

1. Let  $\mathcal{B}$  be a combinatorial class and suppose that  $\mathcal{B}$  has generating function

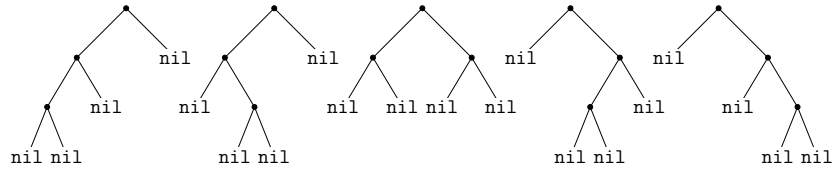
$$B(x) = \frac{1 + x^5}{1 - 3x - x^2}.$$

Determine the radius of convergence of this generating function. What bounds does this give for the counting sequence of  $\mathcal{B}$ ?

2. Let  $\mathcal{S}$  be the combinatorial class of all sums whose terms are integers greater than one (order of terms matters); thus,  $\mathcal{S}$  contains for example the empty sum “”, the single-term sum “2”, and the sums “2 + 2 + 3”, “2 + 3 + 2”, and “2 + 2 + 3”. We define the size of a sum to be its value; e.g.,  $\text{size}_{\mathcal{S}}(\text{“2 + 3 + 2”}) = 7$ . Let  $\mathcal{S}_1 = \{\text{“2”}, \text{“3”}, \dots\}$  be the combinatorial class of all single-term sums from  $\mathcal{S}$ .

- (a) Show that the generating function of  $\mathcal{S}_1$  is  $\frac{x^2}{1-x}$ .  
 (b) Show that  $\mathcal{S}$  is isomorphic to  $\{\text{“”}\} \cup \mathcal{S}_1 \times \mathcal{S}$ .  
 (c) Translate (b) to an algebraic relation for the generating function  $S(x)$  of  $\mathcal{S}$ .

3. Binary trees are recursively defined as follows: A *binary tree* either is the empty tree `nil`, or consists of the root vertex, the left son, and the right son, where both sons are also binary trees. The number of vertices of `nil` is 0, and the number of vertices of a non-empty binary tree with left son  $l$  and right son  $r$  is  $1 + n_l + n_r$ , where  $n_l$  is the number of vertices of  $l$  and  $n_r$  is the number of vertices of  $r$ . Thus, these are the binary trees with three vertices:



Show that for every non-negative integers  $n$ , the number of binary trees with  $n$  vertices is equal to the number of correctly matched strings of opening and closing brackets of length  $2n$ .