1. Determine the probability distributions corresponding to the random graphs $G(3, 2/3)$ and $G'(3, 2)$; that is, for each graph H with vertex set $\{1, 2, 3\}$, determine the probability that $G(3, 2/3) = H$ and that $G'(3, 2) = H$.

2. Let $n \geq 1$ and $m \geq 0$ be integers such that $m \leq \binom{n}{2}$. Consider the following random process: We start with n isolated vertices labelled 1, 2, …, n. For $i = 1, 2, \ldots, m$, we choose a pair $\{u_i, v_i\}$ uniformly at random among the $\binom{n}{2} - (i-1)$ pairs of vertices different from $\{u_1, v_1\}$, …, $\{u_{i-1}, v_{i-1}\}$. Show that the random graph with vertex set $\{1, \ldots, n\}$ and with edge set $\{\{u_1, v_1\}, \ldots, \{u_m, v_m\}\}$ is equal to $G'(n, m)$; that is, for every graph H with vertex set $\{1, \ldots, n\}$, the probability that $G'(n, m) = H$ is the same as the probability that the graph arising from this process is equal to H.

3. Let $n \geq 4$ and $m \geq 6$ be integers such that $m \leq \binom{n}{2}$, and let $p = \frac{m}{\binom{n}{2}}$. Compute and compare the expected number of cliques of size 4 in $G(n, p)$ and in $G'(n, m)$. Hint: Make use of the equality

$$\frac{\binom{M-k}{m-k}}{\binom{M}{m}} = \prod_{i=0}^{k-1} \frac{m-i}{M-i} = \Big(\frac{m}{M}\Big)^k \prod_{i=0}^{k-1} \Big(1 - \frac{i(M-m)}{m(M-i)}\Big).$$

4. Consider the following random process. We start with the value 0. In each step, we increase the value by 1 with probability 1/3 and decrease it by 1 with probability 2/3. The process stops once the value becomes negative. Show that the probability that this process runs for at least $2s$ steps is at most

$$2^{-2H(1/2\|2/3)s} = 1.125^{-s}.$$

Use this to show that if we run this process n times, the probability that it always runs for at most $O(\log n)$ steps goes to 1 as n increases. How does this relate to the content of the lecture?

5. Let $c > 0$ be a real number and let d be a non-negative integer. Show that the probability that a fixed vertex of $G(n+1, c/n)$ has degree exactly d goes to

$$\frac{c^d e^{-c}}{d!}$$

as n increases.

For the last two exercises, you can use the following facts. For real numbers $p, q \in [0, 1]$, we define

$$H(q\|p) = q \log_2 \frac{q}{p} + (1-q) \log_2 \frac{1-q}{1-p}.$$

As a remark, you can compare this to the definition of the entropy function. More importantly, note that $H(q\|p) \geq 0$, with equality if and only if $p = q$. For all integers $k \geq 0$ and $n \geq k$ such that $\frac{k}{n} \leq p$, the following inequality holds:

$$\frac{1}{n+1} 2^{-H(k/n\|p)\cdot n} \leq \binom{n}{k} p^k (1-p)^{n-k} \leq \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \leq 2^{-H(k/n\|p)\cdot n}.$$

If you are interested, as an exercise for what we learned in Lesson 3, you can try to prove this inequality. Moreover, for every integer $d \leq n$ and every $p \in [0,1]$, we have

$$\frac{(np)^d}{d!} \cdot (1 - d^2/n) \cdot (1-p)^n \leq \binom{n}{d} p^d (1-p)^{n-d} \leq \frac{(np)^d e^{-p(n-d)}}{d!}.$$