1. Let G be a 3-regular graph. Show that if G has exactly one 3-edge-coloring (up to renaming of colors), then G has a Hamiltonian cycle (a cycle containing all vertices of G).

2. Let G be a planar graph and let φ be a 5-coloring of G. For distinct colors i and j, let e_{ij} denote the number of edges between vertices of colors i and j, and let n_i denote the number of vertices of color i. Show that there exist distinct colors i and j such that

$$n_i + n_j - e_{ij} \geq \tfrac{1}{10}|V(G)|.$$

Hint: Express $\sum_{1\le i<j\le 5}(n_i+n_j-e_{ij})$ in terms of the number of vertices and edges of G, and apply Euler's formula.

3. Use the previous exercise to show that there exists a constant $c > 1$ such that every planar graph G has at least $c^{|V(G)|}$ different 5-colorings. Hint: Consider Kempe chains for colors i and j.

4. Let $G_1, \ldots, G_k$ be graphs such that $\chi(G_i) = i$ for $i \in \{1, \ldots, k\}$. Let G be the graph obtained from the disjoint union of $G_1, \ldots, G_k$ by, for each k-tuple $(v_1, \ldots, v_k) \in V(G_1) \times \cdots \times V(G_k)$ of their vertices, adding a vertex adjacent exactly to v_1, v_2, ..., and v_k. Show that

   - $\chi(G) = k + 1$, and
   - if $G_1, \ldots, G_k$ are triangle-free, then G is triangle-free.

5. Show that for every positive integer k, there exists a function f_k such that the following claim holds: Every graph G that does not contain $K_{1,k}$ as an induced subgraph has chromatic number at most $f_k(\omega(G))$. Hint: Show that G actually must have bounded maximum degree.

6. Let I be a set of intervals of real numbers, and let G be the graph with vertex set I and with two vertices adjacent if and only if the corresponding intervals intersect. Show that G is chordal.