1. Consider the message space $\{a, b, c, d\}$, where $P[a] = 0.6$, $P[b] = P[c] = 0.15$, and $P[d] = 0.1$. Find a prefix-free binary encoding of this space with minimum average length. What is the entropy of this message space?

2. Let M_1 and M_2 be message spaces, and let $M = M_1 \times M_2$ be the message space of messages (m_1, m_2), where m_1 is sampled from M_1 and m_2 is independently sampled from M_2 (thus, $\Pr[(m_1, m_2)] = \Pr[m_1] \cdot \Pr[m_2]$). Show that $H(M) = H(M_1) + H(M_2)$.

3. Consider a text consisting of n letters, where each letter is `a` with probability 30%, `b` with probability 50%, or `c` with probability 20% (and the letters are independent). What can you say about the minimum possible average length of a prefix-free binary encoding of this text?

4. You are playing the game of "guess a word" with your friend: He chooses a word, and you need to ask him a series of yes / no questions to determine which word he chose. You are very good at this game, and so for any set S of words, you can formulate a question which has positive answer exactly for the words belonging to S. You also know for each admissible word w the probability p_w that your friend will choose this word. What can you say about the smallest possible number of questions you will on average need to determine the word?

5. Let M be a message space with at least two elements such that all elements of M have non-zero probability, let f be a prefix-free binary encoding of M with minimum possible average length α, and let $\ell = \max\{|f(m)| : m \in M\}$.

   - Show that there exist two distinct messages $a, b \in M$ such that $|f(a)| = |f(b)| = \ell$, $f(a)$ and $f(b)$ differ exactly in the last bit, and every $m \in M \setminus \{a, b\}$ satisfies $\Pr[m] \geq \Pr[a]$ and $\Pr[m] \geq \Pr[b]$.
   - Let M' be the message space obtained from M by replacing the messages a and b by a single message q such that $\Pr[q] = \Pr[a] + \Pr[b]$, and let α' be the minimum possible average length of a prefix-free encoding of M'. Show that $\alpha = \alpha' + \Pr[a] + \Pr[b]$.
   - Use these observations to design an algorithm to find a prefix-free encoding of M with minimum possible average length.

6. For a positive integer k, let M_k be the message space containing for every $i \in \{0, \ldots, k-1\}$ exactly 2^i distinct messages with probability $\frac{1}{k2^i}$.

   - What is the entropy of this message space?
   - Show that M_k has a (not prefix-free) encoding with average length $(k-1)/2$.
   - What does this tell you about the difference between average lengths of prefix-free and non-prefix-free encodings?