1. Use (ordinary) generating functions and Lagrange inversion to find a formula for the number of rooted trees with n vertices where each vertex has at most three children (the order of children matters).

2. Let b_n denote the number of ways how to partition the set $\{1, \ldots, n\}$ to (any number) of non-empty subsets. E.g., $b_3 = 5$ counts the partitions

   - $\{1\}$, $\{2\}$, $\{3\}$;
   - $\{1, 2\}$, $\{3\}$;
   - $\{1, 3\}$, $\{2\}$;
   - $\{1\}, \{2, 3\}$; and
   - $\{1, 2, 3\}$.

   Show that the exponential generating function $B(x) = \sum_{n \geq 0} b_n \cdot \frac{x^n}{n!}$ satisfies $B(x) = e^{e^x - 1}$. Hint: b_n is also the number of graphs with vertex set $\{1, \ldots, n\}$ whose components are cliques.

3. Let i_n denote the number of involutions of the set $\{1, \ldots, n\}$, i.e., permutations π of this set such that $\pi^2 = \text{id}$. E.g., $i_3 = 4$ counts the permutations

   - $\pi(1) = 1$, $\pi(2) = 2$, $\pi(3) = 3$;
   - $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 3$;
   - $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 1$; and
   - $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 2$.

   Determine an explicit formula for the exponential generating function $I(x) = \sum_{n \geq 0} i_n \cdot \frac{x^n}{n!}$. Hint: Consider the directed graph with edges $(x, \pi(x))$ for $x \in \{1, \ldots, n\}$.

4. Let p_n be the number of all permutations of the set $\{1, \ldots, n\}$ and let c_n be the number of directed cycles with vertex set $\{1, \ldots, n\}$. Let $P(x)$ and $C(x)$ be the corresponding generating functions. Show that $P(x) = \frac{1}{1-x}$ and use the exponential formula "in reverse" to determine an explicit formula for $C(x)$.

5. Let t_n be the number of rooted trees with n vertices such that each non-leaf vertex has exactly two children, where the order of the children *does not* matter (to clarify, we define $t_0 = 0$), and let $T(x) = \sum_{n \geq 0} t_n x^n$. Show that this generating function satisfies $T(x) = x(1 + \frac{1}{2}(T^2(x) + T(x^2)))$.