1. Let $R(\alpha, \omega)$ be the smallest integer n such that every graph with at least n vertices contains an independent set of size α or a clique of size ω. Determine the values $R(\alpha, 1)$, $R(\alpha, 2)$, $R(1, \omega)$, and $R(2, \omega)$, and show that for $\alpha, \omega \geq 2$, we have

$$R(\alpha, \omega) \leq R(\alpha - 1, \omega) + R(\alpha, \omega - 1).$$

2. Use this to show that

$$R(\alpha, \omega) \leq \binom{\alpha + \omega - 2}{\alpha - 1}$$

holds for all positive integers α and ω. What bound does this give you for $R_2(k)$?

3. Generalize the idea from the previous two exercises to give a bound on $R_c(k)$ for any number $c \geq 3$ of colors.

4. Let $\varphi : \binom{V}{3} \to \{1, \ldots, c\}$ be a c-coloring of 3-tuples of vertices from some set V. Let us run the following procedure as long as there are any vertices left: We choose a vertex v arbitrarily, and let $X \subseteq V \setminus \{v\}$ be the largest set such that φ assigns the same color to all triples $\{x, y, v\}$ for distinct $x, y \in X$. Delete all the vertices not in X, and repeat. Show that for every positive integer m, if the initial size of V is at least $1 + R_c(1 + R_c(1 + R_c(\ldots(1)\ldots)))$ (nested m times), then this procedure will run for at least m rounds.

5. Use this to prove that for all integers $c, k \geq 2$, there exists an integer n such that $n \to (k)^3_c$ (i.e., that Theorem 55 from the lecture notes holds for $s = 3$).

6. Show that for every positive integer k, there exists n such that any 2-coloring φ of the edges of $K_{n,n}$ contains a monochromatic subgraph isomorphic to $K_{k,k}$. Hint: Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be the vertices forming the two parts of $K_{n,n}$. Consider the complete graph with vertex set $v_1, \ldots, v_n$ and with each edge $v_i v_j$ (where $i < j$) receiving the color $(\varphi(x_i y_j), \varphi(x_j y_i))$.

7. Let G be the infinite complete bipartite graph with parts formed by vertices $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ and let us color each edge $x_i y_j$ blue if $i \leq j$ and red otherwise. Show that G contains a monochromatic subgraph isomorphic to $K_{k,k}$ for arbitrarily large finite k, but there are no infinite sets $X \subseteq \{x_1, x_2, \ldots\}$ and $Y \subseteq \{y_1, y_2, \ldots\}$ such that all edges xy with $x \in X$ and $y \in Y$ have the same color.