centers of opposite edges, then r fixes exactly the colorings for which the two opposite faces not incident with the pierced edges, as well as the pairs of faces incident with the pierced edges, have the same color, and thus $|\mathrm{Fix}_a(r)| = k^3$.*
- *Finally, if r is one of the eight rotations by ± 120 degrees along an axis passing through opposite vertices, then r fixes exactly the colorings for which the three faces incident with each of the pierced vertices have the same color, and thus $|\mathrm{Fix}_a(r)| = k^2$.*

By Burnside's lemma, it follows that the number of colorings is

$$\frac{1}{|R_{cube}} \sum_{r \in R_{cube}} |\mathrm{Fix}_a(r)| = \frac{k^6 + 3k^4 + 12k^3 + 8k^2}{24}.$$

Example 7. *Let us count the number of non-isomorphic graphs on 5 vertices. Here we consider the symmetric group Sym_5 with the action a defined in Example 2. Let us discuss the permutations $\pi \in \mathrm{Sym}_5$ according to their cycle structure:*

- *If π has five cycles of length* 1*, i.e., π is the identity, then π fixes all* $2^{\binom{5}{2}} = 2^{10}$ *graphs on* 5 *vertices.*

- *There are* 10 *permutations with one cycle* (ab) *of length two and three cycles of length* 1*. In a graph fixed by such a permutation, the neighborhood of a uniquely determines the neighborhood of b (they must be the same outside of* $\{a, b\}$*), and thus we only have freedom to select edge/non-edge status for the remaining* 7 *pairs of vertices. Thus, such a permutation fixes* 2^7 *of the graphs.*

- *There are* 15 *permutations with two cycles* (ab) *and* (cd) *of length two and one of length one. In a fixed graph, ac is an edge iff bd is, ad is an edge iff bc is, and the fifth vertex has the same adjacencies to a and b and to c and d, leaving us with* 6 *choices; hence, there are* 2^6 *fixed graphs.*

- *There are* 20 *permutations with one cycle* (abc) *of length three and two cycles of length one. If fixed by this permutation, ab is an edge iff bc and ac are, and for the remaining two vertices, each of them is adjacent to either all of a, b, c or none. Hence, there are* 4 *choices and* 2^4 *fixed graphs.*

- *There are* 20 *permutations with one cycle* (abc) *of length three and one cycle* (de) *of length two. The situation is similar to the previous case, but either all edges between a, b, c and d, e are present, or none is. Hence, there are three choices and* 2^3 *fixed graphs.*

- *There are* 30 *permutations with one cycle* $(abcd)$ *of length four and one cycle of length one. In the fixed graphs, ab, bc, cd, de are either all edges or all non-edges, ad and bc are either both edges or both non-edges, and the fifth vertex is adjacent to either all or none of a, b, c, d. This leaves three choices and* 2^3 *graphs.*

- *Finally, there are* 24 *permutations with one cycle of length five. We have one choice for the edges of the cycle and one choice for the diagonals, giving* 2^2 *fixed graphs.*

By Burnside's lemma, the number of non-isomorphic graphs on 5 *vertices is*

$$\frac{1}{5!}(2^{10} + 10 \cdot 2^7 + 15 \cdot 2^6 + 20 \cdot 2^4 + 20 \cdot 2^3 + 30 \cdot 2^3 + 24 \cdot 2^2) = 34.$$

3 Polya enumeration formula

We can also combine the theory we developed with the theory of generating functions. Let us start with a way how to represent the symmetries in a generating function. Let B be a set of size n. For a permutation π of B with c_1 cycles of length 1, c_2 cycles of length 2, …, c_n cycles of length n, let $x^{c(\pi)}$ be defined as $x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n}$. Let G be a subgroup of Sym_n, i.e., a set of permutations of n elements closed on composition and inverse. The *cycle index* Z_G of G is the following polynomial in variables $x_1, \ldots, x_n$.

$$Z_G(x_1,\ldots,x_n) = \frac{1}{|G|}\sum_{\pi\in G} x^{c(\pi)}.$$

Example 8. *For a permutation π of the set $\{1,\ldots,n\}$, we can define a permutation π' on the set $\binom{\{1,\ldots,n\}}{2}$ of pairs of its elements by setting $\pi'(\{i,j\}) = \{\pi(i),\pi(j)\}$. Let $\mathrm{Sym}'_n = \{\pi' : \pi \in \mathrm{Sym}_n\}$. Let us determine the cycle index of Sym'_5. The group Sym_5 contains:*

- *One permutation π with five cycles of length 1 (the identity). For this permutation, π' is also the identity and has 10 cycles of length 1. Hence, this contributes x_1^{10} to the cycle index.*
- *10 permutations π with one cycle (ab) of length two and three cycles of length 1. Then the permutation π' fixes the pair $\{a,b\}$ and all pairs disjoint from $\{a,b\}$, giving four cycles of length 1, and for each $c \notin \{a,b\}$, the pairs $\{a,c\}$ and $\{b,c\}$ form a cycle of length two. This contributes $10x_1^4x_2^3$ to the cycle index.*
- *15 permutations π with two cycles (ab) and (cd) of length two and one of length one. The permutation π' fixes the pairs $\{a,b\}$ and $\{c,d\}$ (two cycles of length one), and the remaining pairs are contained in four cycles of length two. This contributes $15x_1^2x_2^4$ to the cycle index.*
- *20 permutations π with one cycle (abc) of length three and two cycles of length one. The pair of elements distinct from a, b, and c is fixed by π', while the remaining pairs are in cycles of length three, contributing $20x_1x_3^3$ to the cycle index.*
- *20 permutations π with one cycle (abc) of length three and one cycle (de) of length two. In π', the pair $\{d,e\}$ is fixed, the pairs contained in $\{a,b,c\}$ form a cycle of length three, and the remaining pairs form a cycle of length six, contributing $20x_1x_3x_6$ to the cycle index.*

- 30 *permutations with one cycle* $(abcd)$ *of length four and one cycle of length one. In* π', *the diagonal pairs contained in* $\{a, b, c, d\}$ *form a two cycle of length two and the remaining pairs form two cycles of length four, contributing* $30x_2x_4^2$ *to the cycle index.*

- 24 *permutations with one cycle of length five. In* π', *the pairs are split to two cycles of length five, contributing* $24x_5^2$ *to the cycle index.*

Therefore,

$$Z_{\mathrm{Sym}_5'} = \frac{1}{120}(x_1^{10} + 10x_1^4x_2^3 + 15x_1^2x_2^4 + 20x_1x_3^3 + 20x_1x_3x_6 + 30x_2x_4^2 + 24x_5^2).$$

We view the elements of B as boxes to which we can arrange objects of certain sizes, with the size of the arrangement being the sum of the sizes of the objects in the boxes. Suppose $F(x) = f_0 + f_1x_1 + f_2x_2 + \ldots$ is a generating function for the objects; i.e., to each box, we can choose to put one of f_0 objects of size 0, or one of f_1 objects of size 1, etc. Let t_m denote the number of such arrangements of size m, where two arrangements that differ only by a permutation belonging to G are considered to be the same. Letting a be the action of G on the set A_m of arrangements of size m and using Burnside's lemma, we have

$$t_m = \frac{1}{|G|}\sum_{\pi \in G} |\mathrm{Fix}_a(\pi)|.$$

For π to fix an arrangement, we need to put the same object to all boxes in each of the cycles of π. I.e., for a cycle of length ℓ, we need to choose one object, and if the size of the chosen object is s, this cycle will in total contribute ℓs to the size of the whole arrangement. It is convenient to see this as being able to choose one of f_0 objects of size 0, or one of f_1 objects of size ℓ, or one of f_2 objects of size 2ℓ, etc.; the generating function of this sequence is $f_0 + f_1x^\ell + f_2x^{2\ell} + \ldots = F(x^\ell)$. Now we need to combine such independent choices in each cycle; the independent choices correspond to the product of generating functions, and thus

$$|\mathrm{Fix}_a(\pi)| = [x^m] \prod_{b \text{ cycle of } \pi} F(x^{\ell(b)}).$$

The product on the right-hand side is the same as $x^{c(\pi)}$ with $F(x)$ substituted for x_1, $F(x^2)$ for x_2, ... Hence,

$$t_m = [x^m]\frac{1}{|G|}\sum_{\pi \in G}\prod_{b \text{ cycle of } \pi} F(x^{\ell(b)}) = [x^n]Z_G(F(x), F(x^2), F(x^3), \ldots, F(x^n)).$$

Thus, we actually obtain a generating function for the arrangements up to the symmetries; defining $T(x) = \sum_{n\geq 0} t_n x^n$, we have

$$T(x) = Z_G(F(x), F(x^2), F(x^3), \ldots, F(x^n)).$$

Example 9. *Let us count again the number of non-isomorphic graphs with* 5 *vertices, this time classified by the number of edges. I.e., let* t_m *denote the number of non-isomorphic graphs with* 5 *vertices with* m *edges. We can view the graphs as arrangements into boxes: We have one box for each pair of vertices, and we need to decide whether we put an edge to it (an object contributing* 1 *to the size), or a non-edge (an object contributing* 0 *to the size). Hence, the generating function for the objects is* $F(x) = 1 + x$, *and*

$$\begin{aligned}
T(x) &= Z_{\mathrm{Sym}_5'}(1+x, 1+x^2, \ldots, 1+x^10) \\
&= \frac{1}{120}((1+x)^{10} + 10(1+x)^4(1+x^2)^3 + 15(1+x)^2(1+x^2)^4 + \\
&\qquad 20(1+x)(1+x^3)^3 + 20(1+x)(1+x^3)(1+x^6) + \\
&\qquad 30(1+x^2)(1+x^4)^2 + 24(1+x^5)^2) \\
&= 1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 + 4x^7 + 2x^8 + x^9 + x^{10}.
\end{aligned}$$

For example, this tells us there are 6 *pairwise non-isomorphic graphs on* 5 *vertices with* 4 *edges.*