# Erdős-Stone theorem

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We will often need the following technical lemma, which enables us to get rid of vertices of small degree.

**Lemma 1.** *For every $c \geq 0$, $\varepsilon > 0$ and sufficiently large $n$, if $G$ is an $n$-vertex graph and $\|G\| \geq (c+\varepsilon)\frac{n^2}{2}$, then $G$ has an induced subgraph $G_0$ with $n_0 \geq \sqrt{\varepsilon/4} \cdot n$ vertices and minimum degree at least $(c+\varepsilon/2)n_0$.*

*Proof.* Without loss of generality, we can assume $c+\varepsilon \leq 1$, as otherwise the assumptions cannot hold for any graph $G$. Consider the induced subgraph $G_0$ obtained by the following algorithm. Initialize $G_0 := G$, and while there exists $v \in V(G_0)$ of degree less than $(c+\varepsilon/2)|G_0|$, set $G_0 := G_0 - v$.

Let $n_0 = |G_0|$. Clearly $G_0$ has minimum degree at least $(c+\varepsilon/2)n_0$. Moreover,

$$\begin{aligned}\|G\| &\leq \|G_0\| + (c+\varepsilon/2)\sum_{a=n_0+1}^{n} a\\ &\leq \binom{n_0}{2} + (c+\varepsilon/2)\sum_{a=1}^{n} a\\ &\leq \frac{n_0^2}{2} + (c+\varepsilon/2)\frac{n^2+n}{2}.\end{aligned}$$

Since $\|G\| \geq (c+\varepsilon)\frac{n^2}{2}$ and $c+\varepsilon/2 < 1$, we have

$$\begin{aligned}\frac{n_0^2}{2} &\geq \frac{\varepsilon}{2}\cdot\frac{n^2}{2} - \frac{n}{2}\\ &\geq \frac{\varepsilon}{4}\cdot\frac{n^2}{2}\end{aligned}$$

for $n \geq 4/\varepsilon$, and thus $n_0 \geq \sqrt{\varepsilon/4}\cdot n$. □

In particular, Lemma 1 implies that it suffices to prove Erdős-Stone theorem only for graphs with large minimum degree.

**Lemma 2.** *For every integer* $r \geq 1$ *and a real number* $\beta > 0$*, every* $n$*-vertex graph of minimum degree at least* $(1 - 1/r + \beta)n$ *contains* $T_{r+1}(\Omega(\log n))$ *as a subgraph.*

*Proof.* We prove the claim by induction on $r$. Note that $G$ contains $T_r(mr)$ as a subgraph for some integer $m = \Theta(\log n)$ (for $r \geq 2$ this follows by the induction hypothesis, while for $r = 1$ this is trivial); let $K$ be the vertex set of this subgraph. Without loss of generality, we can assume

- $n \gg r, 1/\beta$, since otherwise $1 = \Omega(\log n)$ and trivially $T_{r+1}(1) \subseteq G$;
- $|K| = mr \leq \frac{1}{2}\log_2 n$.

Let $U \subseteq V(G) \setminus K$ be the set of vertices with more than $(1 - 1/r + \beta/2)|K|$ neighbors in $K$. Let $q$ be the number of edges of $G$ between $K$ and $V(G) \setminus K$. Since $G$ has minimum degree at least $(1 - 1/r + \beta)n$, we have

$$q \geq |K|\big((1 - 1/r + \beta)n - |K|\big).$$

On the other hand, the vertices not belonging to $U$ have at most $(1 - 1/r + \beta/2)|K|$ neighbors in $K$, and thus

$$q \leq |U||K| + (1 - 1/r + \beta/2)n|K| = |K|\big((1 - 1/r + \beta/2)n + |U|\big).$$

Therefore $|U| \geq \frac{\beta}{2}n - |K|$, and since $|K| = \Theta(\log n)$, assuming $n$ is sufficiently large, we have $|U| \geq \frac{\beta}{3}n$.

Every vertex $u \in U$ has less than $(1/r - \beta/2)|K| \leq m - (\beta r/2) \cdot m$ non-neighbors in $K$, and thus $u$ has more than $m' = \lfloor (\beta r/2) \cdot m \rfloor$ neighbors in each part of the $r$-partite subgraph $T_r(mr)$ with vertex set $K$; hence, this $T_r(mr)$ contains a subgraph $T_r(m'r)$ with vertex set $K_u$ such that $u$ is adjacent to all vertices of $K_u$ in $G$. The number of distinct subgraphs of $T_r(m'r)$ in $T_r(mr)$ is at most $2^{mr} \leq \sqrt{n}$. Therefore, there exists such a subgraph with vertex set $Z$ such that $K_u = Z$ holds for at least $|U|/\sqrt{n} \geq \frac{\beta}{3}\sqrt{n}$ vertices $u \in U$. Since $m' = \Theta(\log n)$, for sufficiently large $n$ we have $\frac{\beta}{3}\sqrt{n} \geq m'$. Then $Z$ together with the vertices $u \in U$ such that $K_u = Z$ forms a subgraph $T_{r+1}(m'(r+1))$ in $G$. □

**Corollary 3** (Erdős-Stone)**.** *For every integer* $r \geq 1$ *and real number* $\varepsilon > 0$*, every* $n$*-vertex graph with at least* $(1 - 1/r + \varepsilon)\frac{n^2}{2}$ *edges contains* $T_{r+1}(\Omega(\log n))$ *as a subgraph.*

**Corollary 4.** *For every integer* $r \geq 1$ *and* $\varepsilon > 0$*, there exists* $c$ *such that*

$$\mathrm{ex}(n; F) \leq (1 - 1/r + \varepsilon)\frac{n^2}{2}$$

*for every graph* $F$ *with chromatic number greater than* $r$ *and for every* $n \geq c^{|F|}$*.*

The assumption that $n \geq \exp(|F|)$ cannot be eliminated.

**Lemma 5.** *For every positive real number* $\varepsilon \leq 1/20$ *and every integer* $m \geq 2$*, there exists a graph with* $n = \left\lfloor \left(\frac{1}{2\varepsilon}\right)^{m/2} \right\rfloor$ *vertices and at least* $\varepsilon \frac{n^2}{2}$ *edges not containing* $K_{m,m}$ *as a subgraph.*

*Proof.* Let $G$ be a random $n$-vertex graph in which every pair forms an edge independently with probability $p = 2\varepsilon$. We have

$$\mathbb{E}[\|G\|] = p\binom{n}{2}.$$

Let $t$ be the number of appearances of $K_{m,m}$ in $G$. We have

$$\mathbb{E}[t] \leq \binom{n}{m}^2 p^{m^2} \leq n^{2m} p^{m^2} = \left(n^2 p^m\right)^m \leq 1.$$

Let $G'$ be the graph obtained from $G$ by deleting one edge from every $K_{m,m}$ subgraph. Then $G'$ avoids $K_{m,m}$ and

$$\mathbb{E}[\|G'\|] \geq \mathbb{E}[\|G\| - t] \geq p\binom{n}{2} - 1 = p\frac{n^2}{2} - p\frac{n}{2} - 1 \geq \varepsilon\frac{n^2}{2}.$$

□