List coloring and Gallai trees

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1 List coloring and choosability

A *list assignment* for a graph G is a function L that to each vertex $v \in V(G)$ assigns a *list* $L(v)$ of colors. An L-coloring of G is a proper coloring φ such that $\varphi(v) \in L(v)$ for all $v \in V(G)$. The *choosability* $\chi_l(G)$ of G is the minimum integer k such that G can be L-colored for every assignment L of lists of size at least k.

Observation 1. *Every graph G satisfies*

$$\chi(G) \le \chi_l(G).$$

If G is d-degenerate, then

$$\chi_l(G) \le d + 1.$$

Choosability matches the chromatic number for many graphs (cycles, cliques, ...). However, the choosability can be arbitrarily large compared to the chromatic number, as the following result shows.

Lemma 2. *For every positive integer a,*

$$\chi_l(K_{a,a^a}) = a + 1.$$

Proof. For every n, the bipartite graph $K_{a,n}$ is a-degenerate, implying that $\chi_l(K_{a,n}) \le a + 1$. Hence, it suffices to show that there exists an assignment of lists of size a to K_{a,a^a} from that the graph cannot be colored.

Let A and B be the parts of K_{a,a^a}, where $|A| = a$ and $|B| = a^a$. Let $A = \{v_1, \ldots, v_a\}$. Since $|B|$ is equal to the number of sequences of numbers $\{1, \ldots, a\}$ of length a, we can label vertices of B as $u_{i_1,\ldots,i_a}$ for $1 \le i_1, \ldots, i_a \le a$. Let us give vertices of A pairwise disjoint lists, say $L(v_i) = \{(i, 1), (i, 2), \ldots, (i, a)\}$ for $1 \le i \le a$. We give vertices of B different lists, each of them intersecting the list of each vertex of A in exactly one color; say $L(u_{i_1,\ldots,i_a}) = \{(1,i_1),(2,i_2),\ldots,(a,i_a)\}$ for $1 \le i_1,\ldots,i_a \le a$. We claim that K_{a,a^a} is not L-colorable; indeed, if we give vertices of A colors $(1,c_1)$, $(2,c_2)$, $\ldots$, (a,c_a), then all colors in the list of the vertex $u_{c_1,c_2,\ldots,c_a}$ are used on its neighbors, and thus this vertex cannot be colored. □

On the other hand, we have the following positive result on choosability of bipartite graphs.

Lemma 3. *For every positive integer n,*

$$\chi_l(K_{n,n}) \le \lfloor \log_2 n \rfloor + 2.$$

Proof. Let $c = \lfloor \log_2 n \rfloor + 2$ and let L be any assignment of lists of size at least c to vertices of $K_{n,n}$. Let A and B be the parts of $K_{n,n}$. For each color, we flip a fair coin and according to the result we delete it either from the lists of all vertices of A or from the lists of all vertices of B. Afterwards, the lists of vertices of A are disjoint from the lists of vertices of B, and thus if these lists are non-empty, we can properly color $K_{n,n}$. The probability that a list of a vertex $v \in V(K_{n,n})$ becomes empty is at most 2^{-c}. Hence, the expected number of empty lists is at most

$$2^{-c}|V(K_{n,n}|) = \frac{2n}{2^{\lfloor \log_2 n \rfloor + 2}} < 1.$$

Hence, with non-zero probability, all the lists are non-empty, and thus $K_{n,n}$ can be L-colored. Since the choice of L was arbitrary, it follows that $\chi_l(K_{n,n}) \le c$ as required. □

Note that

$$\chi_l(K_{n,n}) \ge \frac{\log_2 n}{\log_2 \log_2 n}$$

by Lemma 2; a more involved argument shows that actually $\chi_l(K_{n,n}) = \Omega(\log n)$.

2 Planar graphs

By Observation 1, all planar graphs are 6-choosable. Thomassen proved that they are actually 5-choosable, strengthening the 5-color theorem. In fact, he proved the following stronger claim.

Theorem 4. *Let G be a plane graph, let p_1p_2 be an edge contained in the boundary of its outer face, and let L be a list assignment for G satisfying the following.*

- $|L(v)| \geq 5$ *for every vertex* $v \in V(G)$ *not incident with the outer face.*
- $|L(v)| \geq 3$ *for every vertex* $v \in V(G)$ *incident with the outer face and distinct from* p_1 *and* p_2.
- $|L(p_1)|, |L(p_2)| \geq 1$, *and if* $|L(p_1)| = |L(p_2)| = 1$, *then* $L(p_1) \neq L(p_2)$.

Then G *is* L*-colorable.*

Proof. We proceed by induction on the number of vertices of G. The case $|V(G)| = 2$ is trivial, hence assume $|V(G)| \geq 3$.

We can assume G is connected, otherwise we apply induction to each component of G. Furthermore, we can assume G is 2-connected. Otherwise, $G = G_1 \cup G_2$ for proper induced subgraphs G_1 and G_2 of G intersecting in exactly one vertex v. We can assume that $p_1p_2 \in E(G_1)$. By the induction hypothesis, there exists an L-coloring φ_1 of G_1. Let $L'(v) = \{\varphi_1(v)\}$ and $L'(x) = L(x)$ for all $x \in V(G_2) \setminus \{v\}$. Then G_2 with the list assignment L' satisfies the assumptions of the theorem (with v and one of its neighbors playing the role of p_1p_2), and thus G_2 has an L'-coloring φ_2. The colorings φ_1 and φ_2 together give an L-coloring of G.

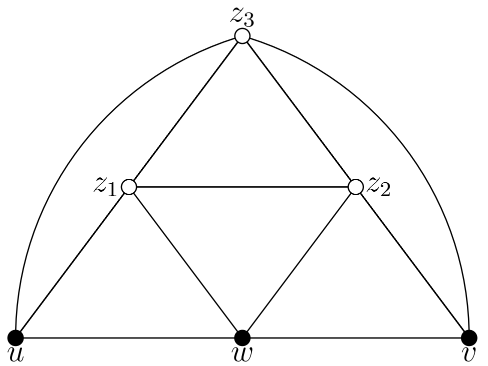
Since G is 2-connected, its outer face is bounded by a cycle C. We can assume that the cycle C is induced. Otherwise, if C has a chord v_1v_2, then $G = G_1 \cup G_2$ for proper induced subgraphs G_1 and G_2 of G intersecting exactly in v_1v_2. We can assume that $p_1p_2 \in E(G_1)$. By the induction hypothesis, there exists an L-coloring φ_1 of G_1. Let $L'(v_1) = \{\varphi_1(v_1)\}$, $L'(v_2) = \{\varphi_1(v_2)\}$ and $L'(x) = L(x)$ for all $x \in V(G_2) \setminus \{v_1, v_2\}$. Then G_2 with the list assignment L' satisfies the assumptions of the theorem (with v_1v_2 playing the role of p_1p_2), and thus G_2 has an L'-coloring φ_2. The colorings φ_1 and φ_2 together give an L-coloring of G.

We can also assume that $|L(p_1)| = |L(p_2)| = 1$, as otherwise we can throw away extra colors from their lists. Let $C = p_1p_2p_3 \ldots p_k$. By the assumptions, $|L(p_3)| \geq 3$, and thus there exist two distinct colors $c_1, c_2 \in L(p_3) \setminus L(p_2)$. Let $t \geq 3$ be the maximum integer such that $\{c_1, c_2\} \subseteq L(p_i)$ for $3 \leq i \leq t$. Let $v = p_{t+1}$ if $t < k$ and $v = p_1$ if $t = k$. By the choice of t and since $|L(p_1)| = 1$, we can assume that $c_2 \notin L(v)$. Let $G' = G - \{p_3, \ldots, p_t\}$ and let L' be the list assignment for G' such that $L'(p_2) = L(p_2)$, $L'(v) = L(v)$, $L'(x) = L(x)$ if $x \in V(G') \setminus \{v, p_2\}$ has no neighbor in $\{p_3, \ldots, p_t\}$, and $L'(x) = L(x) \setminus \{c_1, c_2\}$ otherwise. Since the outer face of G is bounded by an induced cycle C, if $x \in V(G') \setminus \{v, p_2\}$ has a neighbor in $\{p_3, \ldots, p_t\}$, then $x \notin V(C)$, and thus $|L(x)| \geq 5$ and $|L'(x)| \geq 3$; furthermore, such a vertex x is contained in the boundary of the outer face of G'. We conclude that G' with the list assignment L' satisfies the assumptions of the theorem, and thus G' has an L'-coloring φ' by the induction hypothesis. By the choice of L', none of the neighbors of $\{p_3, \ldots, p_t\}$ in G' except for v can be given color c_1 or c_2. Recall also that $c_2 \notin L(v)$, and thus $\varphi'(v) \neq c_2$. Hence, we can extend φ' to an L-coloring of G by giving p_t, p_{t-2}, ... the color c_2 and p_{t-1}, p_{t-3}, ... the color c_1. □

However, in contrast to the Four Color Theorem, not all planar graphs are 4-choosable.

Lemma 5. *There exists a planar graph G that is not 4-choosable.*

Proof. Let G_{uwv} be the following graph.



Let $L_{a,m,b}$ (with distinct $a, m, b \notin \{11, 12\}$) be the list assignment such that $L_{a,m,b}(z_1) = \{a, m, 11, 12\}$, $L_{a,m,b}(z_2) = \{m, b, 11, 12\}$, and $L_{a,m,b}(z_3) = \{a, b, 11, 12\}$. Then a precoloring of (u, w, v) by colors (a, m, b) cannot be extended to an $L_{a,m,b}$-coloring of G_{uwv}.

Let G_{uv} be the graph formed by two copies of G_{uwv} sharing the path uwv. Let $L_{a,b}$ (with distinct $a, b \notin \{9, 10, 11, 12\}$) be the list assigment matching $L_{a,9,b}$ in one of the copies, $L_{a,10,b}$ in the other copy, and with $L_{a,b}(w) = \{a, b, 9, 10\}$. Then a precoloring of (u, v) by colors $\{a, b\}$ cannot be extended to an $L_{a,b}$-coloring of G_{uv}.

Let G be the graph formed by 16 copies of G_{uv} sharing the vertices u and v. Let $L(u) = \{1, 2, 3, 4\}$, $L(v) = \{5, 6, 7, 8\}$, and let L match $L_{a,b}$ for $a \in \{1, 2, 3, 4\}$ and $b \in \{5, 6, 7, 8\}$ on the 16 copies of G_{uv}. Then G is not L-colorable. □

3 Degree choosability

We want to obtain a list version of Brooks' theorem.

Theorem 6 (Brooks). *Let G be a connected graph of maximum degree at most Δ. If G is not Δ-colorable, then either $G = K_{\Delta+1}$, or $\Delta = 2$ and G is an odd cycle.*

A *degree assignment* to a graph G is a list assignment such that $|L(v)| \geq \deg(v)$ for all $v \in V(G)$.

Lemma 7. *Let G be a connected graph and let L be a degree assignment for G. If G is not L-colorable, then $|L(v)| = \deg(v)$ for all $v \in V(G)$.*

Proof. If $|L(v)| > \deg(v)$, then let $v_1, \ldots, v_n$ be a listing of vertices of G in non-increasing order according to their distance from v; hence, $v_n = v$ and for $1 \leq i \leq n-1$, the vertex v_i has a neighbor v_j with $j > i$ (the neighbor of v_i on a shortest path from v_i to v). Let us greedily L-color $v_1, \ldots, v_n$ in order. For $1 \leq i \leq n-1$, at least one neighbor of v_i has not been colored yet, and thus at most $\deg(v_i) - 1 < |L(v_i)|$ colors need to be avoided. At v_n, at most $\deg(v_n) < |L(v_n)|$ colors need to be avoided. Hence, in both cases, we can give v_i a color from its list different from the colors of its neighbors. □

Corollary 8. *Let G be a connected graph and let L be a degree assignment for G. If G is not L-colorable, $uv \in E(G)$ and u is not a cutvertex in G, then $L(u) \subseteq L(v)$.*

Proof. Otherwise, there exists a color $c \in L(u) \setminus L(v)$. Let $G' = G - u$; since u is not a cutvertex, G' is connected. Let $L'(x) = L(x) \setminus \{c\}$ for all neighbors x of u and $L'(x) = L(x)$ for all non-neighbors x. Note that the list size decreases only for neighbors x of u for which $\deg_{G'}(x) = \deg_G(x) - 1$, and thus L' is a degree assignment for G'. Furthermore, $|L'(v)| = |L(v)| \geq \deg_G(v) > \deg_{G'}(v)$, and thus G' is L'-colorable by Lemma 7. We can extend this coloring to an L-coloring of G by giving v color c. □

Note that if neither u nor v is a cutvertex, then Corollary 8 implies $L(u) \subseteq L(v)$ and $L(v) \subseteq L(u)$, and thus $L(u) = L(v)$.

Corollary 9. *Let G be a 2-connected graph and let L be a degree assignment for G. Then G is not L-colorable if and only if G is a clique or an odd cycle and all vertices of G have the same list of length equal to the degree of vertices of G.*

Proof. The "if" part is trivial. For the "only if" part, suppose that G is not L-colorable. Since G is 2-connected, Corollary 8 implies that any two adjacent vertices of G have the same list, and consequently all the vertices of G have the same list, say $\{1, \ldots, d\}$, where $d \leq \Delta(G)$. It follows that G is not d-colorable, and thus either $G = K_{d+1}$ or $d = 2$ and G is an odd cycle by Theorem 6. □

A *Gallai tree* is a connected graph T such that every 2-connected block of T is either a clique or an odd cycle. Suppose $B_1, \ldots, B_k$ are the blocks of a Gallai tree T, and let $S_1, \ldots, S_k$ be sets of colors satisfying the following conditions:

- For $1 \le i \le k$, if B_i is a clique, then $|S_i| = |V(B_i)| - 1$, and if B_i is an odd cycle, then $|S_i| = 2$.
- For $1 \le i < j \le k$, if $B_i \cap B_j \neq \emptyset$, then $S_i \cap S_j = \emptyset$.

For $v \in V(T)$, let $L(v) = \bigcup_{v \in B_i} S_i$. If a list assignment L can be expressed in this way, we say that L is a *blockwise uniform* assignment for T.

Theorem 10 (Gallai)**.** *Let G be a connected graph and let L be a degree assignment for G. Then G is not L-colorable if and only if G is a Gallai tree and L is blockwise uniform.*

Proof. It is easy to see that a Gallai tree cannot be colored from a blockwise uniform assignment, and thus it suffices to prove the "only if" part. We do the proof by induction on the number of vertices of G.

By Corollary 9, the claim holds when G is 2-connected. Hence, suppose that G is not 2-connected. First, we prove that G is a Gallai tree. Let B be a block of G. Since G is not 2-connected, there exists a leaf block B' of G distinct from B. Let v be a vertex of B' which is not a cutvertex, and let $G' = G - v$. Let c be any color in $L(v)$ and let $L'(x) = L(x) \setminus \{c\}$ for all neighbors x of v and $L'(x) = L(x)$ for all other vertices x of G'. Note that L' is a degree assignment for G' and that G' is not L'-colorable, as otherwise we can extend the coloring to an L-coloring of G by giving v the color c. By the induction hypothesis, G' is a Gallai tree. Note that B is also a block of G', and thus B is a clique or an odd cycle. As the choice of B was arbitrary, all blocks of G are cliques or odd cycles, and thus G is a Gallai tree.

Let $B_1, \ldots, B_k$ be the blocks of G, where without loss of generality B_k is a leaf block. Let z be the cutvertex of B_k and let v be any other vertex of B_k, and let $S_k = L(v)$; by Corollary 8, we conclude that all non-cut vertices of B_k have list S_k, and $S_k \subseteq L(z)$. By Lemma 7, if B_k is a clique then $|S_k| = |B_k| - 1$, and if B_k is an odd cycle, then $|S_k| = 2$. Let $G' = B_1 \cup \ldots \cup B_{k-1}$, let $L'(x) = L(x)$ for $x \in V(G') \setminus \{z\}$ and $L'(z) = L(z) \setminus S_k$. Note that L' is a degree assignment for G' and that G' is not L'-colorable, as otherwise the coloring would extend to an L-coloring of G by using the colors in S_k to color $B_k - z$. By the induction hypothesis, L' is blockwise uniform as shown by sets $S_1, \ldots, S_{k-1}$. But then the sets $S_1, \ldots, S_{k-1}, S_k$ show that L is blockwise uniform. □

Corollary 11. *Let G be a $(c+1)$-critical graph and let S be the set of vertices of G of degree c. Then each component of $G[S]$ is a Gallai tree.*

Proof. Consider any component C of $G[S]$. Since G is $(c+1)$-critical, $G - C$ has a c-coloring φ. Let L be the list assignment to $G[C]$ in which for each $v \in C$, the list $L(v)$ consists of those of colors $\{1, \ldots, c\}$ that are not used by φ on the neighhbors of v. If v has k neighbors in $V(G) \setminus V(C)$, then $\deg_{G[C]}(v) = \deg_G(v) - k = c - k$ and $|L(v)| \geq c - k$, and thus L is a degree assignment for $G[C]$. An L-coloring of $G[C]$ together with φ would give a c-coloring of G; since G is $(c + 1)$-critical, we conclude that $G[C]$ is not L-colorable, and by Theorem 10, $G[C]$ is a Gallai tree. □