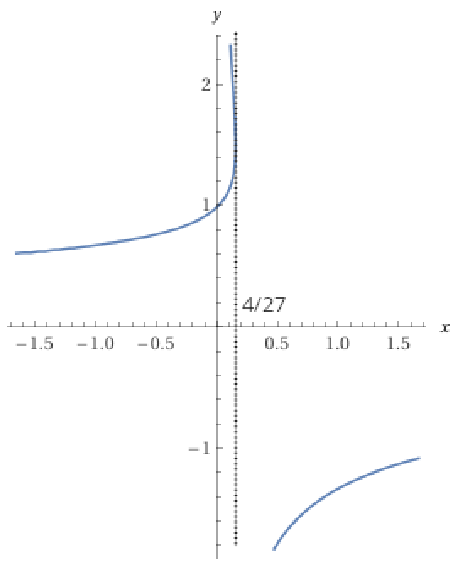
1. Show that
$$\binom{-n}{k} = (-1)^k \binom{k+n-1}{n-1}$$
holds for all integers $n \geq 1$ and $k \geq 0$.

2. Suppose that the sequence $d_0, d_1, d_2, d_3, \ldots$, where $d_0 = 1$, $d_1 = 5$, and $d_2 = 17$, satisfies the recurrence $d_n = 7d_{n-1} - 16d_{n-2} + 12d_{n-3}$ for $n \geq 3$. Find the generating function of this sequence and use it to determine an explicit formula for d_n.

3. Ternary trees are recursively defined as follows: A *ternary tree* either is the empty tree `nil`, or consists of the root vertex, the left son, the middle son, and the right son, where all sons are also ternary trees. The number of vertices of `nil` is 0, and the number of vertices of a non-empty ternary tree with left son l, middle son m, and right son r is $1+n_l+n_m+n_r$, where n_l, n_m, and n_r are the numbers of vertices of l, m, and r, respectively. Let t_n be the number of ternary trees with n vertices and let $T(x)$ be the generating function for this sequence. Show that $T(x) = 1 + xT^3(x)$.

4. Here is the plot of points (x, y) such that $y = 1 + xy^3$.



Use the information from this plot to determine the radius of convergence of $T(x)$. What does this tell you about t_n?

5. Suppose $A(x) = \sum_{n\geq 0}^{\infty} a_n x^n$ has non-zero but finite radius of convergence R, and that $a_n \geq 0$ for all n. Show that for $n \geq 10$, we have
$$a_n \leq 3A(R \cdot (1 - 1/n)) \cdot (1/R)^n$$

(it can be useful to know that for $n \geq 10$, we have $(1 - 1/n)^n \geq 1/3$). What upper bound does this give for t_n?

6. Using a different approach, it is actually possible to show that

$$t_n = \frac{1}{n}\binom{3n}{n-1}$$

for every $n \geq 1$. What bounds on t_n does the entropy estimate of the binomial coefficients give?