1. Show that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} x^n$$

holds for every real number x such that $|x| < 1$.

2. Let b_n be the number of ways how to divide the set $\{1, \ldots, n\}$ into (any number of) non-empty parts. For instance, $b_3 = 5$ counts the partitions

   - $\{1\}$, $\{2\}$, $\{3\}$
   - $\{1\}$, $\{2,3\}$
   - $\{1,2\}$, $\{3\}$
   - $\{1,3\}$, $\{2\}$
   - $\{1,2,3\}$

   It can be shown that

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} \cdot x^n = e^{e^x - 1}$$

   holds for every real number x. Using this fact, prove that for every non-negative integer n, we have

$$b_n \leq n \cdot \left(e^{\frac{1}{l(n)} + l(n) - 1} \right)^n ,$$

   where $l(n)$ is the unique real number such that $l(n)e^{l(n)} = n$. If you cannot quite get to this exact expression, you can submit a solution leading to any other reasonable upper bound.

3. Show that

$$\binom{ck}{k} \leq \left(\frac{c^c}{(c-1)^{c-1}} \right)^k$$

holds for all integers $c \geq 2$ and $k \geq 0$.