Circular coloring

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1   Motivation: scheduling

Suppose we have a list of tasks, each taking one unit of time to accomplish. The tasks can be performed in parallel, however some of the tasks conflict: they cannot be worked on at the same time. This is expressed by a *conflict graph* G, whose vertices are the tasks and edges join pairs of conflicting tasks. There are several scheduling problems one can consider in this setting:

- Suppose the tasks are indivisible—once you start working on a task, you must finish it (and it takes one unit of time, during which you cannot work on the conflicting tasks). Hence, during each unit of time, you can finish in parallel tasks corresponding to an independent set in G, and thus the minimum time needed to finish all tasks is exactly the chromatic number $\chi(G)$ of G.

- Suppose the tasks are divisible—you can interrupt your work on a task at any moment and continue later, and the task is finished once you devoted one unit of time to it in total. In this case, the minimum time needed to finish all tasks is exactly the fractional chromatic number $\chi_f(G)$ of G.

- In this lecture, we will consider the *circular chromatic number* of G. This corresponds to the case that the tasks are indivisible, but they are to be performed repeatedly; we now ask to create a periodic schedule in which all tasks are performed for one consecutive unit of time every t units of time, and we ask what is the minimum t for that this is possible.

For example, suppose G is the 5-cycle $v_1 \ldots v_5$. Then there exists a schedule with period 2.5: for any integer i, v_1 is worked on in times $[2.5i, 2.5i+1)$, v_2 in times $[2.5i + 1, 2.5i + 2)$, v_3 in times $[2.5i + 2, 2.5i + 3) = [2.5i +$

$2, 2.5(i + 1) + 0.5)$, v_4 in times $[2.5i + 0.5, 2.5i + 1.5)$, and v_5 in times $[2.5i + 1.5, 2.5i + 2.5) = [2.5i + 1.5, 2.5(i + 1))$.

2 Definition

Note that instead of an infinitely repeating schedule, we can consider a schedule consisting of intervals in a circle. For a non-negative real number t, a *circular t-coloring* of a graph G is a function ι that assigns to each vertex v of G a half-open unit interval $\iota(v)$ in a circle of circumference t, such that $\iota(u) \cap \iota(v) = \emptyset$ for all $uv \in V(G)$. Equivalently, we can record just the starting points of the intervals: this gives a function φ that assigns to each vertex v of G a point $\varphi(v)$ of the circle, such that the distance between $\varphi(u)$ and $\varphi(v)$ (measured along the circle) is at least 1 whenever $uv \in E(G)$. Another equivalent way to view this definition is as follows: suppose we cut the circle at the point 0 and map the circle to the interval $[0, t)$. Then a circular t-coloring is a function $\varphi : V(G) \to [0, t)$ satisfying $1 \leq |\varphi(u) - \varphi(v)| \leq t - 1$ for all $uv \in E(G)$.

Definition 1. *The* circular chromatic number $\chi_c(G)$ *of a graph G is the infimum of real numbers t such that G has a circular t-coloring.*

As we will see later, any graph G has a circular $\chi_c(G)$-coloring, and thus we could write "minimum" instead of "infimum" in the definition.

It is convenient to consider the following discretized version of circular coloring. For any integers $a \geq 0$ and $b \geq 1$, a *circular (a/b)-coloring* of a graph G is a function $\psi : V(G) \to \{0, 1, \ldots, a - 1\}$ such that $b \leq |\psi(u) - \psi(v)| \leq a - b$ for all $uv \in E(G)$. Clearly, $\varphi(v) = \psi(v)/b$ is a circular t-coloring of G for $t = a/b$, and thus if G has a circular (a/b)-coloring, then $\chi_c(G) \leq a/b$.

Lemma 1. *Let G be a graph.*

(a) *If G has a circular t-coloring and $t \leq a/b$ for integers $a \geq 0$ and $b \geq 1$, then G has a circular (a/b)-coloring.*

(b) $\chi_c(G)$ *is infimum of $\left\{\frac{a}{b} : a, b \in \mathbf{Z}, a \geq 0, b \geq 1, G \text{ has a circular } (a/b)\text{-coloring}\right\}$.*

(c) $\chi_f(G) \leq \chi_c(G) \leq \chi(G) \leq \chi_c(G) + 1$.

Proof. Let φ be a circular t-coloring of G. Let $\psi(v) = \lfloor \varphi(v)b \rfloor$ for all $v \in V(G)$. Since $\varphi(v) < t$, we have $\psi(v) < tb \leq a$, and since $\psi(v)$ is an integer, $\psi(v) \in \{0, \ldots, a - 1\}$. Consider any edge uv of G, where w.l.o.g. $\varphi(v) + 1 \leq \varphi(u) \leq \varphi(v) + t - 1$. Then $\psi(u) - \psi(v) \geq \lfloor (\varphi(v) + 1)b \rfloor - \lfloor \varphi(v)b \rfloor = b$ and

$\psi(u) - \psi(v) \leq \lfloor (\varphi(v) + t - 1)b \rfloor - \lfloor \varphi(v)b \rfloor \leq \lfloor \varphi(v)b + a - b \rfloor - \lfloor \varphi(v)b \rfloor = a - b$.
Consequently, ψ is a circular (a/b)-coloring of G, and (a) holds. The claim (b) clearly follows from (a).

For a function $\theta : V(G) \to \{0, 1, \ldots, a-1\}$, note that θ is a circular $(a/1)$-coloring iff θ is a proper coloring. Consequently, G has a circular $(\chi(G)/1)$-coloring, and thus $\chi_c(G) \leq \chi(G)$. Suppose that ψ is a circular (a/b)-coloring of G. The function $\gamma : V(G) \to 2^{\{0,\ldots,a-1\}}$ defined by $\gamma(v) = \{\psi(v), (\psi(v) + 1) \bmod a, \ldots, (\psi(v) + b - 1) \bmod a\}$ for all $v \in V(G)$ is an $(a : b)$-coloring of G, and thus $\chi_f(G) \leq a/b$; by (b), we conclude that $\chi_f(G) \leq \chi_c(G)$. Also, by (a) G has a circular $(\lceil a/b \rceil/1)$-coloring, which as we observed is a proper coloring by $\lceil a/b \rceil$ colors, and thus $\chi(G) \leq \lceil a/b \rceil < a/b + 1$; by (b), we conclude that $\chi(G) \leq \chi_c(G) + 1$. Hence, (c) holds. $\qquad\square$

Note also that if G has no edges, then $\chi_c(G) = \chi(G) = 1$, if $E(G) \neq \emptyset$ and G is bipartite, then $\chi_c(G) = \chi(G) = 2$, and if G is not bipartite and contains a cycle of length $2k + 1$, then $\chi_c(G) \geq \chi_c(C_{2k+1}) = 2 + 1/k$.

3 Circular chromatic number and orientations

Let G be a graph and let $\vec{G}$ be an orientation of G. For a walk $W = v_0 v_1 \ldots v_k$ in G, we define $W^+ = |\{i \in \{0, \ldots, k-1\} : (v_i, v_{i+1}) \in E(\vec{G})\}|$ and $W^- = |\{i \in \{0, \ldots, k-1\} : (v_{i+1}, v_i) \in E(\vec{G})\}|$. For a cycle C in G (viewed as a closed walk tracing C in either of the two possible directions), we define $\mathrm{bal}(C) = \max(|C|/C^+, |C|/C^-)$; when C is a directed cycle in $\vec{G}$, then $\mathrm{bal}(C) = \infty$. Let us define $\mathrm{bal}(\vec{G})$ as the maximum of $\mathrm{bal}(C)$ over all cycles C in G; if $E(G) = \emptyset$ define $\mathrm{bal}(\vec{G}) = 1$, and if $E(G) \neq \emptyset$ and G is a forest, then define $\mathrm{bal}(\vec{G}) = 2$. Let us define $\mathrm{bal}(G)$ as the minimum of $\mathrm{bal}(\vec{G})$ over all orientations of G. Note that $\mathrm{bal}(G)$ is always finite, since G has an acyclic orientation.

Theorem 2. *For any graph G,*

$$\chi_c(G) = bal(G).$$

Proof. If ψ is a circular (a/b)-coloring of G, then let $\vec{G}$ be the orientation of G such that $(u, v) \in E(\vec{G})$ iff $\psi(u) < \psi(v)$. Consider any cycle $C = v_1 v_2 \ldots v_k$ in G, and set $v_{k+1} = v_1$. For $i = 1, \ldots, k$, if $(v_i, v_{i+1}) \in E(\vec{G})$, then $\psi(v_{i+1}) \geq \psi(v_i) + b$, and if $(v_{i+1}, v_i) \in E(\vec{G})$, then $\psi(v_{i+1}) \geq \psi(v_i) + b - a$. Summing these inequalities, we obtain $\psi(v_1) \geq \psi(v_1) + |C|b - C^- a$, and thus $|C|/C^- \leq a/b$. Traversing the cycle C in the opposite direction, we conclude that $|C|/C^+ \leq a/b$, and thus $\mathrm{bal}(C) \leq a/b$. Since this holds for all cycles in

G, we have $\mathrm{bal}(G) \leq \mathrm{bal}(\vec{G}) \leq a/b$, and by Lemma 1(b), we conclude that $\mathrm{bal}(G) \leq \chi_c(G)$.

Let us now consider an orientation $\vec{G}$ of G such that $\mathrm{bal}(\vec{G}) = \mathrm{bal}(G)$. W.l.o.g., G is connected and not a tree. Note that $\mathrm{bal}(G) \geq 2$ is a rational number and we can write $\mathrm{bal}(G) = a/b$ for integers $b \geq 1$ and $a \geq 2b$. Let v_0 be an arbitrary vertex of G and let T be a spanning tree of G rooted in v_0. Let $\tau_T : V(G) \to \mathbf{Z}$ be the (unique) function such that $\tau_T(v_0) = 0$ and for any $v \in V(G) \setminus \{v_0\}$, if u is the parent of v in T, then

$$\tau_T(v) = \begin{cases} \tau_T(u) + b & \text{if } (u,v) \in E(\vec{G}) \\ \tau_T(u) + b - a & \text{if } (v,u) \in E(\vec{G}). \end{cases}$$

Let $\tau(T) = \sum_{v \in V(G)} \tau_T(v)$. Let us choose the spanning tree T such that $\tau(T)$ is maximum.

We claim that $\psi(v) = \tau_T(v) \bmod a$ for all $v \in V(G)$ defines a circular (a/b)-coloring of G. To see that, it suffices to prove that $b \leq |\tau_T(u) - \tau_T(v)| \leq a - b$ for all $uv \in E(G)$. By symmetry, we can assume that $(u,v) \in E(\vec{G})$.

If v is not an ancestor of u in T, then consider the spanning tree $T' = T - vv' + uv$, where v' is the parent of v in T. By the choice of T, we have $\tau(T) \geq \tau(T')$, and we conclude that $\tau_T(v) \geq \tau_{T'}(v) = \tau_T(u) + b$. If v is an ancestor of u in T, then let P be the path in T from v to u, and let C be the cycle consisting of P and the edge uv. By the definition of τ_T, we have

$$\tau_T(u) = \tau_T(v) + bP^+ + (b-a)P^- = \tau_T(v) + b(C^+ - 1) + (b-a)C^-$$
$$= \tau_T(v) + b(|C| - C^- - 1) + (b-a)C^- = \tau_T(v) + b|C| - aC^- - b.$$

Since $\mathrm{bal}(C) \leq a/b$, we have $|C|/C^- \leq a/b$, and thus $b|C| \leq aC^-$. Consequently, $\tau_T(u) \leq \tau_T(v) - b$, and we again conclude that $\tau_T(v) \geq \tau_T(u) + b$.

If u is not an ancestor of v, then let u' be the parent of u in T; since $\tau(T) \geq \tau(T - uu' + uv)$, we have $\tau_T(u) \geq \tau_T(v) + b - a$. If u is an ancestor of v in T, then let P be the path in T from u to v, and let C be the cycle consisting of P and the edge vu. By the definition of τ_T, we have

$$\tau_T(v) = \tau_T(u) + bP^+ + (b-a)P^- = \tau_T(u) + bC^+ + (b-a)(C^- - 1)$$
$$= \tau_T(u) + b(|C| - C^-) + (b-a)(C^- - 1) = \tau_T(u) + b|C| - aC^- + a - b.$$

Since $\mathrm{bal}(C) \leq a/b$, we have $|C|/C^- \leq a/b$, and thus $b|C| \leq aC^-$. Consequently, $\tau_T(v) \leq \tau_T(u) + a - b$ and $\tau_T(u) \geq \tau_T(v) + b - a$ in this case as well.

We conclude that $b \leq \tau_T(v) - \tau_T(u) \leq a - b$ for any $(u,v) \in E(\vec{G})$, and thus ψ defined above is a circular (a/b)-coloring of G. Consequently, $\chi_c(G) \leq a/b = \mathrm{bal}(G)$, and thus $\chi_c(G) = \mathrm{bal}(G)$. $\qquad \square$

Corollary 3. *For any graph G, $\chi_c(G)$ is a rational number and $\chi_c(G) = a/b$ for some integers $a \geq 0$ and $b \geq 1$ such that $a \leq |V(G)|$ and $b \leq |V(G)|/2$.*

In particular, G has a circular (a/b)-coloring such that $\chi_c(G) = a/b$. Consequently, in the definition of the circular chromatic number as well as in Lemma 1(b), we can write "minimum" instead of "infimum". Furthermore, the problem of deciding whether $\chi_c(G) \leq t$ for any real number t is in NP, since it suffices guess a circular (a/b)-coloring for some integers a and b such that $1 \leq b \leq |V(G)|/2$ and $0 \leq a \leq \min(|V(G)|, bt)$.

Corollary 4. *For any graph G, $\chi(G) = \lceil \chi_c(G) \rceil$.*

Proof. If $\chi_c(G)$ is an integer, then by Corollary 3 and Lemma 1(a), G has a circular $(\chi_c(G)/1)$-coloring, which as we observed before is a proper $\chi_c(G)$-coloring of G, and thus $\chi(G) \leq \chi_c(G)$. Since $\chi_c(G) \leq \chi(G)$ by Lemma 1(c), we conclude that $\chi(G) = \chi_c(G) = \lceil \chi_c(G) \rceil$.

If $\chi_c(G)$ is not an integer, then by Lemma 1(c), $\chi_c(G) \leq \chi(G) \leq \chi_c(G) + 1$, and since $\chi(G)$ is an integer, it follows that $\chi(G) = \lceil \chi_c(G) \rceil$. $\qquad\square$

4 Circular chromatic number of projective planar quadrangulations

Previously, we have seen that non-bipartite quadrangulations of projective plane have chromatic number 4. Using Theorem 2, we can easily show that this is the case even for the circular chromatic number.

Theorem 5. *If G is a non-bipartite quadrangulation of projective plane, then $\chi_c(G) = 4$.*

Proof. Let G be a non-bipartite quadrangulation of projective plane; note that G is 2-connected. By generalized Euler's formula, it is easy to see that G is 3-degenerate, and thus $\chi_c(G) \leq \chi(G) \leq 4$. Hence, it suffices to prove that $\chi_c(G) \geq 4$.

Consider any orientation $\vec{G}$ of G. Since G is not bipartite, there exists an odd cycle K in G; since $|K|$ is odd, we have $K^+ \neq K^-$. Since G quadrangulates the surface, all contractible cycles in G have even length, and thus K is non-contractible. Let $\vec{G}_0$ be the graph obtained from $\vec{G}$ by cutting the projective plane along the cycle K. Note that $\vec{G}_0$ is a directed graph drawn in a disk bounded by a cycle K_0 of length $2|K|$, and $K_0^+ = 2K^+ \neq 2K^- = K_0^-$. Hence, $\mathrm{bal}(K_0) > 2$.

Let C be a cycle in $\vec{G}_0$ such that $\mathrm{bal}(C) > 2$ and the disk Δ bounded by C is minimal; such a cycle exists by the previous paragraph. If C is not a facial

cycle, then since G is 2-connected, there exists a path P joining two vertices of C and drawn in Δ. Let C_1 and C_2 be the two cycles in $C+P$ different from C, and let Q_1 and Q_2 be the paths in C joining the endpoints of P. Note that $C_1^+ = Q_1^+ + P^+$, $C_1^- = Q_1^- + P^-$, $C_2^+ = Q_2^+ + P^-$, and $C_2^- = Q_2^- + P^+$. If $\mathrm{bal}(C_1) = 2$ and $\mathrm{bal}(C_2) = 2$, then $C_1^+ = C_1^-$ and $C_2^+ = C_2^-$, and $C^+ = Q_1^+ + Q_2^+ = C_1^+ - P^+ + C_2^+ - P^- = C_1^- - P^- + C_2^- - P^+ = Q_1^- + Q_2^- = C^-$, and thus $\mathrm{bal}(C) = 2$, which is a contradiction. Hence, either $\mathrm{bal}(C_1) > 2$ or $\mathrm{bal}(C_2) = 2$; but the disks bounded by C_1 and C_2 are proper subsets of Δ, which contradicts the choice of C.

Hence, C is a facial cycle, and thus $|C| = 4$. This implies that $\mathrm{bal}(C) \in \{2, 4, \infty\}$, and since $\mathrm{bal}(C) > 2$, we conclude that $\mathrm{bal}(C) \geq 4$. Hence, $\mathrm{bal}(\vec{G}) \geq 4$ for any orientation $\vec{G}$ of G. It follows that $\mathrm{bal}(G) \geq 4$, and by Theorem 2, we conclude that $\chi_c(G) \geq 4$. $\qquad\square$