1. Let $G$ be a graph with $n$ vertices and $m$ edges. For $i \in \{1,2,3\}$, let $t_i$ be the number of subsets of $V(G)$ of size three inducing a subgraph with exactly $t_i$ edges (thus, $t_3$ is the number of triangles in $G$).

   - By double-counting the number of pairs $(e,v)$ such that $e \in E(G)$, $v \in V(G)$, and the edge $e$ is *not* incident with the vertex $v$, show that $m(n-2) = t_1 + 2t_2 + 3t_3$.
   - By double-counting the number of pairs $(v,D)$ such that $v$ is a vertex of $G$ and $D$ is a set consisting of two neighbors of $v$, show that

   $$\sum_{v \in V(G)} \binom{\deg v}{2} = t_2 + 3t_3.$$

   Use these identities to show that if $G$ is triangle-free, then

   $$m = \frac{n^2}{4} - \frac{t_1 + \sum_{v \in V(G)} (\deg v - n/2)^2}{n}.$$

2. Show that for all positive integers $k$ and $d$,

   - every graph of minimum degree at least $k-1$ contains as subgraphs all trees with at most $k$ vertices, and
   - every $n$-vertex graph with more than $(d-1)n$ edges contains a subgraph of minimum degree at least $d$.

   Use this to show that $\mathrm{ex}(T;n) \leq (|V(T)|-2)n$ holds for every tree $T$ with at least two vertices. What lower bound can you provide for $\mathrm{ex}(T;n)$?

3. Recall that $\alpha(G)$ is the maximum size of an independent set in a graph $G$ and that

   $$\overline{d}(G) = \tfrac{1}{|V(G)|} \sum_{v \in V(G)} \deg v = \frac{2|E(G)|}{|V(G)|}$$

   is the average degree of $G$. Using Turán's theorem, show that

   $$\alpha(G) \geq \frac{|V(G)|}{\overline{d}(G)+1}$$

   holds for every graph $G$.