Dependent random choice

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From the first lecture:

Theorem 1. *If F is a bipartite graph with one of the parts of size a, then*

$$\mathrm{ex}(n; F) = O(n^{2-1/a}).$$

We want to generalize this to graphs in which one of the parts only contains vertices of degree at most a. Idea: Let us find a large set B of vertices such that every a vertices of B have many common neighbors. How to find such a set? Let us select several vertices at random and choose B as the set of their common neighbors. If some vertices of B had few common neighbors, we would have only a small probability that we hit all of them with the random choice. More precisely, this gives us the following lemma.

Lemma 2. *Let G be an n-vertex graph and let a, b, m, and t be positive integers. If*

$$\|G\| \geq (b + m^t n^{a-t})^{1/t} n^{2-1/t},$$

then there exists a set $B \subseteq V(G)$ of size at least b such that every a vertices of B have at least m common neighbors.

Proof. Let us select the vertices $v_1, \ldots, v_t$ uniformly independently at random, and let B_0 be the set of their common neighbors. For every vertex v, the probability that $v \in B_0$, that is, that $v_1, \ldots, v_t$ are neighbors of v, is equal to $n^{-t} \deg^t(v)$. Hence,

$$\begin{aligned} E[|B_0|] = n^{-t} \sum_{v \in V(G)} \deg^t(v) &\geq n^{1-2t} \Big(\sum_{v \in V(G)} \deg v \Big)^t \\ &> n^{1-2t} \|G\|^t \geq b + m^t n^{a-t}. \end{aligned}$$

What is the probability that an a-tuple $u_1, \ldots, u_a$ of vertices with less than m common neighbors belongs to B_0? We would have to hit these common neighbors with $v_1, \ldots, v_t$, and thus the probability is less than $m^t n^{-t}$. The expected value of the number of such a-tuples in B_0 therefore is less than

$$n^a m^t n^{-t} = m^t n^{a-t}.$$

For each such a-tuple, let us delete one of its vertices from B_0, and let B denote the resulting set. This ensures that every a-tuple of vertices of B has at least m common neighbors and

$$E[|B|] > E[|B_0|] - m^t n^{a-t} > b.$$

□

Let us first state a simple corollary.

Theorem 3. *If F is a bipartite graph such that the vertices in one of its parts have degree at most a, then*

$$\mathrm{ex}(n; F) = O(n^{2-1/a}).$$

Proof. Let G be an n-vertex graph not containing F as a subgraph. Then there is no set $B \subseteq V(G)$ of size $|F|$ such that each a vertices of B has at least $|F|$ common neighbors: Otherwise, let f be an arbitrary injective function mapping the unrestricted part of F to B. Then, let us take one by one the vertices v belonging to the part of F containing only vertices of degree at most a, and choose $f(v)$ among the common neighbors of $f(N_F(v))$ which are not yet contained in the image of f. Then $f : V(F) \to V(G)$ shows that F is a subgraph of G, which is a contradiction.

By Lemma 2 with $b = m = |F|$ and $t = a$, it follows.

$$\||G\|| < (|F| + |F|^a)^{1/a} n^{2-1/a} \leq 2|F| n^{2-1/a}.$$

□

Sometimes, it is useful to choose larger t, especially if we want to find a subgraph whose size depends on n.

Lemma 4. *For every $c \geq 2$ and a sufficiently large n, the following claim holds. If an n-vertex graph G has at least $3n^2/c$ edges, then it contains the 1-subdivision of the complete graph with $\lfloor \sqrt{n/c^3} \rfloor$ vertices.*

Proof. The existence of the 1-subdivision of K_p is implied by the presence of p vertices such that any two of them have at least $p + \binom{p}{2} \leq p^2$ common neighbors. Hence, if an n-vertex graph G does not contain the 1-subdivision of K_p, then Lemma 2 (with $a = 2$, $b = p$, $m = p^2$) implies the following inequality for every positive integer t:

$$\|G\| < (p + p^{2t}n^{2-t})^{1/t}n^{2-1/t} = (pn^{-1} + p^{2t}n^{1-t})^{1/t}n^2.$$

For $p = \lfloor\sqrt{n/c^3}\rfloor$, we have

$$\|G\| < (n^{-1/2}c^{-3/2} + nc^{-3t})^{1/t}n^2.$$

If $t < \frac{\log cn}{2\log c}$, then $n^{-1/2}c^{-3/2} < nc^{-3t}$, and thus

$$\|G\| < 2n^{1/t}c^{-3}n^2.$$

Let us set $t = \lfloor\frac{\log n}{2\log c}\rfloor$, so that for sufficiently large n we have $t^2 \geq \log n/\log(3/2)$. It follows that

$$\|G\| < 2n^{1/t}c^{-3}n^2 < 2n^{1/(t+1)}n^{1/t^2}c^{-3}n^2 \leq 3n^2/c.$$

Hence, if G has at least $3n^2/c$ edges, then it contains the 1-subdivision of the complete graph with $\lfloor\sqrt{n/c^3}\rfloor$ vertices. □

We can actually improve this result: When representing the vertex subdividing the i-th edge, we do not need to have p^2 common neighbors, it suffices to have at least i common neighbors outside of B. For this purpose, let us give a variation on Lemma 2.

Lemma 5. *Let G be a $2n$-vertex graph with at least n^2/c edges, and let $b \leq \frac{\sqrt{2n}}{4c}$ be a non-negative integer. Then there exists a set $B \subseteq V(G)$ of size b such that for every $i \geq 1$, less than i pairs of vertices of B have less than i common neighbors in $V(G) \setminus B$.*

Proof. Consider a partition of vertices of G into parts V_1 and V_2 of size n such that at least half of the edges of G has one end in V_1 and the other end in V_2 (consider a random bipartition). Let G_1 be the bipartite subgraph of G created by deleting the edges inside V_1 and inside V_2. By symmetry, we can assume $\sum_{v\in V_1}\deg^2_{G_1}(v) \leq \sum_{v\in V_2}\deg^2_{G_1}(v)$.

Let us choose vertices $v_1, v_2 \in V_1$ uniformly independently at random and let $B_0 \subseteq V_2$ be the set of their common neighbors. As in the proof of Lemma 2, we have

$$E[|B_0|] = n^{-2}\sum_{v\in V_2}\deg^2_{G_1}(v) \geq n^{-3}\Big(\sum_{v\in V_2}\deg_{G_1}(v)\Big)^2 \geq \frac{1}{4}n^{-3}\|G\|^2 \geq \frac{n}{4c^2}.$$

For a pair $T = \{x_1, x_2\} \subseteq V_2$ with $t > 0$ common neighbors in V_1, let us define $w(T) = 1/t$; note that if $x_1, x_2 \in B_0$, then $t \geq 1$, since x_1 and x_2 are adjacent to v_1 by the definition of B_0. Let W denote the set of all pairs of vertices in V_2 that have at least one common neighbor. Let $Y = \sum_{T \in \binom{B_0}{2}} w(T)$; then

$$E[Y] = \sum_{T \in W} w(T) \Pr[T \subseteq B_0] = \sum_{T \in W} w(T) \frac{(1/w(T))^2}{n^2} = n^{-2} \sum_{T \in W} w^{-1}(T).$$

Each pair of vertices of V_2 contributes the number of their common neighbors in V_1 to the last sum. Hence, we can instead express it by counting for each vertex of V_1 the number of pairs of its neighbors. Therefore,

$$\begin{aligned} E[Y] &= n^{-2} \sum_{v \in V_1} \binom{\deg_{G_1}(v)}{2} < \frac{1}{2n^2} \sum_{v \in V_1} \deg^2_{G_1}(v) \\ &\leq \frac{1}{2n^2} \sum_{v \in V_2} \deg^2_{G_1}(v) = E[|B_0|]/2. \end{aligned}$$

Therefore, we have $E[|B_0| - Y] > E[|B_0|]/2$, and thus there exists a choice of B_0 such that $|B_0| > Y + E[|B_0|]/2$. In particular, $|B_0| > E[|B_0|]/2 \geq \frac{n}{8c^2}$ and $|B_0| > Y$. Let B be a random subset of B_0 of size b. Then

$$E\Big[\sum_{T \in \binom{B}{2}} w(T)\Big] = \frac{\binom{b}{2}}{\binom{|B_0|}{2}} Y \leq \frac{b^2 |Y|}{|B_0|^2} < \frac{b^2}{|B_0|} \leq 1.$$

Therefore, there exists such a set B of size b such that any two vertices of B have a common neighbor in V_1 and $\sum_{T \in \binom{B}{2}} w(T) < 1$. For $i \geq 2$, if $T \in \binom{B}{2}$ is a pair of vertices with less than i common neighbors outside of B, then $w(T) > 1/i$, and thus $\binom{B}{2}$ contains less than i such pairs. □

Let $T_1, \ldots, T_{\binom{b}{2}}$ be the pairs of vertices of B sorted according to the number of their common neighbors outside of B. Then the vertices of T_i have at least i common neighbors outside of B, and as we argued before, this suffices to obtain the 1-subdivision of K_b in G.

Corollary 6. *Every graph with $2n$ vertices and at least n^2/c edges contains the 1-subdivision of K_b for $b = \lfloor \frac{\sqrt{2n}}{4c} \rfloor$.*

The analysis of a suitably chosen random graph shows that the dependence of b on c is asymptotically optimal.

Next, we aim to generalize Theorem 3 to all a-degenerate bipartite graphs. To this end, we need a variant of Lemma 2 with two subsets such that each a-tuple of vertices in any one of them has many common neighbors in the other subset.

Lemma 7. *Let $a, m \geq 2$ be integers and let G be an n-vertex graph with at least $2n^{2-\frac{1}{8a}}$ edges, for large enough n. Then there exist sets $B_1, B_2 \subset V(G)$ of size at least m such that for $i \in \{1,2\}$, every a-tuple of vertices of B_i has at least m common neighbors in B_{3-i}.*

Proof. Let $t = 4a$, $b = \lfloor n^{1/2} \rfloor$, $a' = \lceil 7a/2 \rceil$. Then

$$\|G\| \geq 2n^{2-\frac{1}{8a}} \geq (b + m^t n^{a'-t})^{1/t} n^{2-1/t}$$

for sufficiently large n. By Lemma 2, there exists a set $B_1 \subseteq V(G)$ of size at least $b \geq m$ such that every a'-tuple of vertices of B_1 has at least m common neighbors.

Now choose $t_1 = a' - a$ vertices T_1 from B_1 uniformly independently at random, and let B_2 be the set of their common neighbors (clearly $|B_2| \geq m$). The probability that B_2 contains an a-tuple of vertices with less than m common neighbors in B_1 is less than

$$\begin{aligned} n^a \left(\frac{m}{b}\right)^{t_1} &\leq n^a \left(\frac{2m}{n^{1/2}}\right)^{t_1} \\ &= (2m)^{t_1} n^{a-t_1/2} = (2m)^{t_1} n^{(3a-a')/2} \leq (2m)^{t_1} n^{(3-7/2)a/2} \\ &= (2m)^{t_1} n^{-a/4} \leq 1 \end{aligned}$$

for sufficiently large n. Therefore, there exists a choice of B_2 such that each a-tuple of vertices of B_2 has at least m common neighbors in B_1. Moreover, each a-tuple of vertices in B_1 can be extended to an a'-tuple by adding T_1; this a'-tuple has at least m common neighbors and by the definition all of them belong to B_2. □

Corollary 8. *If F is an a-degenerate bipartite graph, then*

$$\mathrm{ex}(n; F) = O\big(n^{2-\frac{1}{8a}}\big).$$

Proof. We apply Lemma 7 to a graph with n vertices and $\Omega(\big(n^{2-\frac{1}{8a}}\big)$ edges, with $m = |F|$, obtaining sets B_1 and B_2. Suppose $v_1, \ldots, v_{|F|}$ are vertices of F in the order such that for $i = 1, \ldots, |F|$, v_i has at most a neighbors in $\{v_1, \ldots, v_{i-1}\}$. Let $p(i) \in \{1,2\}$ be the number of the part of the bipartition of F containing v_i. Then for $i = 1, \ldots, |F|$ in order, we assign v_i to a vertex in $B_{p(i)}$ chosen as the not-yet-used common neighbor of the vertices to which we have previously assigned the preceding neighbors of v_i; such a vertex exists, since v_i has at most a preceding neighbors and the corresponding $(\leq a)$-tuple vertices of $B_{3-p(i)}$ has at least $|F|$ common neighbors in $B_{p(i)}$. □