Coloring of triangle-free graphs and the Rosenfeld counting method

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There are many constructions that show that triangle-free graphs can have arbitrarily large chromatic number. Let us give one that shows that being triangle-free does not help to get better bound on the chromatic number even for degenerate graphs.

Lemma 1. *For every integer $d \ge 1$, there exists a d-degenerate graph G_d of girth at least six and of chromatic number $d+1$.*

Proof. We prove the claim by induction on d. For $d = 1$, we can take $G_1 = K_2$. Suppose now that $d \ge 2$. The graph G_d is obtained as follows. We start with an independent set K of size $d(|V(G_{d-1})| - 1) + 1$ and with $\binom{|K|}{|V(G_{d-1})|}$ copies of G_{d-1}. For each subset S of K of size $|V(G_{d-1})|$, we then choose a distinct copy G_S of G_{d-1} and add a perfect matching between S and $V(G_S)$. Observe that G_d has girth at least six, as any cycle is either contained in a copy of G_{d-1}, or contains edges of at least two copies of G_{d-1} and passes through K. Moreover, G_d is d-degenerate, as each copy of G_{d-1} is $(d-1)$-degenerate and the degrees of its vertices are increased only by 1 in G_d.

Suppose for a contradiction that G_d can be colored using at most d colors. By the pigeonhole principle, some set $S \subseteq K$ of size $|V(G_{d-1})|$ is colored using just one color. But then the copy G_S of G_{d-1} would be colored using only the remaining $d-1$ colors, which is a contradiction. $\square$

However, we can get an improvement in terms of the maximum degree. We give a relatively recent proof of Martinsson, which uses an idea based on a counting argument of Rosenfeld. To illustrate the method, let us start with a simpler example. Recall that a *star coloring* is a proper coloring in which any two color classes induce a star forest (or equivalently, at least three colors appear on any 4-vertex path).

Theorem 2. *For any positive integer Δ, any graph of maximum degree at most Δ has a star coloring by at most $\lceil 13\Delta^{3/2}\rceil$ colors.*

Proof. Let $k = \lceil 13\Delta^{3/2}\rceil$ and $\beta = k/3$. Let $\mathcal{C}(G)$ be the set of all star colorings of G using colors $\{1,\ldots,k\}$. We will prove that if G is a graph of maximum degree at most Δ, then for every $v \in V(G)$, we have

$$\frac{|\mathcal{C}(G)|}{|\mathcal{C}(G-v)|} \geq \beta.$$

This clearly implies $\mathcal{C}(G) \neq \emptyset$, and in fact that G has at least $\beta^{|V(G)|}$ star colorings by at most k colors. We prove the claim by induction on the number of vertices of G. When $|V(G)| = 1$, we have $|\mathcal{C}(G)| = k > \beta$ and $|\mathcal{C}(G-v)| = 1$, and thus the claim holds. Hence, we can assume that $|V(G)| > 1$.

Let $\mathcal{C}_v(G)$ be the set of proper k-colorings of G whose restriction to $G-v$ is a star coloring. Note that any coloring in $\mathcal{C}(G-v)$ can be extended to a coloring in $\mathcal{C}_v(G)$ by choosing a color of v different from the color of the neighbors of v, and this can be done in at least $k-\Delta$ ways. Hence,

$$|\mathcal{C}_v(G)| \geq (k-\Delta)|\mathcal{C}(G-v)|.$$

For a 4-vertex path P containing v, let $\mathcal{C}_P$ be the set of colorings in $\mathcal{C}_v$ that use only two colors on P. Letting $\mathcal{P}$ be the set of all 4-vertex paths in G containing v, we have

$$\mathcal{C}(G) = \mathcal{C}_v \setminus \bigcup_{P\in\mathcal{P}} \mathcal{C}_P.$$

Hence, we need to bound $|\mathcal{C}_P|$. Note that each coloring in $\mathcal{C}_P$ is obtained from a star coloring of $G - V(P)$ by choosing the two colors used on P, and thus (using the induction hypothesis),

$$|\mathcal{C}_P| \leq k^2|\mathcal{C}(G-V(P))| \leq \frac{k^2}{\beta^3}|\mathcal{C}(G-v)|.$$

Putting these bounds together, we have

$$\begin{aligned}
|\mathcal{C}(G)| &\geq |\mathcal{C}_v| - \sum_{P\in\mathcal{P}} |\mathcal{C}_P| \\
&\geq (k-\Delta)|\mathcal{C}(G-v)| - |\mathcal{P}| \cdot \frac{k^2}{\beta^3}|\mathcal{C}(G-v)| \\
&= \left(k - \Delta - 2\Delta^3 \cdot \frac{k^2}{\beta^3}\right)|\mathcal{C}(G-v)| \\
&\geq \beta|\mathcal{C}(G-v)|.
\end{aligned}$$

□

Let us now give a (somewhat more involved) argument for the chromatic number of triangle-free graphs.

Theorem 3. *For every $\Delta \geq 10^{10}$, every triangle-free graph of maximum degree at most Δ has chromatic number at most $\lceil 4\Delta/\log\Delta \rceil$.*

Proof. Let $k = \lceil 4\Delta/\log\Delta \rceil$ and $\ell = \frac{\Delta^{1-\log 2}}{\log\Delta}$. For a graph G, let $\mathcal{C}(G)$ be the set of all proper k-colorings of G. We will show that if G is a triangle-free graph of maximum degree at most Δ, then for every $v \in V(G)$, we have

$$\frac{|\mathcal{C}(G)|}{|\mathcal{C}(G-v)|} \geq \ell.$$

This clearly implies $\mathcal{C}(G) \neq \emptyset$, and in fact that G has at least $\ell^{|V(G)|}$ k-colorings. We prove the claim by induction on the number of vertices of G. When $|V(G)| = 1$, we have $|\mathcal{C}(G)| = k > \ell$ and $|\mathcal{C}(G-v)| = 1$, and thus the claim holds. Hence, we can assume that $|V(G)| > 1$.

For a vertex $x \in V(G)$ and a partial coloring φ of G, let $a(G, \varphi, x)$ denote the number of colors in $\{1, \ldots, k\}$ that do not appear on the neighbors of x in φ. Note that each coloring $\varphi \in \mathcal{C}(G-v)$ extends to a proper k-coloring of G in exactly $a(G, \varphi, v)$ ways, and thus

$$\frac{|\mathcal{C}(G)|}{|\mathcal{C}(G-v)|} = \frac{\sum_{\varphi\in\mathcal{C}(G-v)} a(G, \varphi, v)}{|\mathcal{C}(G-v)} = E[a(G, \varphi, v)],$$

where the expectation is over a k-coloring φ of $G - v$ chosen uniformly at random. Hence, we need to prove that

$$E[a(G, \varphi, v)] \geq \ell.$$

Let $t = \frac{\ell}{\log\Delta} \geq 2$; we say a neighbor u of v is *φ-poor* if $a(G-v, \varphi, u) \leq t$. Consider any k-coloring ψ of $G-v-u$. If $a(G-v, \psi, u) > t$, then ψ does not extend to any k-coloring φ of $G-v$ such that u is φ-poor; if $a(G-v, \psi, u) \leq t$, then ψ extends to exactly $a(G-v, \psi, u) \leq t$ k-colorings φ of $G-v$ such that u is φ-poor. Hence, using the induction hypothesis for $G - v$ and u, the number of k-colorings of $G - v$ such that u is φ-poor is at most

$$t \cdot |\mathcal{C}(G-v-u)| \leq \frac{t}{\ell}|\mathcal{C}(G-v)|,$$

and thus

$$\Pr[u \text{ is } \varphi\text{-poor}] \leq \frac{t}{\ell}.$$

Let $q(\varphi)$ denote the number of φ-poor neighbors of v. Then

$$E[q(\varphi)] \le \frac{t}{\ell} \cdot \deg v \le \frac{t\Delta}{\ell} \le \frac{k}{4},$$

and thus

$$\Pr\left[q(\varphi) > \frac{k}{2}\right] \le \frac{1}{2}.$$

We say that v is *φ-rich* if v has at most $\frac{k}{2}$ φ-poor neighbors. Hence,

$$\Pr[v \text{ is } \varphi\text{-rich}] \ge \frac{1}{2}.$$

Consequently,

$$E[a(G,\varphi,v)] \ge \Pr[v \text{ is } \varphi\text{-rich}] \cdot E[a(G,\varphi,v) | v \text{ is } \varphi\text{-rich}] \ge \frac{1}{2} E[a(G,\varphi,v) | v \text{ is } \varphi\text{-rich}],$$

and thus it suffices to show that

$$E[a(G,\varphi,v) | v \text{ is } \varphi\text{-rich}] \ge 2\ell.$$

Note that since G is triangle-free, the neighborhood $N(v)$ of v is an independent set, and thus whether v is φ-rich depends only on the restriction of φ to $V(G) \setminus N[v]$. Hence, it suffices to show that for every k-coloring ψ_0 of $G - N[v]$ such that v is ψ_0-rich and ψ_0 can be extended to a k-coloring of $G - v$,

$$E[a(G,\varphi,v) | \varphi \text{ extends } \psi_0] \ge 2\ell.$$

Let R be the set of the neighbors of v that are not ψ_0-poor. Observe it it suffices to show that for any k-coloring ψ of $G - R - v$ extending ψ_0, we have

$$E[a(G,\varphi,v) | \varphi \text{ extends } \psi] \ge 2\ell.$$

Let A be the set of colors that ψ does not use on the neighbors of v; since v is ψ-rich, we have $|A| \ge k/2$. For each $u \in R$, let L_u be the set of colors not used by ψ on the neighbors of u. For $c \in A$, let X_c denote the event that φ

does not use c on the neighbors of v. Observe that

$$
\begin{aligned}
&E[a(G,\varphi,v)|\varphi \text{ extends } \psi] \\
&\qquad = \sum_{c\in A} \Pr[X_c|\varphi \text{ extends } \psi] \\
&\qquad = \sum_{c\in A} \prod_{u\in R: c\in L_u} (1-1/|L_u|) \\
&\qquad \geq |A| \left(\prod_{c\in A} \prod_{u\in R: c\in L_u} (1-1/|L_u|) \right)^{1/|A|} && \textit{(AG inequality)} \\
&\qquad = |A| \left(\prod_{u\in R} \prod_{c\in L_u\cap A} (1-1/|L_u|) \right)^{1/|A|} \\
&\qquad \geq |A| \left(\prod_{u\in R} (1-1/|L_u|)^{|L(u)|} \right)^{1/|A|} \\
&\qquad \geq |A| \left(\prod_{u\in R} (1-1/t)^{t} \right)^{1/|A|} \geq |A|(1-1/t)^{t\Delta/|A|} && \textit{(since } |L_u| \geq t \textit{ and } |R| \leq \Delta\textit{)} \\
&\qquad \geq |A| 4^{-\Delta/|A|} \geq \frac{k}{2} 4^{-2\Delta/k} \geq \frac{k}{2\cdot 4^{\frac{1}{2}\log\Delta}} && \textit{(since } t\geq 2 \textit{ and } |A| \geq k/2\textit{)} \\
&\qquad \geq 2\frac{\Delta^{1-\log 2}}{\log\Delta} = 2\ell.
\end{aligned}
$$

This finishes the proof. $\square$

Corollary 4. *If G is a triangle-free graph with n vertices and maximum degree Δ, then G has an independent set of size*

$$\Omega\left(\max\left(n\log\Delta/\Delta, \sqrt{n\log n}\right)\right).$$

Hence, the Ramsey number $R(3,m)$ is $O\left(\frac{m^2}{\log m}\right)$.

Proof. Since G has chromatic number $O(\Delta/\log\Delta)$, the largest color class has size $\Omega(n\log\Delta/\Delta)$. If $\Delta = \Omega(\sqrt{n\log n})$, then the neighborhood of a vertex of maximum degree is an independent set of size $\Omega(\sqrt{n\log n})$. Otherwise, $\Delta = O(\sqrt{n\log n})$ and $\Omega(n\log\Delta/\Delta) = \Omega(\sqrt{n\log n})$. Hence, if $n \geq c\frac{m^2}{\log m}$ for a sufficiently large constant c, then G has an independent set of size $\Omega(\sqrt{n\log n}) \geq m$. $\square$

All these bounds are tight (but proving this is non-trivial).