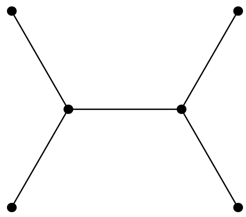
1. Let T be the following tree:



   Show that $K_{1,4} \preceq_m G$ if and only if $K_{1,4} \preceq_t G$ or $T \preceq_t G$.

2. Show that for every graph H, there exists a 3-regular graph G such that $H \preceq_m G$.

3. Show that for every graph H of maximum degree at least four, there exists a graph G such that $H \preceq_m G$ but $H \not\preceq_t G$.

4. For a graph G, let G^+ denote the graph obtained from G by adding a new vertex adjacent to all vertices of G. Let H be another graph, and let $H_1, H_2, \ldots, H_n$ be all graphs that can be obtained from H by deleting a single vertex. Show that $H \not\preceq_m G^+$ if and only $H_1, H_2, \ldots, H_n \not\preceq_m G$.

5. A graph G is *outerplanar* if G can be drawn in the plane (without crossings) so that all vertices of G are incident with the outer face. Show (using Wagner's theorem) that a graph G is outerplanar if and only if $K_4, K_{2,3} \not\preceq_m G$.

6. Let us say that a graph G is *dense* if $|V(G)| \geq 4$ and $|E(G)| \geq 2|V(G)| - 2$. Let G_0 be a minor-minimal dense graph, i.e., the graph G_0 is dense and no proper minor of G_0 is dense.

   - Show that $|E(G_0)| = 2|V(G_0)| - 2$ and $\delta(G_0) \leq 3$. Hint: What happens when you delete an edge?
   - Show that if $|V(G_0)| \geq 5$, then $\delta(G_0) = 3$. Hint: What happens when you delete a vertex of degree at most two?
   - Show that if $|V(G_0)| \geq 5$, then every edge of G_0 is contained in at least two triangles. Hint: What happens when you contract an edge?
   - Show that for every vertex $v \in V(G_0)$ of degree three, the neighbors of v induce a clique. Hint: Use the previous point.

   Using these observations, show that every dense graph contains K_4 as a minor.