The extremal function for bipartite graphs

Zdeněk Dvořák

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From the first lecture:

Theorem 1. *If F is bipartite and one of its parts has size a, then*

$$\mathrm{ex}(n; F) = O(n^{2-1/a}).$$

A probabilistic lower bound:

Lemma 2. *For all integers a and b such that $2 \le a \le b$ and for sufficiently large n, we have*

$$\mathrm{ex}(n; K_{a,b}) \ge \frac{1}{24} n^{2-\beta},$$

where

$$\beta = \frac{a+b-2}{ab-1}.$$

Proof. Let G_0 be a random graph on n vertices containing each edge independently with probability $p = \frac{1}{2} n^{-\beta}$. Then

$$\mathbb{E}[\|G_0\|] = p\binom{n}{2} \ge \frac{p}{3} n^2 = \frac{1}{6} n^{2-\beta}.$$

Let t be the number of appearances of $K_{a,b}$ in G_0. We have

$$\mathbb{E}[t] \le n^{a+b} p^{ab} \le \frac{1}{8} n^{2-\beta}$$

Let G be the graph obtained from G_0 by deleting an edge from every appearance of $K_{a,b}$. Then G does not contain $K_{a,b}$ as a subgraph and

$$\mathbb{E}[\|G\|] \ge \mathbb{E}[\|G_0\|] - \mathbb{E}[t] \ge \frac{1}{24} n^{2-\beta}.$$

□

Remark: $\beta = 1/a + \frac{(a-1)^2}{a(ab-1)} > 1/a$.

We now aim to prove a tight bound for $K_{3,3}$, and more generally for $K_{a,b}$ with $b \gg a$.

Let p be a prime and m a positive integer. For $x \in GF(p^m)$, let us define the norm of x as $N(x) = x \cdot x^p \cdot x^{p^2} \cdots x^{p^{m-1}}$.

Lemma 3. *The norm has the following properties.*

- $N(xy) = N(x)N(y)$ *for every* $x, y \in GF(p^m)$.
- $N(x) = 0$ *if and only if* $x = 0$.
- $N(x) \in GF(p)$ *for every* $x \in GF(p^m)$.

Proof. The first two properties are trivial. In $GF(p^m)$, we have $x^{p^m} = x$ for every $x \in GF(p^m)$, and thus $N(x)^p = N(x)$. The roots of the polynomial $y^p - y$ are exactly the elements of $GF[p]$, and thus $N(x) \in GF[p]$. □

Let us now define a graph $H_{p,m}$. The vertices of $H_{p,m}$ are the pairs (x, y) for $x \in GF(p^m)$ and $y \in GF(p) \setminus \{0\}$; the vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $N(x_1 + x_2) = y_1 y_2$. For every (x_1, y_1) and $x_2 \neq -x_1$, the neighbor (x_2, y_2) with the first element of the pair equal to x_2 is uniquely determined: It must be $y_2 = y_1/N(x_1 + x_2)$. Therefore, the graph $H_{p,m}$ is $(p^m - 1)$-regular. Furthermore, $H_{p,m}$ has $n = p^m(p-1)$ vertices and

$$\|H_{p,m}\| = \frac{1}{2}(p^m - 1)n = \frac{1}{2}n^{2-\frac{1}{m+1}} + \frac{m}{2(m+1)}n^{2-\frac{2}{m+1}} + O\big(n^{2-\frac{3}{m+1}}\big).$$

Theorem 4. *The graph* $H_{p,1}$ *does not contain* $K_{2,2}$ *as a subgraph, and thus for every* $b \geq 2$*, we have*

$$\mathrm{ex}(n; K_{2,b}) \geq \frac{1}{2}n^{3/2} + \frac{1}{4}n + O(n^{1/2})$$

for infinitely many values of n.

Proof. For $m = 1$ we have $N(x) = x$. A common neighbor (x, y) of vertices $(a_1, b_1) \neq (a_2, b_2)$ must satisfy the equations

$$\begin{aligned} x + a_1 &= b_1 y \\ x + a_2 &= b_2 y. \end{aligned}$$

Substituting for y from the first equation to the second one, we have

$$x + a_2 = \frac{b_2}{b_1}(x + a_1),$$

and thus

$$(1 - b_2/b_1)x = a_1 b_2/b_1 - a_2.$$

If $b_1 \neq b_2$, then this equation has a unique solution giving x, and x then uniquely determines y. If $b_1 = b_2$, then $a_1 \neq a_2$ and the right-hand side is non-zero, and thus the equation has no solution.

Therefore, the system has at most one solution. It follows that any two vertices have at most one common neighbor, and thus the graph does not contain $K_{2,2}$ as a subgraph. $\square$

Theorem 5. *The graph $H_{p,2}$ does not contain $K_{3,3}$ as a subgraph, and thus for every $b \geq 3$, we have*

$$\mathrm{ex}(n; K_{3,b}) \geq \frac{1}{2} n^{5/3} + \frac{1}{3} n^{4/3} + O(n)$$

for infinitely many values of n.

Proof. A common neighbor (x, y) of distinct vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3) must satisfy the following equations.

$$\begin{aligned} N(x + a_1) &= b_1 y \\ N(x + a_2) &= b_2 y \\ N(x + a_3) &= b_3 y. \end{aligned}$$

If this system has a solution, then a_1, a_2, and a_3 are pairwise distinct ($a_i = a_j$ would imply $b_i = b_j$, contradicting the assumption $(a_i, b_i) \neq (a_j, b_j)$).

Substitute for y from the first equation to the second and third one, and divide the resulting equations by $N(a_i - a_1)$ for $i = 2, 3$. We obtain the following system.

$$\begin{aligned} N(1/(x + a_1) + (a_2 - a_1)^{-1}) &= \frac{b_2}{b_1 N(a_2 - a_1)} \\ N(1/(x + a_1) + (a_3 - a_1)^{-1}) &= \frac{b_3}{b_1 N(a_3 - a_1)} \end{aligned}$$

Therefore, it suffices to prove that if $c_2 \neq c_3$, then any system of form

$$\begin{aligned} N(z + c_2) &= d_2 \\ N(z + c_3) &= d_3 \end{aligned}$$

has at most two solutions. Note that $N(z + c_i) = (z + c_i)(z + c_i)^p = (z + c_i)(z^p + c_i^p)$. Therefore, it suffices to show that if $c_2 \neq c_3$ and $c_2' \neq c_3'$, then the system of equations (with unknowns z and z')

$$\begin{aligned} (z + c_2)(z' + c_2') &= d_2 \\ (z + c_3)(z' + c_3') &= d_3 \end{aligned}$$

has at most two solutions. Indeed, subtracting the equations, we obtain

$$z' = \frac{1}{c_2 - c_3}((c_3' - c_2')z + d_2 - d_3 + c_3c_3' - c_2c_2'),$$

and thus z uniquely determines z'. Substituting to the first equation, we obtain the following quadratic equation for z:

$$\frac{c_3' - c_2'}{c_2 - c_3}z^2 + k_1 z + k_2 = 0$$

Since the coefficient at z^2 is non-zero, the equation has at most two solutions. □

For the general case, we use the following claim without proof.

Theorem 6. *In every field, the system*

$$\begin{aligned}
(z_1 - a_{1,1})(z_2 - a_{1,2}) \cdots (z_t - a_{1,t}) &= b_1 \\
(z_1 - a_{2,1})(z_2 - a_{2,2}) \cdots (z_t - a_{2,t}) &= b_2 \\
&\cdots \\
(z_1 - a_{t,1})(z_2 - a_{t,2}) \cdots (z_t - a_{t,t}) &= b_t
\end{aligned}$$

such that $a_{i,j} \neq a_{i',j}$ for every $i \neq i'$ and j, has at most $t!$ solutions.

Remark: to see why this is plausible, consider the case $b_1 = \ldots = b_t = 0$.
Similarly to Theorem 5, we can then prove the following result.

Theorem 7. *The graph $H_{p,m}$ does not contain $K_{m+1,m!+1}$ as a subgraph, and thus for every $b \geq m! + 1$ we have*

$$\mathrm{ex}(n; K_{m+1,b}) \geq \frac{1}{2}n^{2-\frac{1}{m+1}} + \frac{m}{2(m+1)}n^{2-\frac{2}{m+1}} + O\big(n^{2-\frac{3}{m+1}}\big)$$

for infinitely many values of n.