

1. Show that every 3-regular Hamiltonian graph is 3-edge-colorable.
2. Let G be the graph whose vertices are the subsets of $\{1, \dots, 5\}$ of size two, with two vertices adjacent iff the corresponding subsets are disjoint. Show that G is isomorphic to the Petersen graph. Use this to argue that for any paths $v_1v_2v_3v_4$ and $v'_1v'_2v'_3v'_4$ in the Petersen graph, there exists an automorphism φ of the Petersen graph such that $\varphi(v_1) = v'_1$, $\varphi(v_2) = v'_2$, $\varphi(v_3) = v'_3$, and $\varphi(v_4) = v'_4$.
3. Show that the Petersen graph only contains cycles of lengths 5, 6, 8, and 9, and in particular it is not Hamiltonian. Use this to prove that the Petersen graph is not 3-edge-colorable.
4. Show that if each two non-adjacent vertices x and y of a graph G satisfy $\deg x + \deg y \geq |V(G)| - 1$, then G has a Hamiltonian path. Also show that the lower bound cannot be decreased to $|V(G)| - 2$.
5. Show that for every positive integer k , there exists a k -connected graph G_k with at least three vertices and $\alpha(G_k) = k + 1$ such that G_k is not Hamiltonian.
6. Let G be a k -connected graph with at least three vertices. Show that if $\alpha(G) \leq k$, then G is Hamiltonian. Hint: Let C be a cycle in G , let v be a vertex outside of C , and let P_1, P_2, \dots, P_k be paths in G from v to $V(C)$ such that the paths intersect only in v and intersect $V(C)$ only in their endpoints v_1, \dots, v_k . Let us go around the cycle C (in either of the two possible directions) and for $i \in \{1, \dots, k\}$, let u_i be the vertex immediately preceding v_i in the cycle. Show that if either
 - there exists $i \in \{1, \dots, k\}$ such that $u_iv \in E(G)$, or
 - there exist distinct $i, j \in \{1, \dots, k\}$ such that $u_iu_j \in E(G)$,
 then G contains a cycle longer than C .