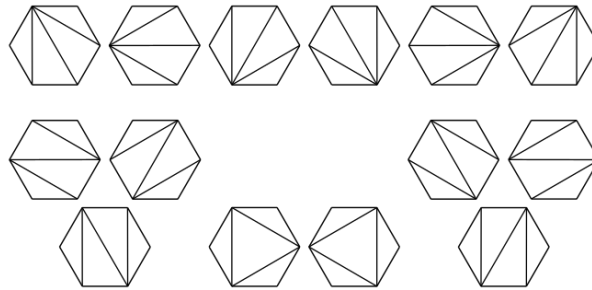


- For  $n \geq 1$ , let  $p_n$  be the number of non-decreasing sequences of integers  $x_1, \dots, x_n$  such that  $x_1 = 1$ ,  $x_n = n$ , and  $x_i \leq i$  for every  $i \in \{2, \dots, n-1\}$ . For instance,  $p_4 = 5$  counts the following sequences:  $1, 1, 1, 4$ ;  $1, 1, 2, 4$ ;  $1, 1, 3, 4$ ;  $1, 2, 2, 4$ ; and  $1, 2, 3, 4$ . Show that  $t_n$  is equal to the number of correctly matched strings of opening and closing brackets of length  $2n - 2$  by finding a bijection.
- For  $n \geq 3$ , let  $q_n$  be the number of ways how we can cut a convex  $n$ -gon into triangles by  $n - 3$  non-crossing line segments. The following picture illustrates all possibilities contributing to  $q_6 = 14$ :



Show that  $q_n$  is equal to the number of correctly matched strings of opening and closing brackets of length  $2n - 4$  by finding a bijection.

- For a positive integer  $m$ , let  $\mathcal{S}_m$  be the combinatorial class of strings composed a single letter such that the length of the string is divisible by  $m$  (with the size being defined as the length of the string). Find an explicit formula for the generating function of  $\mathcal{S}_m$ .
- For a non-negative integer  $n$ , let  $p_n$  be the number of ways one can pay  $n$  CZK using (any number of) 1, 2, and 5 CZK coins. Show that  $p_0, p_1, p_2, \dots$  is equal to the counting sequence of  $\mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_5$ , and use this observation to find an explicit formula for the generating function  $P(x)$  of this sequence. What is the radius of convergence of  $P(x)$ , and what bound does this give for the sequence  $p_0, p_1, \dots$ ?
- Show that there exist complex numbers  $r_1, \dots, r_k$  such that  $|r_1| = \dots = |r_k| = 1$ , positive integers  $d_1, \dots, d_k \leq 2$ , and polynomials  $p, p_1, \dots, p_k$  such that

$$P(x) = \frac{p(x)}{(1-x)^3} + \sum_{i=1}^k \frac{p_i(x)}{(1-r_i x)^{d_i}}.$$

Determine the value  $p(1)$  (hint: multiply this expression by  $(1-x)^3$ , and simplify the left-hand side using the explicit formula for  $P(x)$ ).

6. In the following lecture, we will see that for every positive integer  $k$ ,  $(1 - x)^{-k}$  is the generating function of a sequence  $q_0, q_1, q_2, \dots$  such that  $q_n = \frac{n^{k-1}}{(k-1)!} + O(n^{k-2})$ . Use this fact to show that  $p_n = \frac{n^2}{20} + O(n)$ .