KG1 exam requirements

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You should be able to solve all homework and tutorial problems (and their small modifications), with the following exceptions:

- Exercises 5 and 6 from Tutorial 2.
- Problem 3 from Homework 4.
- Exercises 5 and 6 from Tutorial 4.
- Exercises 4, 5, and 6 from Tutorial 6.
- Exercises 5, 6, and 7 from Tutorial 7.
- Exercises 4 and 5 from Tutorial 8.
- Problem 2 from Homework 9.
- Exercise 4 from Tutorial 9.

Part I

Introduction to combinatorics

Lesson 1

Goals of combinatorial counting and introduction to generating functions

- Understand the possible kinds of solutions to a combinatorial problem: exact formula, recurrence, estimates (asymptotically precise, Θ, even less precise), bijections, generating functions.
- Be able to find recurrences for simple combinatorial problems and to use recurrences to prove upper and lower bounds.
- Be able to find bijections between simple combinatorial problems.
- Understand the definition of a generating function, be able to determine generating functions of simple sequences and to determine the coefficients of rational generating function.
- Be able to determine the generating function of a sequence based on a linear recurrence.
- Be able to use generating functions to find an explicit formula for a sequence given by linear recurrence.

Lesson 2

Theory underlying generating functions and operations on generating functions

- Be able to prove that the formulas derived using the generating function method are actually valid, e.g., that if we apply it to a sequence defined by a linear recurrence, then the values produced by the resulting formula are guaranteed to be equal to the terms of the sequence. Understand the role of the Safety Theorem in this context.
- Understand the definitions of a combinatorial class and its counting sequence and generating function.
- Be able to use the Safety Theorem to determine the radius of convergence of a sequence given by its generating function and to correctly state exponential upper and lower bounds for the terms of the sequence based on the radius of convergence. Be able to show how these bounds follow from the definition of the radius of convergence. Understand the limitations of these bounds, be able to give an example where the coefficients differ from $(1/R)^n$ by more than just a constant factor.
- Understand the operation of product of combinatorial classes. Be able to state and prove its effect on the generating functions. Be able to use this interpretation to determine generating functions for simple combinatorial problems, e.g., number of correctly matched strings of brackets of given length.

Lesson 3

Generalized binomial formula. Estimates from generating functions. Entropy and binomial coefficient estimates.

- Know the definition of generalized binomial coefficients and be able to simplify the resulting expressions.
- Know the statement of generalized binomial formula and be able to use it to determine the coefficients of generating functions similar to the one arising in counting the number of correctly matched strings of brackets.
- Know the definition of Catalan numbers and be able to use them to express solutions to combinatorial problems (e.g., by finding bijections to "standard" problems of this form, such as the number of correctly matched strings of brackets, or through generating functions).
- Be able to provide bounds on coefficients of a generating function A (for a non-negative sequence) based on values of $\frac{A(x)}{x^n}$, and be able to prove the validity of such a bound.
- Know and be able to apply the bounds $(n/e)^n \leq n! \leq en(n/e)^n$,

  $$\frac{1}{n+1}2^{nH(k/n)} \leq \binom{n}{k} \leq \sum_{i=0}^{k}\binom{n}{i} \leq 2^{nH(k/n)} \text{ when } k \leq n/2,$$

  and

  $$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Part II

Coding theory

Lesson 4

Information and entropy

- Be able to define the entropy of a finite probability space and compute it in simple cases, including in the case the space is a product of several spaces (i.e., the space consist of tuples of independently sampled elements). Be able to show that the entropy of the product of two probability spaces is the sum of their entropies.
- Understand the definition and motivation behind prefix-free codes, be able to explain the role of entropy in this context (Theorems 22 and 24 without proofs).
- Be able to use this theory to derive a near-optimal strategy for the "guessing game" with known distribution on possible solutions.

Lesson 5

Error-correcting codes: Definitions and bounds

- Know and understand the basic definitions relating to error-correcting codes (length, size, message length, distance, rate, relative distance), the definition of Hamming distance. Be able to determine these parameters for simple codes. Understand the relationship between the distance of the code and its usefulness in correcting / detecting errors (Observation 30).
- Understand the effect of simple operations (truncation, addition of a parity bit) on these parameters.
- Know and be able to prove Singleton bound and Hamming bound, including their asymptotic versions (for rate and relative distance).
- Know the greedy construction of a code of distance d and length n (Gilbert-Varshamov code) and be able to give a lower bound on the message length of this code and compare it with the Singleton and Hamming bounds.

Lesson 6

Linear codes. Hamming and Reed-Solomon codes.

- Know and understand the definition of a linear code, its basis and check matrix, and be able to determine the parameters of the code from the basis or the check matrix.
- Be able to explain the use of the check matrix in error detection.
- Be able to prove that the distance of a linear code is equal to the minimum number of 1-bits in its non-zero words, and to the minimum number of columns of the check matrix that sum to zero (Lemma 44).
- Be able to describe the construction of Hamming and Reed-Solomon codes and determine their parameters.

Part III

Extremal, Ramsey, and probabilistic graph theory

Lesson 7

Introduction to extremal graph theory (and double-counting arguments)

- Be able to use double-counting arguments to solve simple combinatorial problems.
- Mantel's theorem (Theorem 48) including the proof.
- Be able to state and explain Turán's and Erdős-Stone theorems.
- An upper bound on the number of edges in a C_4-free graph (Theorem 51) including the proof, and a lower-bound construction using the point-line incidence graph in $\mathbb{F}^2$.

Lesson 8

Introduction to Ramsey theory

- Be able to state and prove the finite Ramsey theorem for pairs with any fixed number of colors (Theorem 53), understand its interpretation in terms of independence and clique number of a graph.
- Lower bound on Ramsey numbers (Lemma 54) including the proof.
- Know other versions (for tuples, infinite) without proof, be able to use them to prove simple Ramsey-like statements (e.g., existence of many points in convex position in a large set of points in the plane).

Lesson 9

Introduction to random graph theory

- Be able to explain basic models of random graphs (Erdős-Rényi, configuration model) and compute simple probabilities and expectations in graphs from these models (expected number of triangles, probability that the graph from this model is equal to a given graph, probability that a fixed vertex has a given degree, etc.)
- Be able to prove simple concentration results (the value of a parameter is highly likely to be close to its expected value), e.g., Lemma 58.
- Be able to explain shortcomings of these models when applied to real-world situations (degree distribution, clustering).

Part IV

Connectivity in graphs

Lesson 10

Vertex and edge connectivity

- Know the basic definitions (cuts, k-connectedness, connectivity and the edge variants of these concepts), be able to relate the minimum degree, edge-connectivity and vertex connectivity (with proofs).
- Understand the effects of removal of an edge or a vertex on edge-connectivity or a vertex connectivity (with proofs / examples), see the first problem from the homework assignment for this lecture.
- Menger's theorem in various versions (Theorems 64, 65, 67) and their proofs using flows in networks.

Lesson 11

Matchings in bipartite graphs

- Be able to state, prove, and use Hall's theorem.
- Be able to prove that every regular bipartite graph has a perfect matching.
- Be able to explain the stable matching problem and the Gale-Shapley algorithm used to solve it, including the proof of the correctness of the algorithm.

Lesson 12

Expanders

- Know and understand the definitions of expanders and edge-expanders and the relationships between them.
- Understand basic properties of expanders, including the proof that they have logarithmic diameter.
- Know and understand the definitions of the Laplacian matrix and of spectral expanders. Be able to explain the relation between spectral expanders and edge-expanders (Theorems 79 and 80) without proofs.