Graph polynomials

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In this lecture, it will be convenient to allow the graphs to have loops (even multiple loops at a single vertex) and parallel edges. For an edge e of G such that e is not a loop, by G/e we mean the graph obtained by contracting the edge e, i.e., deleting it and identifying its ends to a single vertex. Note that in this way, new parallel edges may be created, and the edges of G parallel to e are turned into loops incident with the vertex resulting from the contraction.

1 Chromatic polynomial

How many proper colorings using k colors does a graph G have (let us denote this number by $\pi_G(k)$)? This is of course hard to determine, but we can give a simple recursive formula (whose evaluation would take exponential time).

- If G has no edges, then we can color its vertices independently, and thus $\pi_G(k) = k^{|V(G)|}$.
- Otherwise, consider any edge e of G.
  - If e is a loop, then G has no proper coloring, and thus $\pi_G(k) = 0$.
  - Otherwise, note that every proper coloring of G is also a proper coloring of $G - e$. On the other hand, a proper coloring of $G - e$ is not proper in G if and only if both ends of e have the same color c; but such a coloring naturally corresponds to a proper coloring of G/e where the vertex arising from the contraction has color c. Consequently,
  $$\pi_G(k) = \pi_{G-e}(k) - \pi_{G/e}(k).$$

Note that the terms on the right-hand side refer to graphs with fewer than $|E(G)|$ edges, and thus we can evaluate the formula by recursively expanding them until we end with one of the basic cases.

Note that for a fixed graph G, the basic case $\pi_G(k) = k^{|V(G)|}$ is a polynomial of degree $|V(G)|$ in the variable k, and in the general case, we are summing two terms coming from the recursion. Hence, by induction we easily see that the following claim holds.

Observation 1. *For any graph G, $\pi_G(k)$ is a polynomial in k of degree at most $|V(G)|$.*

Furthermore, the polynomial π_G is uniquely determined, even though the formula we used to compute it does not prescribe the exact choice of the considered edge (and thus we can compute it in many different ways). Indeed, a polynomial of degree d is uniquely determined by its values in d points, and the values $\pi_G(1), \ldots, \pi_G(|V(G)|)$ are the numbers of colorings of G by 1, 2, ..., $|V(G)|$ colors in order, and thus they only depend on G. We say that π_G is the *chromatic polynomial* of G.

Exercise 2. *Show that $\pi_{K_n}(k) = k(k-1)\cdots(k-n+1)$ and that for any tree T with n vertices, $\pi_T(k) = k(k-1)^{n-1}$.*

The study of the chromatic polynomial is motivated by the idea that it enables us to use algebra and analysis to argue about graph coloring. Moreover, we can ask a number of questions. To what extent does the chromatic polynomial capture the properties of the graph G (are there any other graph parameters determined by p_G? Do the coefficients of p_G or the values of p_G in points other than positive integers have any meaning?

We can also ask whether there any other natural polynomials associated with graphs. This turns out to be the case; let us give two examples.

2 Flow polynomial

Let $\vec{G}$ be an arbitrary orientation of an undirected graph G. For a finite Abelian group $\mathbb{A}$, an $\mathbb{A}$-*flow* in G is a function $f : E(\vec{G}) \to \mathbb{A}$ satisfying the flow conservation condition, i.e., for every $v \in V(G)$ we have

$$\sum_{e=(u,v)\in E(\vec{G}} f(e) = \sum_{e=(v,u)\in E(\vec{G}} f(e).$$

Let us remark that the choice of the orientation of G is not important (reversing the orientation of an edge e can be compensated by replacing the flow $f(e)$ on e by $-f(e)$).

Of particular interest are the *nowhere-zero* flows, i.e., the flows satisfying $f(e) \neq 0$ for every $e \in E(\vec{G})$. It can be seen that if G is a plane connected graph, then the number of k-colorings of G is equal to k times the number of nowhere-zero $\mathbb{Z}_k$-flows in the dual $G^\star$ of G. Hence, nowhere-zero flows are a dual concept to graph colorings.

Moreover, we can determine the number $C_G(\mathbb{A})$ of nowhere-zero $\mathbb{A}$-flows in G by a formula similar to the one used in the definition of the chromatic polynomial.

- If G has no edges, there is only one (trivial) nowhere-zero $\mathbb{A}$-flow; $C_G(\mathbb{A}) = 1$.
- Otherwise, consider an edge $e \in E(G)$.
  - If e is a loop, then we can set the flow on e arbitrarily (different from 0), without affecting the flow conservation. Hence, $C_G(\mathbb{A}) = (|\mathbb{A}| - 1)C_{G-e}(\mathbb{A})$.
  - Otherwise, a nowhere-zero $\mathbb{A}$-flow in G/e can be uniquely turned into an $\mathbb{A}$-flow in G by setting the value on e so that the flow conservation condition holds at both ends of e (it can be easily seen that if it holds at one end, it will also automatically hold at the other end). The resulting flow is not necessarily nowhere-zero, as the value on e can be 0; however, if that is the case, then the flow also corresponds to a nowhere-zero $\mathbb{A}$-flow in $G-e$. Therefore, we have
  $$C_G(\mathbb{A}) = C_{G/e}(\mathbb{A}) - C_{G-e}(\mathbb{A}).$$

Note that the only dependence of $C_G(\mathbb{A})$ on the group $\mathbb{A}$ comes from the case where G contains a loop, and in this case only the size of $\mathbb{A}$ appears in the formula. Hence, defining $C_G(k) = C_G(\mathbb{Z}_k)$, the following claim holds.

Observation 3. *For any graph G, $C_G(k)$ is a polynomial in k of degree at most $|E(G)|$. Moreover, for any finite Abelian group $\mathbb{A}$, the number of nowhere-zero $\mathbb{A}$-flows in G is $C_G(|\mathbb{A}|)$.*

We call C_G the *flow polynomial* of G.

3 Reliability polynomial

For a connected graph G and $p \in [0, 1]$, let $R_G(p)$ be the probability that if we delete each edge independently at random with probability p, then the graph remains connected. E.g., in case G is a computer network and p gives a probability that one of the connections is broken, $R_G(p)$ gives the probability that any two computers can still communicate. We can compute $R_G(p)$ as follows:

- If G has no edges, then it has only one vertex (since we assume G is connected), and thus $R_G(p) = 1$.
- Otherwise, let e be an edge of G.
  - If e is a loop, then deleting it does not affect the connectivity of the graph, and thus $R_G(p) = R_{G-e}(p)$.
  - If e is a bridge, then deleting it would disconnect the graph. Moreover, observe that if h is an edge of a graph H, then H is connected if and only if H/e is connected. Hence, $R_G(p) = (1-p)R_{G/e}(p)$.
  - Otherwise, both deleting e and keeping it (which is equivalent to contracting it as we observed in the previous point) may result in G being connected; hence, we have

$$R_G(p) = pR_{G-e}(p) + (1-p)R_{G/e}(p).$$

Again, the inductive argument shows that $R_G(p)$ is a polynomial in p of degree at most $|E(G)|$. We say that R_G is the *reliability polynomial* of G.

4 Tutte polynomial

Motivated by the examples we have seen so far, one can try to find the most general polynomial that can be defined in terms of edge deletions and contractions. It turns out any such polynomial can be expressed in terms of *Tutte polynomial* in two variables.

For a graph G and a set $A \subseteq E(G)$, let us define $\kappa(G)$ to be the number of components of G and $\kappa_G(A)$ to be the number of components of the graph with vertex set $V(G)$ and edge set A. Note that

$$\kappa_G(A) \geq \min(\kappa(G), |V(G)| - |A|),$$

and let $r_G(A) = \kappa_G(A) - \kappa(G)$ and $c_G(A) = \kappa_G(A) + |A| - |V(G)|$. The *Tutte polynomial* $T_G(x, y)$ of a graph G is defined by

$$T_G(x, y) = \sum_{A \subseteq V(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)}.$$

Note that if G has no edges, then $T_G(x, y) = 1$. Otherwise, we can compute $T_G(x, y)$ by a deletion-contraction recurrence.

Lemma 4. *If e is an edge of a graph G, then*

$$T_G(x,y) = \begin{cases} xT_{G/e}(x,y) & \text{if } e \text{ is a bridge} \\ yT_{G-e}(x,y) & \text{if } e \text{ is a loop} \\ T_{G-e}(x,y) + T_{G/e}(x,y) & \text{otherwise.} \end{cases}$$

Proof. Suppose first e is a bridge and $A \subseteq E(G)\setminus\{e\}$. Contracting e does not affect the number of components while adding it to a subgraph of G decreases the number of components, and thus $r_G(A)-1 = r_G(A\cup\{e\}) = r_{G/e}(A)$ and $c_G(A\cup\{e\}) = c_G(A) = c_{G/e}(A)$. Hence,

$$\begin{aligned} T_G(x,y) &= \sum_{A\subseteq V(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\ &= \sum_{A\subseteq V(G)\setminus\{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A\cup\{e\})}(y-1)^{c_G(A\cup\{e\})}\Big) \\ &= \sum_{A\subseteq V(G/e)} \Big((x-1)^{r_{G/e}(A)+1}(y-1)^{c_{G/e}(A)} + (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)}\Big) \\ &= x \sum_{A\subseteq V(G-e)} (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)} = xT_{G/e}(x,y). \end{aligned}$$

Suppose next e is a loop and $A \subseteq E(G)\setminus\{e\}$. Deleting e does not affect the number of components, and thus $r_G(A) = r_G(A\cup\{e\}) = r_{G-e}(A)$ and $c_G(A\cup\{e\})-1 = c_G(A) = c_{G-e}(A)$. Hence,

$$\begin{aligned} T_G(x,y) &= \sum_{A\subseteq V(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\ &= \sum_{A\subseteq V(G)\setminus\{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A\cup\{e\})}(y-1)^{c_G(A\cup\{e\})}\Big) \\ &= \sum_{A\subseteq V(G-e)} \Big((x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} + (x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)+1}\Big) \\ &= y \sum_{A\subseteq V(G-e)} (x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} = yT_{G-e}(x,y). \end{aligned}$$

Finally, suppose that e is neither a loop nor a bridge and $A \subseteq E(G)\setminus\{e\}$. Then $r_G(A) = r_{G-e}(A)$, $r_G(A\cup\{e\}) = r_{G/e}(A)$, $c_G(A) = c_{G-e}(A)$ and

$c_G(A \cup \{e\}) = c_{G/e}(A)$. Hence,

$$\begin{aligned}
T_G(x,y) &= \sum_{A \subseteq V(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\
&= \sum_{A \subseteq V(G) \setminus \{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A \cup \{e\})}(y-1)^{c_G(A \cup \{e\})} \Big) \\
&= \sum_{A \subseteq V(G) \setminus \{e\}} \Big((x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} + (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)} \Big) \\
&= T_{G-e}(x,y) + T_{G/e}(x,y).
\end{aligned}$$

□

Note that T_G is unique, as we have defined it just from the properties of the graph G, independently of the formulas from Lemma 4. On the other hand, from the recursive formula we can e.g. see that the coefficients of Tutte polynomial are nonnegative, something which is not easily seen from the definition. It is natural to ask whether the coefficients have some combinatorial meaning, and this turns out to be the case—they count the number of spanning forests of G with certain properties (the prescribed number of internally and externally active edges).

5 The universal polynomial

Comparing the recursive formula for Tutte polynomial with the ones for say the chromatic polynomial or the reliability polynomial, it may seem that it is not general enough to express them—the basic case of Tutte polynomial is independent of the number of vertices, and the deletion-contraction recurrence has fixed coefficients. However, this is easy to fix. Let us define

$$U_G(n,b,l,d,c) = n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} T_G(b/c, l/d).$$

Lemma 5. *For any graph G,*

- *If G has no edges, then $U_G = n^{|V(G)|}$.*
- *Otherwise, let e be an edge of G.*
  - *If e is a bridge, then $U_G = b \cdot U_{G/e}$.*
  - *If e is a loop, then $U_G = l \cdot U_{G-e}$.*
  - *Otherwise, $U_G = d \cdot U_{G-e} + c \cdot U_{G/e}$.*

Proof. If G has no edges, then $T_G = 1$, and thus $U_G = n^{\kappa(G)} = n^{|V(G)|}$. Otherwise, consider an edge e of G.

If e is a bridge, then

$$\begin{aligned} U_G(n,b,l,d,c) &= n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} T_G(b/c,l/d) \\ &= n^{\kappa(G/e)} d^{|E(G/e)|+\kappa(G/e)-|V(G/e)|} c^{|V(G/e)|+1-\kappa(G/e)} (b/c) T_{G/e}(b/c,l/d) \\ &= b \cdot U_{G/e}(n,b,l,d,c). \end{aligned}$$

If e is a loop, then

$$\begin{aligned} U_G(n,b,l,d,c) &= n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} T_G(b/c,l/d) \\ &= n^{\kappa(G-e)} d^{|E(G-e)|+1+\kappa(G-e)-|V(G-e)|} c^{|V(G-e)|-\kappa(G-e)} (l/d) T_{G-e}(b/c,l/d) \\ &= l \cdot U_{G-e}(n,b,l,d,c). \end{aligned}$$

Otherwise,

$$\begin{aligned} U_G(n,b,l,d,c) &= n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} T_G(b/c,l/d) \\ &= n^{\kappa(G)} d^{|E(G)|+\kappa(G)-|V(G)|} c^{|V(G)|-\kappa(G)} (T_{G-e}(b/c,l/d) + T_{G/e}(b/c,l/d) \\ &= n^{\kappa(G-e)} d^{|E(G-e)|+1+\kappa(G-e)-|V(G-e)|} c^{|V(G-e)|-\kappa(G-e)} T_{G-e}(b/c,l/d) \\ &\quad + n^{\kappa(G/e)} d^{|E(G/e)|+\kappa(G/e)-|V(G/e)|} c^{|V(G/e)|+1-\kappa(G/e)} T_{G/e}(b/c,l/d) \\ &= d \cdot U_{G-e}(n,b,l,d,c) + c \cdot U_{G/e}(n,b,l,d,c). \end{aligned}$$

□

In particular, U_G is a polynomial in five variables n, b, l, d, and c. Comparing the formula from Lemma 5 with the definitions, we have

- $\pi_G(k) = U_G(k, k-1, 0, 1, -1) = k^{\kappa(G)}(-1)^{|V(G)|-\kappa(G)} T_G(1-k, 0)$; we are using an extra observation that for a bridge e, the number of k-colorings of G is equal to $k-1$ times the number of k-colorings of G/e.
- $c_G(k) = U_G(1, 0, k-1, -1, 1) = (-1)^{|E(G)|+\kappa(G)-|V(G)|} T_G(0, 1-k)$; we are using an extra observation that the flow on any bridge is 0, and thus if G has a bridge, then it has no nowhere-zero flows.
- For a connected graph G, $R_G(p) = U_G(1, 1-p, 1, p, 1-p) = p^{|E(G)|+1-|V(G)|}(1-p)^{|V(G)|-1} T_G(1, 1/p)$.

6 Properties of Tutte polynomial

Tutte polynomial is multiplicative in the following sense.

Lemma 6. *If G_1 and G_2 intersect in at most one vertex, then $T_{G_1 \cup G_2} = T_{G_1} T_{G_2}$.*

Proof. Note that loops and bridges of G_2 correspond exactly to loops and bridges of $G_1 \cup G_2$ belonging to $E(G_2)$. Using Lemma 4, we can then prove the formula by induction on $|E(G_2)|$. E.g., if $e \in E(G_2)$ is neither a loop nor a bridge, then

$$\begin{aligned} T_{G_1 \cup G_2} &= T_{(G_1 \cup G_2)-e} + T_{(G_1 \cup G_2)/e} = T_{G_1 \cup (G_2 - e)} + T_{G_1 \cup (G_2/e)} \\ &= T_{G_1} T_{G_2 - e} + T_{G_1} T_{G_2/e} = T_{G_1}(T_{G_2-e} + T_{G_2/e}) = T_{G_1} T_{G_2}. \end{aligned}$$

□

Hence, Tutte polynomial of a graph is equal to the product of Tutte polynomials of its 2-connected blocks (and in particular two graphs with the same blocks have the same Tutte polynomial).

The values of Tutte's polynomial in some points have natural combinatorial interpretation:

- $T_G(2,2) = 2^{|E(G)|}$, since $(2-1)^{r_G(A)}(2-1)^{c_G(A)} = 1$ for every $A \subseteq V(G)$.
- $T_G(2,1) =$ number of acyclic subgraphs of G, since $(2-1)^{r_G(A)}(1-1)^{c_G(A)} = 0$ unless $c_G(A) = 0$, in which case it is equal to 1; and $c_G(A) = 0$ iff $\kappa_G(A) = |V(G)| - |A|$, i.e., iff A is the edge set of a forest.
- If G is connected, then $T_G(1,2) =$ number of connected subgraphs of G, since we count the sets $A \subseteq E(G)$ such that $r_G(A) = 0$, i.e., $\kappa_G(A) = \kappa(G) = 1$.
- If G is connected, then $T_G(1,1) =$ number of spanning trees of G, since we count the sets $A \subseteq E(G)$ such that $r_G(A) = c_G(A) = 0$.

Exercise 7. *Show that $T_G(2,0) =$ the number of acyclic orientations of G and $T_G(0,2) =$ the number of strongly connected orientations of G.*

As Tutte polynomial allows evaluating the chromatic polynomial and thus determining the number of 3-colorings, computing Tutte polynomial is hard even for planar graphs. In fact, even determining the value of $T_G(x,y)$ for most values x and y is hard, with the following exceptions: In polynomial time, one can compute $T_G(x, 1/(x-1)+1)$ for any $x \neq 1$, $T_G(1,1)$, $T_G(-1,-1)$, $T_G(0,-1)$, and $T_G(-1,0)$. Moreover, if G is planar, it is possible to compute in polynomial time $T_G(x, 2/(x-1)+1)$ for any $x \neq 1$.

Finally, let us mention the duality for Tutte polynomial. For a connected plane graph G, deleting a non-bridge edge in G results in contracting the corresponding edge in the dual $G^\star$, contracting a non-loop edge results in deleting the corresponding edge in the dual, and bridges in G correspond to loops in $G^\star$ and vice versa. Hence, induction using Lemma 4 gives the following claim.

Lemma 8. *If G is a connected plane graph, then $T_G(x, y) = T_{G^\star}(y, x)$.*

Exercise 9. *Prove that if G is a connected plane graph, then $\pi_G(k) = kC_{G^\star}(k)$. Observe that for a plane 3-regular graph H, we have $C_H(\mathbb{Z}_{\not 2}{}^2) =$ the number of 3-edge-colorings of H, and use this to argue that the following claim is equivalent to the Four Color Theorem: Every 2-edge-connected 3-regular planar graph is 3-edge-colorable.*