Introduction and revision

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October 9, 2020

Notation: $|G|$ number of vertices of G, $\|G\|$ number of edges of G.

Definition 1. *Maximum number of edges of a graph with n vertices that does not contain any subgraph isomorphic to $F_1, \ldots, F_m$:*

$$\mathrm{ex}(n; F_1, \ldots, F_m).$$

Density version:

$$\overline{\mathrm{ex}}(n; F_1, \ldots, F_m) = \frac{\mathrm{ex}(n; F_1, \ldots, F_m)}{\binom{n}{2}}.$$

Asymptotic density:

$$\overline{\mathrm{ex}}(\infty; F_1, \ldots, F_m) = \inf\{\overline{\mathrm{ex}}(n; F_1, \ldots, F_m) : n \in \mathbb{N}\}.$$

Lemma 2. *If $n_1 \geq n_2$, then $\overline{\mathrm{ex}}(n_1; F_1, \ldots, F_m) \leq \overline{\mathrm{ex}}(n_2; F_1, \ldots, F_m)$.*

Proof. Let G be a graph on n_1 vertices not containing $F_1, \ldots, F_n$ and having exactly $\mathrm{ex}(n_1; F_1, \ldots, F_m)$ edges. Let us first randomly uniformly choose $X \subseteq V(G)$ of size n_2, and then an arbitrary unordered pair xy of elements of X. Clearly, every pair of vertices of G has the same probability $1/\binom{n}{2}$ to be chosen as xy, and thus the probability that xy is an edge of G is

$$p = \frac{\|G\|}{\binom{n_1}{2}} = \overline{\mathrm{ex}}(n_1; F_1, \ldots, F_m).$$

On the other hand, $G[X]$ has at most $\mathrm{ex}(n_2; F_1, \ldots, F_m)$ edges, and thus the probability that xy is an edge of $G[X]$ is

$$p_X = \frac{\|G[X]\|}{\binom{n_2}{2}} \leq \frac{\mathrm{ex}(n_2; F_1, \ldots, F_m)}{\binom{n_2}{2}} = \overline{\mathrm{ex}}(n_2; F_1, \ldots, F_m).$$

Consequently,

$$\overline{\text{ex}}(n_1; F_1, \ldots, F_m) = p \leq \max\left\{p_X : X \in \binom{V(G)}{n_2}\right\} \leq \overline{\text{ex}}(n_2; F_1, \ldots, F_m).$$

□

Corollary 3.

$$\overline{\text{ex}}(\infty; F_1, \ldots, F_m) = \lim_{n\to\infty} \overline{\text{ex}}(n; F_1, \ldots, F_m),$$

and for every n_0 *we have*

$$\overline{\text{ex}}(\infty; F_1, \ldots, F_m) \leq \overline{\text{ex}}(n_0; F_1, \ldots, F_m).$$

Asymptotically, for $n \to \infty$*, we have*

$$\text{ex}(n; F_1, \ldots, F_m) = (\overline{\text{ex}}(\infty; F_1, \ldots, F_m) + o(1))\frac{n^2}{2}.$$

Example 4. *Every* 5*-vertex graph without* C_3 *and* C_4 *has at most* 5 *edges, i.e.* $\overline{\text{ex}}(5; C_3, C_4) = 1/2$*. Therefore,* $\text{ex}(n; C_3, C_4) \leq \frac{1}{2}\binom{n}{2}$ *for every* $n \geq 5$ *and* $\overline{\text{ex}}(\infty; C_3, C_4) \leq 1/2$*.*

Remark: As we will see below, $\overline{\text{ex}}(\infty; C_3, C_4) = 0$ *and* $\text{ex}(n; C_3, C_4) = \Theta(n^{3/2})$*.*

Lemma 5. *If* T *is a forest on* $k \geq 3$ *vertices, then* $\text{ex}(n; T) < (k-2)n$*.*

Proof. Suppose for a contradiction that a graph G with $n \geq 1$ vertices and at least $(k-2)n$ edges avoids T, and let us choose such a graph with n minimum. Since $\|G\| > 0$, we have $n \geq 2$. The minimality of $|G|$ implies that G has minimum degree at least $k-1$ (we could delete vertices of degree at most $k-2$ to obtain a smaller counterexample). If H is an arbitrary subgraph of G with less than k vertices, then every vertex of H has a neighbor outside of $V(H)$. Therefore, we can obtain a subgraph isomorphic to T by adding leaves one by one, which is a contradiction. □

Turán graph $T_r(n)$: r-partite graph with n vertices, where the size of any two parts differs by at most 1. Let us define $t_r(n) = \|T_r(n)\|$.

Observation 6.

$$t_r(n) \leq (1 - 1/r)\frac{n^2}{2},$$

with equality iff $r|n$*.*

$$t_r(n) \geq (1 - 1/r)\frac{n^2}{2} - \frac{r}{8},$$

with equality iff r *is even and* $n \equiv r/2 \pmod{r}$*.*

Theorem 7 (Turán theorem)**.** *For every integer $r \geq 1$, we have*

$$\mathrm{ex}(n; K_{r+1}) = t_r(n),$$

and thus $\overline{\mathrm{ex}}(\infty; K_{r+1}) = 1 - 1/r$. Moreover, suppose G is a graph with n vertices and with clique number at most r. If $\|G\| = t_r(n)$, then G is isomorphic to $T_r(n)$.

Proof 1. Suppose $|G| = n$, $\|G\| = \mathrm{ex}(n; K_{r+1})$, and G has clique number at most r. If $v_1, v_2 \in V(G)$ are non-adjacent, then $\deg(v_1) = \deg(v_2)$: If $\deg(v_1) < \deg(v_2)$, then the graph obtained by replacing v_1 by a copy of the vertex v_2 would also have clique number at most r, and it would have more edges than G, a contradiction.

If $v_1, v_2, v_3 \in V(G)$ and $v_1v_2, v_2v_3 \notin E(G)$, then $v_1v_3 \notin E(G)$: By the previous observation, we have $\deg(v_1) = \deg(v_2) = \deg(v_3)$. If $v_1v_3 \in E(G)$, then then graph obtained by replacing v_1 and v_3 by copies of the vertex v_2 would have the clique number at most r and more edges than G.

Therefore, the relation $\sim$ on $V(G)$ defined so that $u \sim v$ iff $uv \notin E(G)$ is an equivalence. The equivalence classes of $\sim$ are independent sets in G and G is complete between any two such classes, and thus G is a complete multipartite graph. Since G has clique number at most r, G has at most r parts. Among such graphs, the graph $T_r(n)$ is the unique graph with the largest number of edges; consequently, G is isomorphic to $T_r(n)$. □

Proof 2. By induction on $|V(G)|$. Suppose $|G| = n$, $\|G\| = \mathrm{ex}(n; K_{r+1})$, and G has clique number at most r. If $n \leq r$, then $G = K_n = T_r(n)$, and thus we can assume $n \geq r+1$. The graph G contains a clique A of size r, as otherwise we could add an edge to G without increasing the clique number above r. Every vertex of $V(G - A)$ has at most $r - 1$ neighbors in A, as otherwise G would contain a clique of size $r + 1$. Using the induction hypothesis on $G - A$, we have

$$\|G\| \leq \|G-A\| + (n-r)(r-1) + \binom{r}{2} \leq t_r(n-r) + (n-r)(r-1) + \binom{r}{2} = t_r(n).$$

If $\|G\| = t_r(n)$, then all the inequalities must hold with equality, and thus every vertex of $V(G-A)$ has exactly $r-1$ neighbors in A and by the induction hypothesis, $G - A$ is isomorphic to $T_r(n - r)$. The vertices in different parts of the multipartite graph $G - A$ must have different neighborhoods in A, as otherwise G would contain a clique of size $r + 1$. It follows that G is isomorphic to $T_r(n)$. □

Theorem 8 (Erdős-Stone theorem)**.** *Every graph F satisfies*

$$\overline{\mathrm{ex}}(\infty; F) = 1 - \frac{1}{\chi(F) - 1}.$$

We will give a proof later. For $\chi(F) \geq 3$, Erdős-Stone theorem gives exact asymptotics of $\mathrm{ex}(n; F)$:

$$\frac{\mathrm{ex}(n; F)}{\left(1 - \frac{1}{\chi(F)-1}\right)\frac{n^2}{2}} = 1 + o(1)$$

as $n \to \infty$. The situation is more complicated for bipartite graphs F, since then Erdős-Stone theorem only gives $\mathrm{ex}(n; F) = o(n^2)$.

Lemma 9. *For all integers $a \leq b$, we have*

$$\mathrm{ex}(n; K_{a,b}) < \frac{\sqrt[a]{a^2 + b}}{2} n^{2-1/a}.$$

Proof. Let G be an n-vertex graph G avoiding $K_{a,b}$ as a subgraph. Let m be the number of $(a+1)$-tuples $(x, v_1, \ldots, v_a)$ of vertices of G such that $xv_1, \ldots, xv_a \in E(G)$. On one hand, for any $x \in V(G)$ we have $\deg^a x$ choices for an a-tuple of its neighbors, giving

$$m = \sum_{x \in V(G)} \deg^a x \geq \frac{\left(\sum_{x \in V(G)} \deg x\right)^a}{n^{a-1}} = \frac{(2\|G\|)^a}{n^{a-1}}.$$

On the other hand, for every a-tuple $(v_1, \ldots, v_a)$ of distinct vertices, we can choose their common neighbor x in less than b ways, as otherwise G would contain $K_{a,b}$. Moreover, the number of $(a+1)$-tuples $(x, v_1, \ldots, v_a)$ where v_1, $\ldots$, v_a are not pairwise distinct is less than $a^2 n^a$, since there are less than a^2 ways how to choose indices $i \neq j$ such that $v_i = v_j$, and n^a ways how to choose x and the vertices v_k for $k \neq i$. Therefore,

$$m < (a^2 + b) n^a.$$

Combining these inequalities, we get

$$\|G\| < \frac{\sqrt[a]{a^2 + b}}{2} n^{2-1/a}.$$

□

Corollary 10. *If F is bipartite and one of its parts has size at most a, then*

$$\mathrm{ex}(n; F) = O(n^{2-1/a}).$$

Lemma 11. *For every prime p, there exists a graph with $2(p^2+p+1)$ vertices and $(p^2+p+1)(p+1)$ edges that does not contain C_4 as a subgraph.*

Proof. Since p is prime, there exists a finite projective plane of order p, with p^2+p+1 points and p^2+p+1 lines. Let G be the incidence graph of this finite projective plane, i.e., the vertices of G are the points and lines, and a point p is adjacent to a line ℓ iff and only if p lies on ℓ. This graph has $2(p^2+p+1)$ vertices and $(p^2+p+1)(p+1)$ edges. Moreover, it does not contain C_4, as otherwise two distinct lines would have intersection greater than 1. □

Corollary 12. *For every $b \geq 2$, we have*

$$\mathrm{ex}(n; K_{2,b}) = \Theta(n^{3/2}).$$

Proof. By Lemma 9, we have $\mathrm{ex}(n; K_{2,b}) \leq \frac{\sqrt{b+4}}{2} n^{3/2} = O(n^{3/2})$. Suppose that $n \geq 16$. Then there exists a prime p such that $\sqrt{n}/4 \leq p \leq \sqrt{n}/2$, and in particular $2(p^2+p+1) \leq n$. Let G be the graph obtained in Lemma 11 together with $n - 2(p^2+p+1)$ isolated vertices. Then G does not contain C_4 as a subgraph, and thus G avoids $K_{2,b}$ as well. Moreover, $|G| \geq n$ and $\|G\| \geq n^{3/2}/64$. Therefore $\mathrm{ex}(n; K_{2,b}) \geq n^{3/2}/64 = \Omega(n^{3/2})$. □

Corollary 13. *Suppose F is a bipartite graph with a part of size at most two and $n \geq |V(F)|$. Then*

$$\mathrm{ex}(n; F) = \begin{cases} -\infty & \text{if } \|F\| = 0 \\ 0 & \text{if } \|F\| = 1 \\ \Theta(n) & \text{if } \|F\| \geq 2 \text{ and } F \text{ is a forest} \\ \Theta(n^{3/2}) & \text{otherwise.} \end{cases}$$

Proof. If F has no edges, then it is a subgraph of every graph with $n \geq |V(F)|$ vertices and $\mathrm{ex}(n; F) = -\infty$. If F has exactly one edge, then it is a subgraph of every graph with $n \geq |V(F)|$ vertices and at least one edge and $\mathrm{ex}(n; F) = 0$. If F is a forest with at least two edges, then F is not a subgraph of either $K_{1,n-1}$ or a maximal matching on n vertices, and thus $\mathrm{ex}(n; F) \geq \lfloor n/2 \rfloor$; together with Lemma 5, this gives $\mathrm{ex}(n; F) = \Theta(n)$. If F is bipartite, not a forest, and has a part of size at most two, then F contains a 4-cycle, and thus Corollary 10 and Lemma 11 imply $\mathrm{ex}(n; F) = \Theta(n^{3/2})$. □