1. Determine $\mathrm{ex}(K_2; n)$, $\mathrm{ex}(2K_2; n)$, and $\mathrm{ex}(K_{1,2}; n)$.

2. Find a graph G with the following property: Adding any possible edge to G creates a triangle, but $|E(G)| < \mathrm{ex}(K_3; |V(G)|)$.

3. Show that

   - if $F_1 \subseteq F_2$, then $\mathrm{ex}(F_1; n) \leq \mathrm{ex}(F_2; n)$ holds for every positive integer n, and that
   - for every graph F and for all integers $n_2 \geq n_1 \geq 2$,

   $$\frac{\mathrm{ex}(F; n_1)}{\binom{n_1}{2}} \geq \frac{\mathrm{ex}(F; n_2)}{\binom{n_2}{2}}.$$

   Hint: Let G be a graph with n_2 vertices and $\mathrm{ex}(F; n_2)$ edges such that $F \not\subseteq G$. Double-count the number of pairs (e, X), where X is a subset of $V(G)$ of size n_1 and e is an edge of $G[X]$.

4. By double-counting the number of tuples $(v, u_1, u_2, \ldots, u_a)$ such that u_1, $\ldots$, u_a are (not necessarily distinct) neighbors of v, show that $\mathrm{ex}(K_{a,b}; n) = O(n^{2-1/a})$ holds for all positive integers a and b. Use this to show that $\mathrm{ex}(F; n) = O(n^{2-1/\lfloor |V(F)|/2 \rfloor})$ holds for every bipartite graph F.

5. Let V be a set of size n and let k, r, and λ be non-negative integers. A system $\mathcal{B} \subseteq \binom{V}{k}$ consisting of b k-element subsets of V (called *blocks*) is a (n, b, r, k, λ)-*design* if

   - each element of V is contained in exactly r blocks, and
   - any pair of distinct elements of V is contained in exactly λ blocks.

   Show that the system $\binom{V}{k}$ of all k-element subsets of V is a (n, b, r, k, λ)-design for some parameters n, b, r, k, and λ, and determine the values of these parameters.

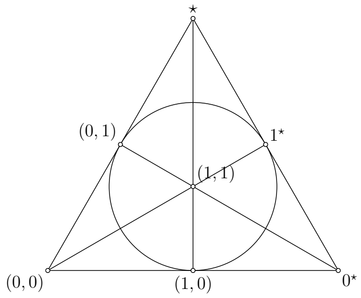
6. Let $\mathbb{F}$ be a finite field of size q, let

   $$V = \{(x, y) : x, y \in \mathbb{F}\} \cup \{x^\star : x \in \mathbb{F}\} \cup \{\star\},$$

   and let $\mathcal{P}_\mathbb{F} \subseteq \binom{V}{q+1}$ be the system consisting of the following sets:

   - $\{(x, ax + b) : x \in \mathbb{F}\} \cup \{a^\star\}$ for all $a, b \in \{F\}$,
   - $\{(a, y) : y \in \mathbb{F}\} \cup \{\star\}$ for all $a \in \{F\}$, and
   - $\{a^\star : a \in \mathbb{F}\} \cup \{\star\}$.

   For example, $\mathcal{P}_{\mathbb{Z}_2}$ consists of the sets of points joined by a straight line or a circle in the following picture:



Show that $\mathcal{P}_{\mathbb{F}}$ is a $(q^2+q+1, q^2+q+1, q+1, q+1, 1)$-design. Remark: $\mathcal{P}_{\mathbb{F}}$ is called the *finite projective plane over* $\mathbb{F}$.

7. Let $\mathcal{B} \subseteq \binom{V}{k}$ be a (n, b, r, k, λ)-design. By double-counting

   - the pairs (v, B) such that $v \in V$, $B \in \mathcal{B}$, and $v \in B$, and
   - the pairs (D, B) such that $D \in \binom{V}{2}$, $B \in \mathcal{B}$, and $D \subseteq B$,

   show that
$$b = \frac{rn}{k} = \frac{\lambda n(n-1)}{k(k-1)}.$$