

1. Show that if G is an n -vertex graph of maximum degree at most two, then G is not an a -expander for any $a > \frac{4}{n-1}$.
2. Let a be a positive real number. Show that if an n -vertex graph G is an a -expander, then for every set $X \subseteq V(G)$ of size less than $\frac{an}{4}$, the graph $G - X$ has a component of size at least $n - (1 + 1/a)|X|$. Hint: Consider the union S of the vertex sets of the components of $G - X$ except for the largest one. If $|S| > n/2$, instead show that we can choose some components of $G - X$ so that the union S' of their vertex sets satisfies $n/4 \leq |S'| \leq n/2$.
3. Let L be the Laplacian matrix of an n -vertex graph G , let $v_1 = \frac{1}{\sqrt{n}}j$, v_2, \dots, v_n be an orthonormal basis formed by eigenvectors of L , and let $\lambda_1 = 0, \lambda_2, \dots, \lambda_n$ be the corresponding eigenvalues. Show that G is connected if and only if the eigenvalues $\lambda_2, \dots, \lambda_n$ are non-zero. Hint: Modify the proof of the observation that all eigenvalues of the Laplacian matrix of a graph are non-negative to show that if $Lv = 0$ for a non-zero vector v , then there exists a component C of G such that all entries of v corresponding to the vertices of C have the same value.
4. Let G be a d -regular graph, let L be its Laplacian matrix, and let λ be an eigenvalue of L . Show that $\lambda \leq 2d$, and that $2d$ is an eigenvalue of L if and only if G is bipartite.
5. Let G be a one-sided (n, n, d, β, γ) -expander of maximum degree at most d , and let A and B be the parts of G . Let M_1, \dots, M_d be pairwise edge-disjoint matchings in G such that $E(G) = E(M_1) \cup \dots \cup E(M_d)$ (as a bonus task, you can try to show that such a system of matchings exists in every bipartite graph of maximum degree at most d).

Suppose we are given real numbers t_1, \dots, t_{2n} and we run the following algorithm: Initially, we assign these numbers to distinct vertices of G arbitrarily. We then perform d rounds. In the r -th round, for each edge $ab \in E(M_r)$, where $a \in A$ and $b \in B$, if the number currently assigned to a is greater than the one assigned to b , we exchange the two numbers.

Note that this algorithm can be performed in time $O(d)$ using $O(n)$ processors in parallel, since for each of the matchings, we can process the edges independently. Show that after we run this algorithm, for every positive integer $k \leq (d - \beta + 1)(\gamma n - 1)$, all but at most $\frac{1}{d - \beta + 1}k$ of the k largest elements of the set $\{t_1, \dots, t_{2n}\}$ are assigned to vertices belonging to B . Hint: For contradiction, consider a set S of $\lfloor \frac{1}{d - \beta + 1}k \rfloor + 1$ vertices of A to which some of the k largest elements would be assigned. What can we say about the numbers assigned to the vertices in $N_G(S)$?