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**DOCTORAL THESIS**

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# Structural Aspects of Graph Coloring

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Abstract: In this thesis, we study the structural and algorithmic properties of graphs embedded or represented in surfaces and with constraints on their faces or cycles.

We derive tools to quantify properties of embedded flows and use these to design an algorithm to decide the extendability of a precoloring of the boundary cycles of near-quadrangulations of the cylinder. We then develop methods to reduce 3-coloring of triangle-free graphs embedded in the torus to 3-coloring near-quadrangulations, obtaining practical algorithms for deciding 3-colorability in linear time, and obtaining a 3-coloring in quadratic time.

We also investigate connection between geometric graph representations and the induced odd cycle packing number (iocp) parameter. We show that wide variety of representable graphs exhibit limited iocp and show that graphs with limited iocp are $\chi$-bounded. We derive an EPTAS for maximum independent set of graphs with limited iocp and linear independence number, as well as QPTAS assuming only limited iocp.

Keywords: graph theory coloring torus algorithms induced odd cycle packing number

# Contents

# 1. Introduction

## 1.1 Coloring

Let $X$ be a set of elements and $\mathcal{C}$ a set of colors. A *coloring* is any map $\psi : X \to \mathcal{C}$. The set $X$ is typically a set of elements of a combinatorial structure $\mathcal{S}$. Using the structure of $\mathcal{S}$, we may then impose requirements on the values of $\psi$ to consider $\psi$ a *valid* coloring. In some cases, the color set $\mathcal{C}$ has its own structure, which also plays role in the constraints on $\psi$.

A typical setting, and the one we work with, is *vertex graph coloring*. For a graph $G$ with vertex set $V(G)$ and edge set $E(G)$, a *vertex coloring* is an assignment of colors to vertices, formally a map $\psi : V(G) \to \mathcal{C}$. a vertex coloring is *proper* if $\psi(u) \neq \psi(v)$ for every edge $uv$. It is clear that for the purposes of proper vertex coloring we may assume our graphs have no parallel edges or loops.

Let us define the *chromatic number* of a graph $G$, denoted as $\chi(G)$, as the smallest integer $k$ such that $G$ has a proper vertex coloring using $k$ colors. The graph $G$ is $k$-colorable if $\chi(G) \leq k$.

A *$k$-coloring* of a graph $G$ is a vertex coloring $\psi : V(G) \to [k]$. Note that $G$ has a $k$-coloring whenever $G$ is $k$-colorable, i.e. the choice of the set of colors is inconsequential. We choose the color set $\{1, \ldots, k\}$ for convenience, to easily refer to colors and to obtain a natural (arbitrary) ordering on the colors. We may assume that every vertex coloring is a $k$-coloring for a suitable integer $k$.

For a graph $G$ and a $k$-coloring $\psi$ of $G$, we define the *color classes* of $G$ as the sets of vertices assigned the same color by $\psi$, that is, for every color $i \in [k]$ we get its color class $\psi^{-1}(i)$. Observe that an equivalent interpretation of $k$-colorability is the lowest integer $k$ such that $G$ can be decomposed into at most $k$ independent sets (color classes), or the complement of $G$ can be decomposed into at most $k$ induced cliques.

## 1.2 Structural Aspects

A graph $G$ is *$k$-critical* if $\chi(G) = k$ and every proper subgraph $H \subsetneq G$ is $(k-1)$-colorable. The following structural characterization of coloring holds.

**Observation.** *a graph $G$ has chromatic number at least $k$ if and only if $G$ contains a $k$-critical graph $H$ as a subgraph.*

This is easily see as either $G$ itself is critical, or by definition there is $G' \subsetneq G$ (obtained by deleting an edge or an isolated vertex) such that $\chi(G') \geq k$. The claim then holds by induction (or rather iteration). An important observation is the following.

**Observation.** *If a graph is $k$-critical, then its minimum degree is at least $k-1$.*

*Proof.* Suppose $G$ is a counterexample of a $k$-critical graph with a vertex $v$ of degree at most $k-2$. Then by the criticality of $G$, $G-v$ has a $(k-1)$-coloring and it can be extended to $G$, as at most $k-2$ distinct colors appear in the neighborhood of $v$. $\square$

A more detailed analysis shows that if $G$ is $(k+1)$-critical, then its vertices of degree $k$ induce a Gallai forest[1]; a graph is a *Gallai forest* if every component of vertex 2-connectivity is a clique or an odd cycle (a connected Gallai forest is a *Gallai tree*). This claim can actually be proven for list-coloring where each vertex is also given a list of colors it may be assigned (the classical coloring is a special case of list coloring).

Using the characterization via Gallai forests, it is relatively easy to show that, for any $k \geq 3$, the average degree of $(k+1)$-critical graphs is strictly greater than $k$, at least $k + \frac{k-2}{k^2+2k-2} \approx k + 1/k$, unless $G = K_{k+1}$, in which case it is exactly $k$.

**Fact** (Gallai [2])**.** *If a graph is $(k+1)$-critical and different from $K_{k+1}$, then its average degree is strictly higher than $k$. In particular, the average degree is at least $k + \frac{1}{3k}$ if $k \geq 3$.*

On a different note, let us consider a greedy coloring algorithm coloring vertices one by one using the lowest free color on each vertex. It is easy to see that if a $k$-coloring $\psi$ exists, then an appropriate order of vertices also produces a $k$-coloring in this greedy manner. First we may assume that each vertex of color $c$ in $\psi$ has neighbors of all colors lower than $c$, as otherwise we can decrease its color. Iteration of this argument eventually produces a $k$-coloring that satisfies the assumption. Applying the greedy approach to vertices ordered by their color in $\psi$ produces exactly $\psi$. Consider a vertex $v$ of (highest) color $k$. The greedy choice of the color of $v$ must be supported by neighbors of $v$ of all lower colors, who are in turn supported by other neighbors. One might suspect that such a high-color vertex $v$ must be surrounded by a rather complex dense structure, possibly even clique-like. While there might be some truth to this intuition, no obvious classes of such dense or enforcing structures exist in general, as we will now explore.

For a graph $G$ we define its Mycielski graph $M(G)$ as follows. Let $V(G) = \{v_1, \ldots, v_n\}$, then $V(M(G)) = V \cup U \cup \{w\} = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\} \cup \{w\}$. Edges of $M(G)$ are of three types, vertices $V$ induce $G$ in $M(G)$, the edge $\{v_i, u_j\} \in E(M(G))$ if and only if $\{v_i, v_j\} \in E(G)$ and $\{u_i, w\} \in E(M(G))$ for all $u_i$. One can consider the vertices $V$ to form a copy of $G$, and the vertices $U$ to be shadows of vertices $V$, mimicking their neighborhoods in $V$.

**Fact.** *For every $k$ there exists a $k$-critical triangle-free graph.*

*Proof.* This fact can be demonstrated using Mycielski's construction. Suppose we have a graph $G$, such that $\chi(G) = k$. Clearly, $\chi(M(G)) \geq k$ as $G \subset M(G)$. First we show that $\chi(M(G)) \geq k+1$. If not, we may obtain a $k$-coloring $\phi$ where $\phi(w) = k$ ($w$ here denotes the special vertex as above) and therefore color $k$ does not appear on $U$. We construct a proper $(k-1)$-coloring of $M(G)[V]$. Whenever $\phi(v_i) = k$, we recolor it using color $\phi(u_i)$. Since $N(v_i) \cap V \subset N(u_i)$, this recoloring avoids monochromatic edges on $V$ (though may be improper coloring of $M(G)$). This contradicts the definition of $k$ as $\chi(G)$. Conversely, using the same coloring on $U$ as on $V$ and a new color on $w$, we have that $\chi(M(G)) = \chi(G) + 1$. A final key observation is that if $G$ is triangle-free, then $M(G)$ is also triangle-free.

Starting with $M_2 = K_2$ of chromatic number 2, we may define an infinite sequence of Mycielski graphs as $M_{i+1} = M(M_i)$ for all $i \geq 2$. All graphs in this sequence are triangle free and as a consequence, there exists an infinite number of graphs of chromatic number at least $k$ with no clique of size at least 3. Each graph of chromatic number at least $k$ of course contains a $k$-critical subgraph, which is also triangle free. $\square$

The graph $M_3$ is $C_5$, $M_4$ is called *Grötzsch graph* and is the smallest triangle-free graph of chromatic number 4. Results similar to the Mycielski construction can be obtained for arbitrary girth. By a probabilistic construction due to Erdös, the following holds.

**Fact** (Erdös [3]). *For every choice of integers $l, k$ there exists a graph $G$ of girth at least $l$ and chromatic number at least $k$.*

Graphs of arbitrary girth and chromatic number can also be explicitely constructed, though the constructions are rather involved [4].

Another way of stating this result is that for every choice of integers $r, k$ there exists a graph $G$ such that for every vertex $v$ of $G$, the vertices at distance at most $r$ induces a tree in $G$, but the chromatic number of $G$ is at least $k$. This also shows that in general one cannot hope to deduce the chromatic number of a graph based on properties of its subgraphs of limited diameter.

An interesting observation is that while a graph of high chromatic number does not necessarily contain a large clique (or any fixed graph with a cycle) as a subgraph, it typically contains a large clique as a minor. The Hadwiger conjecture states the following.

**Conjecture.** *If $\chi(G) \geq k$ then $G$ contains a clique of size $k$ as a minor.*

In other words, a large chromatic number is (perhaps) always accompanied by a clique minor of the corresponding size. While this conjecture is known to hold for all values of $k$ up to 6 [5], the general statement remains as one of the most prominent open problem in vertex graph coloring. In particular the famous 4-color theorem would be a simple consequence of this conjecture. On the other hand, the current proofs of the cases for $k \leq 6$ largely rely on reducing the conjecture to the 4-color theorem. Due to a lack of a transparent proof of a 4-color theorem, this line of reasoning is rather lacking the potential for significant progress without deeper insights into coloring of graphs embedded in surfaces.

The converse of Hadwiger conjecture fails by an arbitrarily big gap. As an example, consider a clique of arbitrary size $k$ and subdivide all of its edges once. The resulting graph contains $K_k$ as a minor, but is bipartite and therefore 2-colorable. Consequently while absence of a fixed clique minor might serve as an easy upper bound on the chromatic number, its presence has no direct implications. Curiously this is contrary to a presence of a clique subgraph implying an easy lower bound on the chromatic number, but its absence does not upper-bound the chromatic number in any way.

If true, in this form or some variation, this conjecture would provide a concrete link between structural properties of a graph and its chromatic number. While such connections are know to exist for various graph classes, such as the classes of graph embeddable into various surfaces, a generally applicable structural characterization of chromatic number remain largely elusive. Various non-aproximability suggest that either there is no such structure to be found, or it is very convoluted and not practically useful. For the reasons above, the conjecture itself seems extremely challenging. Nevertheless, weaker variations are known to hold.

**Fact** (Kostochka [6]). *Every graph with no $K_t$ minor is $\mathcal{O}(t\sqrt{\log t})$-colorable.*

This result is obtained from an argument showing that every graph with no $K_t$ minor is in fact $\mathcal{O}(t\sqrt{\log t})$-degenerate, where the omitted constant is reasonably small. And this degeneracy result is tight. However, a recent breakthrough beating this degeneracy barrier, shows the following.

**Fact** (Norin et al. [7]). *For every $\beta > 0$, every graph with no $K_t$ minor is $\mathcal{O}(t \log t^{\beta})$-colorable.*

And yet another substantial progress was obtained by Postle.

**Fact** (Postle [8]). *Every graph with no $K_t$ minor is $\mathcal{O}(t(\log log(t)))$-colorable.*

It remains as a very interesting open question whether the chromatic number of graphs with no $K_t$ minor can even be bounded by a linear function in $t$.

## 1.3 Coloring on Surfaces

In theory, and especially in this thesis, we often think of graphs as embedded in a surface, such as the plane.

Formally, a *surface* is a 2-dimensional manifold without boundary. By the surface classification theorem, each surface can be assigned a value of genus, and can be constructed from a sphere by attaching a certain number of ears and/or crosscaps, where the number depends on genus of the surface. An *embedding* of a graph $G$ in surface $\Sigma$ is a map assigning vertices of $G$ to points of $\Sigma$ and edges of $G$ to simple curves such that the endpoints of the curves correspond to the mapping of vertices in the natural way, and the curves do not share any points except the endpoints. Other formulations are possible, but the technical details are not of particular importance from the combinatorial point of view.

A more theoretically useful notion is to understand embedding as a combinatorial object rather than continuous map, that is, as a collection of orderings specifying the order of edges around each vertex. We choose to work with this interpretation of embedding.

We say that an embedding is a *2-cell embedding*, when every face is homeomorphic to an open disc (in other words, each face is a 2-cell). Whenever embedding of a graph does not satisfy this assumption, an equivalent embedding into a simpler surface exists. Suppose we are given a 2-cell embedding and interpret it as a combinatorial embedding. It is clearly possible to reconstruct a facial walk of each face. An embedding in the geometric sense can then be reconstructed by composing individually represented faces. While the obtained embedding is clearly not unique, relevant topological properties of the embedding are often preserved. In particular, a connected planar graph with combinatorial embedding is uniquely embedded into the sphere up to continuous deformations of the sphere. We assume that all embedded graph we work with are connected.

The most prominent class of graphs embeddable in a surface are the planar graphs. We say that a graph is *planar* when it is embeddable in the plane, that is, an embedding into the plane exists. When we say that a graph is a *plane graph*, we mean that it is given together with its embedding. Note that a graph is embeddable in the plane if and only if it is embeddable in the sphere.

A *face* of an embedded graph is formally an arc-wise connected component of the surface minus the drawing of the graph. For a graph $G$ embedded in a surface, let denote $F(G)$ the set of faces of $G$. We use the notations $v(G), e(G)$ and $f(G)$ to denote the sizes of the vertex set $V(G)$, of the edge set $E(G)$, and of the face set of $G$ respectively. Depending on the context, by the *length* of a face or a *degree* of a face we mean the length of its facial walk in the number of edges (including repetitions). We use $\deg(x)$ to denote degree of a vertex $x$ or a face $x$, and $\bar{d}(G)$ to denote the average degree of $G$.

For a class of graphs $\mathcal{G}$, we define the chromatic number of $\mathcal{G}$, denoted $\chi(\mathcal{G})$, as the lowest integer $k$ such that every graph in $\mathcal{G}$ is properly vertex $k$-colorable.

The historically most famous problem surrounding coloring and surfaces is the four-color problem. Let $\mathcal{P}$ be the class of planar graphs. The problem can be stated as a question what is the value of $\chi(\mathcal{P})$ and in particular, whether $\chi(\mathcal{P}) = 4$. The original statement of the problem, also known as the map coloring problem, is stated as a problem of coloring maps, formalized in current terminology as face-coloring of planar (bridgeless) graphs. This formulation corresponds exactly to a vertex coloring of duals of bridgeless planar graphs, which are themselves planar graphs.

While obtaining the proof of the fact that indeed $\chi(\mathcal{P}) = 4$ was famously difficult [9, 10, 11], it is not difficult to get close to this result.

To show that $\chi(\mathcal{P}) \leq k$ for some small $k$, let us fix a specific graph $G$ from the class $\mathcal{P}$. We may use the Euler's formula $e(G) = v(G) + f(G) - 2$ and the relation of faces and edges $2e(G) \geq 3f(G)$ obtained from the fact that each face is incident with at least three edges (not necessarily distinct), nless $v(G) \geq 3$ and only one edge exists. By combination of these two relations, we get that $e(G) \leq 3v(G) - 6$. We can now bound the average degree of $G$ as $\bar{d}(G) \leq 2e(G)/v(G) = 6 - \frac{12}{v(G)} < 6$ and conclude that $G$ has a vertex of degree at most 5. Since this holds for any planar graph, and any subgraph of a planar graph is itself planar, we see that planar graphs are 5-degenerate. In general, a graph is *d-degenerate* if any subgraph of $G$ contains a vertex of degree at most $k$.

By a simple observation, if a graph is 5-degenerate, its chromatic number is at most 6. We proceed by induction on the number of vertices of $G$. Pick a $v$ of degree at most 5 and use the induction hypothesis to color $G - v$. The coloring easily extends to a 6-coloring of $G$, as at most 5 colors appear on the neighborhood of $v$, and it therefore has a free color. Consequently, we have that $\chi(\mathcal{P}) \leq 6$. A slightly different way of obtaining the same result is to observe that if $\chi(\mathcal{P}) > k$ then there exists a $(k+1)$-critical planar graph $G$. From criticality, the average degree of $G$ can be lower-bounded in terms of $k$. We conclude that $k < \bar{d}(G) < 6$, and consequently $k \leq 5$.

To improve upon the previous result, the standard tool of choice are Kempe chains, used to recolor the vertices in the inductive argument in $G - v$ to avoid the necessity of using sixth color. This approach shows that $\chi(\mathcal{P}) \leq 5$, however does not generalize for other surfaces, so we omit the details here.

For a general surface $\Gamma$, let $g(\Gamma)$ denote its genus. We define the Euler characteristic of $\Gamma$ as follows. For orientable surfaces, $\gamma(\Gamma) = 2 - 2g(\Gamma)$ and for non-orientable surfaces $\gamma(\Gamma) = 2 - g(\Gamma)$. In particular for the sphere (plane) and the projective plane we have positive values, 2 and 1 respectively, for the torus and the Klein bottle we get value 0 and other surfaces have increasingly negative characteristics.

Heawood proved[12] that any graph $G$ drawn on surface $\Sigma$ is $t$-colorable for any $t$ satisfying $t \geq H(\Sigma) := \lfloor(7 + \sqrt{49 - 24\gamma(\Sigma)}/2\rfloor$ unless $\Sigma$ is the sphere. Incidentally, the assertion holds for the sphere as well, as stated by the Four-Color Theorem. The bound given by Heawood's formula is tight. As proven by [13], the bound is best possible for all surfaces except the Klein bottle, for which the correct bound is 6.

While Heawood's formula gives a tight bound on the possible values of chromatic number of graphs on almost all surfaces, values close to the bound are achieved by only relatively few graphs. An improvement of Heawoods's formula in this sense [14][15] shows that the graphs with chromatic number exactly $H(\Sigma)$ are exactly those containing a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

From now on, assume that every embedded graph we consider is connected and the embedding is a 2-cell embedding. Whenever embedding of a graph does not satisfy this assumption, an equivalent embedding into a simpler surface exists an we switch to such a surface instead.

Using the Euler's characteristic, for a graph $G$ (2-cell-)embedded in a surface $\Gamma$, the general Euler formula states that $e(G) = v(G) + f(G) - \gamma(\Gamma)$. Applying the same arguments as for the plane, we obtain that for $G$ embedded in a surface $\Gamma$, $\bar{d}(G) \leq 6 - \frac{6\gamma(\Gamma)}{v(G)}$.

Suppose a graph $G$ embedded in a general surface $\Gamma$ is $k$-critical for $k \geq 7$ (and $G \neq K_k$). As mentioned earlier, the average degree of $G$ must be strictly greater than $6 + \epsilon$, (where say $\epsilon = \frac{1}{11}$ can be chosen). This is only possible if $\gamma(\Gamma) < 0$ and $v(G)$ is upper-bounded by a function linear in $-\gamma(\Gamma)$, and therefore there are only finitely many $k$-critical graphs for $k \geq 7$ on any fixed surface $\Gamma$.

By a much more complicated analysis, it is possible to show that the same conclusion is true for 6-critical graphs [16]. As a consequence, given a fixed surface $\Gamma$, $k$-colorability for $k \geq 5$ of any $G$ embeddadle in $\Gamma$ can be decided by testing the presence of finitely many $(k + 1)$-critical subgraphs. This can be achieved in linear time with respect to size of $G$ (for more details, see the Section 2.3). Naturally, practical implementation of such algorithm would need access to a list of corresponding critical graphs, which seems to grow exponentially with $-\gamma(\Gamma)$, or spend an immense amount of time searching for them. These lists have however been constructed for only a handful of cases. The lists of 6-critical graphs are explicitly known for the projective plane [17], the torus [18] and the Klein bottle [19, 20].

Similar approach is not possible for testing 4-colorability, as shown by an elegant construction of Fisk[21], who shows that if a triangulation of an orientable surface with exactly two vertices of odd degree is 4-colored, the two odd-degree vertices must receive the same color. In any surface other than the sphere, it is possible to construct infinitely many such triangulations so that the odd vertices are adjacent and thus the triangulations are not 4-colorable.

## 1.4 Coloring of Graphs with High Girth on Surfaces

The problem of deciding whether a graph is $k$-colorable or not is somewhat more approachable when only graphs of high girth are considered. Assuming a graph $G$ has girth at least 5, by the same reasoning as before we obtain that $\bar{d}(G) \leq \frac{10}{3} - \frac{6\gamma(\Gamma)}{v(G)}$; implying that there are only finitely many $k$-critical graphs on a fixed surface $\Gamma$ for any $k \geq 5$.

Furthermore, the chromatic number 3 of embedded graphs of girth at least five has been characterized by a deep theorem of Thomassen [22] who showed that there are only finitely many 4-critical graphs of girth at least 5 on any fixed surface. As a consequence, for any surface $\Gamma$, it is in principle possible to decide $k$-colorability of embedded graph of girth at least five efficiently, as long as $k$ and $\Gamma$ are fixed. There actually turn out to be no 4-critical graphs of girth at least five on the projective plane and the torus [23] and similarly on the Klein bottle [24].

From now on, let us consider the graphs of girth at least 4, in other words triangle-free graphs. By the same reasoning as before, we can show that for any graph $G$ embedded in surface $\Gamma$ the average degree is bounded as $\bar{d}(G) \leq 4 - \frac{2\gamma(\Gamma)}{v(G)}$. Since a $k$-critical graph must have average degree significantly higher than $k-1$, unless it is a clique, it follows that there are only finitely many $k$-critical triangle-free graphs on a fixed surface $\Gamma$ for any $k \geq 5$.

Unlike in the case of girth at least five, triangle-free 4-critical graphs offer a more substative theoretical challenge. By a well known theorem of Grötzsch, every planar triangle-free graph is 3-colorable.

**Fact** (Grötzsch's Theorem [23]). *Every triangle-free planar graph is* 3*-colorable.*

Unfortunately, Grötzsch's theorem cannot be extended to any surface other than the sphere. For instance, the graphs obtained from odd cycles of length five or more by Micielski's construction [25] provide an infinite class of 4-critical graphs embeddable in any surface other than the sphere. This of course means that 3-colorability of triangle-free graphs on a fixed surface $\Gamma$ cannot be in principle decided by testing the presence of specific set of subgraph obstructions.

The only non-planar surface for which the 3-colorability problem for triangle-free graphs is fully characterized in a compact way is the projective plane. Building on earlier work of Youngs[26], Gimbel and Thomassen[27] obtained an elegant characterization stating that a triangle-free graph drawn in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.

Less is known regarding surfaces of higher genus, although it is known that algorithmically exploitable structure of triangle-free 4-critical graphs must exist on any fixed surface. Let us give a rough sketch of the idea.

For a graph $G$ embedded on a surface, let $S(G)$ denote the multiset of lengths of ($\geq 5$)-faces of $G$. We refer to $S(G)$ as the *census* of $G$. Thus, the characterization of triangle-free 4-critical graphs embeddable in the projective plane implies that $S(G) = \emptyset$ for every such graph. The theorem following describes the general case.

**Fact** (Dvořák, Král', Thomas [28])**.** *For any surface* $\Gamma$*, there exists a constant* $c_\Sigma$ *such that every* 4*-critical triangle-free graph* $G$ *embedded in* $\Gamma$ without non-contractible 4-cycles *satisfies* $\sum S(G) \leq c_\Sigma$*.*

In other words, $G$ has only a bounded number of faces of length greater than four and these faces have bounded size. Such a bound does not hold in general if non-contractible 4-cycles are allowed (but it does hold for graphs embedded in the torus).

A detailed treatment of triangle-free 4-critical graphs with non-contractible 4-cycles was given by [29]. In was proved in [30] that for any surface $\Gamma$, a triangle-free graph embedded in $\Gamma$ with large edgewidth is 3-colorable unless $\Gamma$ is non-orientable and the graph contains a quadrangulation with an odd orienting cycle. Dvořák, Král' and Thomas also designed a linear-time algorithm to test 3-colorability of triangle-free graphs embedded in a fixed surface[31].

For the case of torus, there exists a set of template graphs, graphs embeddable in the torus with several special faces, with the property that any 4-critical triangle-free graph embaddable in the torus can be obtained from one of the template graphs by quadrangulating its special faces arbitrarily, and conversely, no graph obtained in this way is 3-colorable. We obtained this set through a sophisticated computer search [32], based on a previous work studying a hierarchy of the critical graph under a reducing operation [33]. We present the theory behind the set of templates in Section 2.3. Similar investigations may be performed for other surfaces, although the lack of non-contractible cycles of length 4 seems necessary without further tools.

Although these templates do not constitute a specific list of obstructions, nor a particularly elegant structural characterization of 4-critical graphs, a reasonable polynomial algorithm can be obtained based on the list of templates. Other consequences follow, such as an upper bound on edge-width and much better constraints on possible values of $S(G)$ whenever $G$ is 4-critical.

Inspired by the properties observed from the knowledge of this set of templates, we eventually designed a practical coloring algorithm which does not rely on the knowledge of the templates, but rather on limited values of parameters of 4-critical graphs which can be proven theoretically, in contrast to a computer search. The construction of this algorithm is described in Sections 2.1 and 2.2.

We collect our brief overview of coloring on surfaces into Table 1.1, showing what type of algorithmic situation occurs while testing $k$-coloring of graphs with given girth. On the right are cases where there are only finitely many obstruction on any surface. Since testing 2-colorability is trivial, only three cases remain wedged between the trivial cases and the cases of finitely many obstructions. The case of 3-colorability of triangle-free graphs is the only one currently known to be polynomial but not characterized by a finite set of obstructions. Out of all surfaces in this case, the torus is the simplest surface with no known exact characterization prior to our investigation. The other remaining case, 4-colorability with no additional restrictions, is a major open issue. This case is widely open and has not been characterized even for very special cases (such as planar graphs with one additional edge).

| $k$-coloring | 2 | 3 | 4 | 5 | $\geq 6$ |
|---|---|---|---|---|---|
| General | triv | $\infty$;NP-c | $\infty$;? | finitely many obstructions | |
| $\Delta$-free | triv | $\infty$;P | finitely many obstructions | | |
| Girth $\geq 5$ | triv | finitely many obstructions | | | |

Table 1.1: An overview of the type of class of $k$-colorability obstructions and the algorithmic complexity of coloring

## 1.5 Duals on Surfaces and Flows

For a graph $G$ embedded in a surface $\Gamma$, let $G^\star$ denote the *dual* of $G$. The vertices of $G^\star$ correspond exactly to faces of $G$; for $f \in F(G)$ we denote the dual vertex of $G^\star$ as $f^\star$. Conversely, the faces of $G^\star$ correspond exactly to vertices of $G$, we use the analogous '$\star$' notation to denote these relations. The edges of $G^\star$ correspond exactly to edges of $G$ in the following way, let $e \in E(G)$ be an edge and $f_1, f_2$ the two faces (not necessarily distinct) on the opposite sides of (the embedding of) $e$, then its dual edge $e^\star \in E(G^\star)$ is exactly $\{f_1^\star, f_2^\star\}$.

From the embedding of $G$ we obtain a natural embedding of $G^\star$; we embed every vertex $f^\star$ of $G^\star$ into its primal face $f$ and draw every edge $e^\star$ in such a way that it only crosses the embedding of $G$ somewhere along the embedding of $e$.

As a technicality, we need to introduce edge orientation into the embedded graphs in order to define flows. One possible approach is to replace each edge with two edges of opposite orientations. Such pair of edges would always exhibit mutually opposite flow, but together represent the same amount of flow from different perspectives. Another approach is to simply fix any arbitrary orientation of edges. In this approach we understand the flow along an edge with mismatched orientation (in respect to the fixed one) as a notational shortcut referring to the negations of the flow along the correct orientation of the edge. In this view flow on each edge represents a unique transfer between vertices.

Both of these approaches are of course equivalent in terms of theoretical strength. We use the latter notation, since it provides a simpler representation of our theory, at least in orientable surfaces. We usually understand all paths and cycles as oriented, by which we do not mean that they need to respect the fixed orientation. However when considering flows on edges of a path or a cycle, those edges that are used by a path or a cycle in the opposite direction than the fixed orientation have the signs of their flow values flipped.

Let $A$ be a finite Abelian group; that is a set of elements with addition-like operation which is associative, commutative and every element of $A$ has an inverse element in $A$. An *edge function* assigning elements of $A$ to (directed) edges is *nowhere-zero* if its range is a subset of $a \setminus \{0\}$, where 0 denotes the neutral element of $A$ (if it exists).

An $A$-*flow* in a graph $G$ is an edge function $f$ assigning elements of $A$ to (directed) edges of $G$ such that for every $v \in V(G)$ we have $\sum_{uv \in E(G)} f(uv) - \sum_{vu \in E(G)} f(uv) = 0$ (Kirchhoff's law).

An $A$-*tension* in a graph $G$ is an edge function $t$ assigning elements of $A$ to (directed) edges of $G$ such that for every directed cycle $C$ in $G$, $\sum_{uv \in E(C)} t(uv) = 0$. Recall that $t(uv) = -t(vu)$ if $uv$ does not match the fixed orientation of $G$. As a consequence, the sum over values of edges of a closed directed walk $W$ is also zero (respecting multiplicities of edges in $W$). We now show the following equivalence between colorings and tensions.

**Observation.** *A graph $G$ has proper coloring using elements of $A$ if and only if $G$ admits a nowhere-zero $A$-tension.*

*Proof.* Let $t$ be a nowhere-zero $A$-tension, $v$ a vertex of $G$ and $\alpha$ an element of $A$, then there exists a proper coloring $\phi_t$ of $G$ such that $\phi_t(v) = \alpha$. The coloring $\phi_t$ may be defined as $\psi(u) = \phi(v) + \sum_{e \in W} t(e)$ where $W$ is any (directed) walk from $v$ to $u$. By definition of tension, $\phi_t$ is independent of the specific choice of $W$.

Let $\psi$ be a vertex coloring of $G$ by the elements of $A$. Let $t_\psi$ be an edge function on $G$ defined as $t_\psi(uv) = \psi(v) - \psi(u)$ for each edge $uv \in E(G)$. The function $t_\psi$ is easily observed to be a nowhere-zero $A$-tension. $\square$

The next step toward the theory of nowhere-zero flows developed by Tutte [34] is to observe a connection between tensions and flows in the dual.

For and edge function $f$ on $G$ embedded in an orientable surface, let us define dual edge function $f^\star$ on $G^\star$ as following. Let $uv \in E(G)$ and let $gh \in E(G^\star)$ be the corresponding edge crossing $uv$ from left to right, as viewed in the direction $uv$. Then we set $f^\star(gh) = f(uv)$.

Note that this definition is consistent only when the embedding surface is orientable, it is not possible to consistently define left-to-right direction for all edges (or more general entities) simultaneously on a non-orientable surface. For our purposes, we consider only orientable surfaces, where consistent orientations for graphs and their duals can be defined. Later we also show some interesting consequences of non-orientability.

in the plane, the relation between flows, tensions and duals is the following. If $t$ is an edge function assigning elements of $A$ to (directed) edges in an embedded graph $G$, then $t$ is $A$-tension on $G$ if and only if $t^\star$ is an $A$-flow in $G^\star$. For any $v^\star \in V(G^\star)$, the set of out-edges incident with $v^\star$ correspond exactly to the facial walk of $v$ in $G$. The conditions on flow $t$ and tension $t^\star$ are satisfied (in the plane) if and only if the values over each such set of edges sum to zero.

**Observation.** *The existence of proper coloring of planar graph $G$ by elements of $A$ is equivalent to the existence of a nowhere-zero $A$-flow in $G^\star$.*

Similar equivalence holds for other surfaces. In orientable surfaces a dual of an $A$-flow is an $A$-tension if it additionally satisfies that it sums to zero over each (oriented) non-contractible cycle (it actually suffices if this holds for generators of the homology group).

Curiously, the structure of $A$ is irrelevant for the coloring, yet seems consequential for flows. It is a simple consequence of the theory of Tutte's polynomials [35] that the structure of $A$ in fact plays no role. The argument can be stated as follows. Let $\chi^\star(G, A)$ denote the number of possible nowhere-zero $A$-flows on $G$. Let $e$ be an edge of $G$. If $e$ is not a loop, then $\chi^\star(G, A) = \chi^\star(G/e, A) - \chi^\star(G-e, A)$ and if $e$ is a loop, then $\chi^\star(G, A) = (|A| - 1)\chi^\star(G - e, A)$. In the former case, consider an $A$-flow in $G/e$ and consider decontraction of $e$. Based on the excesses on the end-vertices of $e$, there is only one possible flow value for $e$ to extend the $A$-flow into $G$. Furthermore, if this values is 0, and it is the only zero element in the extended $A$-flow, then the obtained $A$-flow is equivalent to a nowhere-zero flow in $G - e$. In the latter case, any $A$-flow in $G - e$ can be extended to $G$ by setting the flow on $e$ arbitrarily. In here we view a loop as a short (contracted) cycle, that is, both ends of a loop contribute to the excess of a vertex and their contribution sums to zero. Excluding the element 0, we are left with $(|A| - 1)$ possible extensions over the loop maintaining the nowhere-zero property. Clearly, by expanding these relations, the value $\chi^\star(G, A)$ can be expressed in terms of $|A|$ by reducing the number of edges until only independent set of vertices remains.

We obtain the following corollary.

**Fact.** *A coloring of a plane graph $G$ using $k$ colors exists if and only if a nowhere-zero $A$-flow on $G^\star$ exists, where $A$ is any Abelian group of size $k$.*

Interestingly, any $A$-flow can be expressed using the standard integers. a *$k$-flow* on $G$ is a nowhere-zero flow over integers where flow on each edge is an integer of absolute value smaller than $k$ (analogously to the requirement that the Abelian group contains only $k$ elements). Naturally we require that the excess of every vertex is 0 in the standard integer addition. Clearly, every $k$-flow is a nowhere-zero $\mathbb{Z}_k$-flow up to the natural projection of negative integers into $\mathbb{Z}_k$. The converse is also true, interpreting the values of a nowhere-zero $k$-flow as values of a $k$-flow, the excess of some vertices may be non-zero, but it is always possible to find a path with positive flow on each edge connecting a vertex of positive excess to a vertex of negative excess. If we then lower the flow along such path by $k$, the overall deviation of excesses is reduced, and through iteration, all deviations are eventually removed.

For any graph $G$ we may define the *flow number* of $G$ as the lowest integer $k$ for which there exists a $k$-flow on $G$, or $\infty$ if no such $k$ exists. Alternatively, the flow number of $G$ is lowest integer $k$, such that for any $k' \geq k$ and any Abelian group $A$ of order $k'$, a nowhere-zero $A$-flow exists on $G$.

Let us explore a few specific useful cases of the equivalence between the existence of $k$-coloring of plane graph $G$ and $k$-flows in $G^\star$. Let $G$ be a (loopless) plane graph and let us consider 4-colorability of $G$. We may choose a group $\mathbb{Z}_2^2$. Note that each element in $\mathbb{Z}_2^2$ sums with itself to 0, and the sum of all non-zero elements is also 0. Since all operations are based on $\mathbb{Z}_2$, addition and subtraction coincide. This makes the orientation of edges irrelevant and we may understand $\mathbb{Z}_2^2$-flow as a simple assignment of elements of $\mathbb{Z}_2^2$ to undirected edges. Let us further suppose $G$ is a triangulation and therefore $G^\star$ is *cubic* (3-regular). Let $f$ be a nowhere-zero $\mathbb{Z}_2^2$-flow in $G^\star$. We observe that every vertex of $G^\star$ is incident with three edges, each of a distinct flow value and conclude that $f$ is a proper edge coloring of $G$ using three colors.

The same equivalence can be observed by coloring the edges of $G$ based on the pair of colors on their end-vertices; each edge with the pair of colors $(1, 2)$ or $(3, 4)$ obtains color $\alpha_1$, edge with pair $(1, 3)$ or $(2, 4)$ gets color $\alpha_2$, and the remaining edges get color $\alpha_3$. It is easy to see that every triangle in $G$ contains exactly all three edge-colors and therefore its dual vertex is incident with exactly one edge of each color.

Clearly, it is not possible to color edges of $G^\star$ using less then $\Delta(G^\star)$ colors so that no two edges of the same color share a vertex. By a classical result of Vizing [36], the number of colors needed (*chromatic index*) for any graph $H$ is either $\Delta(H)$ or $\Delta(H) + 1$. The graphs of the first kind form a Vizing's class one, and the rest form Vizing's class two.

A consequence of the relation of nowhere-zero flows and edge-coloring is that every planar graph is 4-colorable if and only if dual of every planar triangulation (which is itself planar) belongs to the Vizing's class one. While many natural classes of graphs are known to be subclasses of the Vizing class one, for example all bipartite graphs, deciding the exact value of chromatic index of a graph (and its Vizing class) in general is an NP-complete problem.

Let $G$ be a (loopless) graph embedded in an orientable surface and let us consider 3-coloring of $G$. We use the group $\mathbb{Z}_3$ with the notation of elements $Z_3 = \{0, 1, -1\}$. a nowhere-zero $\mathbb{Z}_3$-flow in $G^\star$ is then equivalent to an orientation of edges, with the property that for every vertex $f^\star \in V(G^\star)$, the difference $in(f^\star) - out(f^\star)$ is divisible by 3.

Suppose all faces of $G$ are of even length, then $G^\star$ allows a perfectly balanced orientation (every vertex is incident with the same amount of in-edges as the amount of out-edges), which can be obtained by orienting all edges along a closed Euleriean walk in $G^\star$. Such orientation is a special case of nowhere-zero $\mathbb{Z}_3$-flow and therefore $G$ is 3-colorable. This is actually a rather trivial observation in the plane, where the condition of even lengths of all faces actually implies that $G$ is bipartite. However the same argument involving nowhere-zero flows can be applied in higher surfaces as well.

On the other hand, suppose a plane graph $G$ has only two faces of odd length. Then the coloring of $G$ reduces to construction of a set of paths connecting these two faces in $G^\star$, as those are the only dual vertices with unbalanced edge orientations. If sufficient (odd) number of paths can be found, we may delete these (their edges) from $G^\star$, rendering all degrees in $G^\star$ even. It is then straight-forward to finish the nowhere-zero flow and obtain a 3-coloring of $G$. Note that reducing the dual into even-degree graph is not itself sufficient, by the divisibility by 3 condition of the nowehere-zero flow, we need to orient 3 paths from one odd face into the other before the rest of the task reduces to the Eulerian walk. Consequently $G$ is 3-colorable unless there is a small cut separating its odd faces, preventing the existence of these paths in the dual. Similar flavor of characterizations of 3-colorability turns out to be crucial in the study of 4-critical graphs.

Earlier we mentioned a result that a triangle-free graph embedded in the projective plane is 4-critical if and only if it is a non-bipartite quadrangulation of the projective plane. Let us explore why quadrangulations of the projective plane are never 3-colorable, emphasizing the bizzare behavior of nowhere-zero flows in non-orientable surfaces.

Let $\Gamma$ be a non-orientable surface and $\sigma$ a non-contractible curve in $\Gamma$ crossing through exactly one crosscap. Consider a graph $G$ constructed as follows, an odd cycle $C$ is embedded into $\Gamma$ in place of $\sigma$ and the rest of $\Gamma$ is quadrangulated arbitrarily so that every crosscap is contained in a face. The graph $G$ is embeddable into the projective plane, as its embedding uses only one crosscap. Consider such embedding and cut the projective plane along the odd cycle $C$ crossing through crosscap. We obtain a plane drawing (in a disc) with boundary $B$ formed by a cycle of length $2|C|$, formed by concatenation of both sides of $C$ (one of which is reflected because of the non-orientability). Let $uv$ be an edge of $C$. If we walk along $B$ in clockwise direction, we meet two instances of $uv$. Because of the non-orientability either we see its end vertices in the order $u, v$ for both instances or we see the order $v, u$ for both instances. Consider a precoloring of $C$ and the nowhere-zero flow on edges of $B$ given by the precoloring. The flow on both instances of $uv$ in $B$ is oriented either into the disk or out from the disk. This is contra-intuitive, as one would expect that upon gluing $B$ back into $C$, the directions of flow across $C$ should match. We get that the sum of flow across the disk boundary into the interior of the disk is twice some odd value, as there is odd number of edges in $C$, each contributing $+2$ or $-2$ units. Therefore, the outer face bound by $B$ has non-zero excess, acting as a source or a sink. The interior of $B$ is quadrangulated, each 4-face has excess of exactly 0, and therefore there are no sources or sinks of flow other than the outer face. This is clearly not possible. We conclude that the precoloring does not extend to a 3-coloring of $G$.

By the same argument it is possible to construct infinitely many triangle-free 4-critical graphs embeddable in the projective plane. By a much more sophisticated analysis of this idea, the before-mentioned result of Gimbel and Thomassen[27] shows that this is an exact characterization of all triangle-free 4-critical graphs embedded in the projective plane.

In a similar way one can inspect graphs in the torus and in the Klein bottle. In both surfaces we may (appropriately) embed a non-contractible odd cycle and quadrangulate the rest of the surface. If we then cut the surface along the embedded odd cycle, we obtain a cylindrical surface in both cases. In the case of torus the edges on the boundary of the cylinder behave as expected, any flow (on dual edges) leaving the surface over one boundary return over the other boundary. In the case of the cylinder obtained from the Klein bottle, we get the same behavior as in the projective plane case above.

The existence of $k$-flows is in itself a studied area with some prominent questions remaining open. On the other hand, several interesting results are known to hold. We give the following examples.

**Fact.** *The following claims hold:*

- *A graph has a* 2*-flow if and only if its degrees are even.*
- *A cubic graph has a* 3*-flow if and only if it is bipartite.*
- *Every* 4*-edge-connected graph has a* 4*-flow.*
- *Every bridgeless graph has a* 6*-flow [37].*

Each of these results can be translated into a sufficient (and in the first two cases necessary) conditions for the dual of a plane graph $G$ implying colorability of $G$ using an appropriate number of colors.

a few prominent conjectures of Tutte remain open. It is conjectured, that every bridgeless multigraph admits a 5-flow and further admits a 4-flow unless it contains the Petersen graph as a minor. It is also conjectured that every multigraph without a cut consisting of exactly one or three edges admits a 3-flow.

## 1.6 Perfectness and $\chi$-boundedness

For a graph $G$, let $\omega(G)$ denote the size of the largest clique present in $G$ as a subgraph, the *clique number*, and $\alpha(G)$ the size of the largest independent set in $G$, the *independence number*. Clearly, the inequality $\chi(G) \geq \omega(G)$ always holds for any graph $G$, and as witnessed by graphs of arbitrarily high girth and arbitrarily high chromatic number, there is no limit on the gap. A graph $G$ is said to be *perfect* if $\chi(H) = \omega(H)$ for every subgraph of $H$ of $G$.

The property of perfectness holds for several graph classes, prominent examples of which are chordal graphs, cographs, comparability graphs, interval graphs and various related classes. On the other hand, examples of non-perfect graphs are all odd cycles of length at least 5 and their complements; induced subgraphs isomorphic to these are usually referred to as *holes* and *antiholes* respectively. The perfectness is famously characterized by the Strong Perfect Graph Theorem as follows.

**Fact** ([38])**.** *A graph $G$ is perfect if and only if it contains no odd holes and no odd antiholes.*

While perfectness can be exploited in the quest to determine chromatic number of a graph[39], it is a much more generally useful property. In all perfect graphs, the exact values of $\chi(G), \omega(G), \alpha(G)$ can be computed in polynomial time [40] even though all three problems are NP-complete in general, even NP-hard to approximate with any fixed precision [41]. As a consequence, numerous otherwise hard problems can be solved or solutions to them approximated in polynomial time on perfect graphs. Additionally, perfectness itself can be decided in polynomial time[42].

A very useful generalization of perfectness is $\chi$*-boundedness.* a graph class $\mathcal{G}$ is $\chi$-bounded if there exists a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{G}$. In other words, the chromatic number is related to the clique number and can be bounded without dependence on the number of vertices.

While the gap between clique number and chromatic number can be arbitrary, it was conjectured by Esperet[43] that if $\mathcal{G}$ is a hereditary graph class $\chi$-bounded by some function $f$, then $f$ can be upper-bounded by a polynomial of a degree depending only on $\mathcal{G}$. This has recently been shown to be false [44].

Similarly to the relation of critical graphs and chromatic number, a large gap between the clique number and the chromatic number can be, perhaps, refined into specific structures witnessing this discrepancy. It is known, that if $G$ satisfies that $\chi(G) \gg \omega(G)$, then there exists a triangle-free subgraph $H$ of $G$ with a large chromatic number[45]. It is conjectured that the same is true when $H$ is considered to be of arbitrarily large girth. It is know on the other hand, that the same does not hold when $H$ required to be induced [46].

## 1.7 Precoloring Extensions and List Coloring

Suppose we are given a graph with a partial coloring; that is, some vertices have colors already assigned while others do not. It is a natural question whether such precoloring can be extended to a full coloring of all vertices of $G$ by a fixed number of colors. If we understand the precoloring as additional constraints on the coloring, it should come as no surprise that such question can be substantially harder to solve than classical coloring.

As an example of the hardness gap, consider the coloring of interval graphs which is solvable in polynomial time (even linear time, given representation with ordered vertices). On the other hand, we may consider coloring of circular arc graphs; intersection graphs of arcs on a circle. It is know that it is NP-complete [47] to decide the exact value of chromatic number of circular arc graphs. Such coloring of a circular arc graph can be modeled using extension of precoloring on an interval graph. Given representation by arcs, we may cut the circle at any point and unravel the representation onto a line. Any arc incident with the cutting point is now represented by two intervals on either end of the interval representation. Clearly, the task of coloring the circular arc graph is equivalent to coloring the obtained interval graph in such a way that some of the intervals receive the same color. Consequently, precoloring extension in interval graphs is NP-complete, even when restricted to the case that the precolored vertices are contained in two cliques, and thus each color appears at most twice.

When dealing with surfaces and the structure of the precolored vertices is somewhat reasonable, we may turn our attention to the dual and nowhere-zero flows therein. If the precolored vertices are not isolated, some edges of the dual have predetermined flow values and the coloring extension translates to the extension of this preflow. Note that if the precolored vertices induce multiple components, a valid extension of preflow may give a proper vertex coloring, but colors may be permuted differently on each precolored component. In such case, additional properties of the flow are required to extend the coloring properly. For more details refer to Section 2.2. This idea has proven very useful especially in the study of planar graph with precolored outer face.

In the specific case of 3-coloring and triangle-free graphs, let us consider the class of all planar graphs with outer face bound by a cycle of length at most 6, such that some (proper) precoloring of the outer face does not extend into a proper 3-coloring of the whole graph. Gimbel and Thomassen [27] showed that a graph is inclusion-wise minimal in this class if and only if the outer face is of length exactly 6, all other faces are of length 4, there are no separating ($\leq 5$)-cycles and the only precolorings that do not extend assign colors in order 123123 (up to permutation of colors) as viewed along the face.

Using the theory of nowhere-zero flows it is easy to see why the specific case of the result is not colorable; if we view the flow values as integers, the excess of the outer face is either 6 or $-6$ while the excess of all other faces is necessarily 0 in any nowhere-zero flow obtained from any complete coloring extending the given precoloring, which is not possible. On the other hand, removing any internal edge would produce a new 6-face which may attain the appropriate excess, making the existence of the flow at least feasible, pointing towards their minimality. Similarly, if the outer face is of length 5 or 3, its excess is 3 or $-3$, and by parity there has to exist an internal odd face, which may attain the appropriate excess to compensate.

Analysis similar to the thought process above is a common method involved in the study of critical graph on surfaces. a graph embedded into a surface can often be cut into patches interacting only through (ideally short) boundaries. If the colorability of a patch is strongly dependent on the boundary precoloring, then the patch may be reduced or its specific structure may be abstracted away. For example, by the result above (considering triangle-free graphs and 3-coloring), the inside of a separating ($\leq 5$)-cycle, that is, a cycle bounding a planar graph with at least one vertex inside, can simply be ignored when investigating 3-colorability of the whole graph. Similarly, all contractible separating 6-cycles containing a quadrangulation are in a sense equivalent, so their interior may be replaced with a simple quadrangulation with a single vertex of degree 3, forbidding the only two (up to color permutation) boundary precolorings that do not extend inside. In this way, the planar graphs with precolored boundary form a subset of coloring problems that is an essential building block for study of colorability on all surfaces.

In dealing with colorings, we will also briefly touch an important generalization of the precoloring extension, *list coloring*. Suppose we have a list assignment $L$, which assigns to every vertex $v$ a list $L(v)$ of allowed colors. The list coloring problem asks whether it is possible to properly color graph so that color of each vertex is chosen from its list of allowed colors; if so, we say that the graph is $L$-colorable.

We say that $L$ is *assigment of size $s$* if each list is of size at least $s$. a graph is said to be *$s$-choosable* if it is $L$-colorable for every list assignment $L$ of size at least $s$. Clearly, if all of the lists are the same (and of size $k$), the list coloring is equivalent to simple $k$-coloring.

Varying of the sizes of the lists or colors may offer additional freedom. For instance, an odd cycle is $L$-colorable by any assignemnt $L$ of lists of sizes at least 2 unless all lists of $L$ are of size exactly 2 and identical. On the other hand, while it may seem that the most restrictive list assignment (of given size) is the one that assigns the same list to every vertex, that is not the case. There exist bipartite graphs with arbitrarily high choosability [48], while readily colorable from the list assignment assigning the same list of size 2 to every vertex. Since precoloring extension problem can naturally be modeled as a list-coloring, and also the other way around to some degree, we once again see that precoloring extension can be substantially more complex than ordinary coloring.

List coloring is a rich and complex area in itself and is studied heavily. The most interesting results however come from study of special cases. A compelling case of list coloring is when $L$ is *degree assignment*. For a fixed graph $G$ we say that list assignment $L$ is a degree assignment if for every vertex $v$ we have $|L(v)| \geq deg(v)$. By a greedy approach, it is easy to show that if a graph $G$ is not $L$-colorable, where $L$ is a degree assignment, then for every vertex $v$ we have $|L(v)| = deg(v)$. Brooks [1], showed that a connected graph $G$ of maximum degree $\Delta$ is not $\Delta$-colorable if and only if $G$ is a clique or an odd cycle. The same conclusion can be simply lifted to degree assignment coloring, assuming the graph in question is 2-connected. If not, then by study of the cut vertices, we may argue that a connected graph is not list colorable by a degree assignment if and only if it is a Gallai tree, and additionally the color lists follows certain regularities.

Consider now classical coloring of a graph using $k$ colors. Suppose it is possible to precolor all vertices of degree at least $k+1$. We may consider extension of this precoloring to the rest of the vertices, modeled as a list coloring, which falls into the case of degree assignment list coloring. Suppose $G$ is $(k+1)$-critical, with vertex $v$ of degree at most $k$. Removing $v$ from $G$ allows $G$ to be $k$-colored, therefore vertices of degree at least $k+1$ are $k$-colorable, and the vertices of degree at most $k$ must form a subgraph not colorable by a degree assignment and therefore form a Gallai forest. As stated in Section 1.2, this insight is a key idea in the study of critical graphs, and ties the much more abstract list-coloring back to the classical coloring in a rather surprising way.

## 1.8 Outline and Results

In Section 2.1 we focus on developing a theory which will allow us to study colorings, or rather nowhere-zero flows, in the cylinder. We derive a decision algorithm for the problem of 3-colorability which is efficient for near-quadrangulations of the cylinder with precolored boundary cycles. In the case of a negative answer, we also obtain a certificate, using the ideas of duality. Finally we derive sufficient conditions for the existence of 3-coloring based on structural impossibility of the presence of these certifying structures, almost identical with the necessary conditions.

We begin the Section 2.2 by introduction of several key results characterizing pieces of 4-critical graphs. Then, we analyze the algorithm from Section 2.1 when applied to graphs embedded in the torus, formulate a set of obstructions to obtaining a 3-coloring and devise a strategy to avoid these obstructions in a systematic manner. Finally, we show that if a near-quadrangulation of the torus has high enough edge-width, we can avoid all obstructions and obtain a 3-coloring, showing that triangle-free 4-critical graphs embedded in the torus (which are necessarily near-quadrangulations) have bounded edge-width. Both Sections 2.1 and 2.2 are based on the paper

Dvořák, Pekárek: *Coloring near-quadrangulations of the cylinder and the torus* [49]

In Section 2.3 we introduce the concept of reducing 4-faces of 4-critical graphs. In a short detour from our main line of thought, we then show analysis of the process in more detail, leading to alternative computer-assisted enumerative approach of obtaining characterization of 4-critical graphs. The result is a relatively simple, although not particularly concise, characterization of the class of triangle-free 4-critical graphs embedded in the torus via a set of templates. At the end of the section we discuss algorithmic applications of the characterization. Section 2.3 is based on the paper

Dvořák, Pekárek: *Characterization of 4-critical triangle-free toroidal graphs* [32]

Finally, we show that the problem of 3-colorability of triangle-free graphs embedded in a surface can in essence be boiled down to the same problem considering only near-quadrangulations. We apply the results from Section 2.1 and Section 2.2 and obtain an efficient practical algorithm that can decide 3-colorability. Through additional effort, we then bypass non-constructive portions of this construction to design an efficient algorithm which can obtain a specific coloring.

In the remainder of the work we change focus to graphs with limited odd cycle packing number parameter, which is a property found in complements of graph embedded in the plane or represented by shapes embedded in the plane.

In Section 3.1 we discuss the connection between the structural odd cycle packing number parameter and topological properties of representation in the plane, in particular investigating what flavors of graphs obtained from various representations may be expected to have the mentioned parameter limited.

In Section 3.2 we make use of the odd cycle packing number. We begin by introduction into the necessary theory and the concept of $\chi$-boundedness. We derive algorithms (approximately) solving the largest independent set problem and derive several $\chi$-boundedness results including a lower-bound. Section 3.2 is based on the paper

Dvořák, Pekárek: *Induced odd cycle packing number, independent sets, and chromatic number* [50]

# 2. Coloring in the Torus

## 2.1 Cylinder

In this section we investigate the existence of colorings of graphs embedded in the cylinder, with precolored boundaries. The method we employ is interpretation of coloring as a nowhere-zero flow in the dual, and instead of extending precoloring we extend a preflow. The goal is to develop tools to decide 3-colorability in the torus.

Let $G$ be a plane graph and $C$ a cycle in $G$. We view $G$ as drawn in the sphere. By *splitting* $G$ along $C$ we mean the following operation. Let $\lambda$ be the closed curve representing the embedding of $C$ into the sphere $\Sigma$ and let $\Lambda_1, \Lambda_2$ be the closures of the two parts into which $\Sigma$ decomposes. We define graphs $G_1, G_2$ as the subgraphs of $G$ induced by the embedding in $\Lambda_1$ and $\Lambda_2$ respectively. That is, the cycle $C$ is part of both parts $G$ splits into, and $G$ can be obtained back from $G_1, G_2$ by unifying their respective copies of $C$.

Note that both surfaces $\Lambda_1, \Lambda_2$ can be viewed either as closed disks or as spheres with holes, which are topologically equivalent. We consider both $G_1, G_2$ as having planar embedding with one face (bounded by the copy of $C$) being exactly the hole. We call such face a *ring*. Additionally there is a natural mapping between the facial walks of rings in $G_1$ and $G_2$ specifying which verticies are copies of the same original vertex.

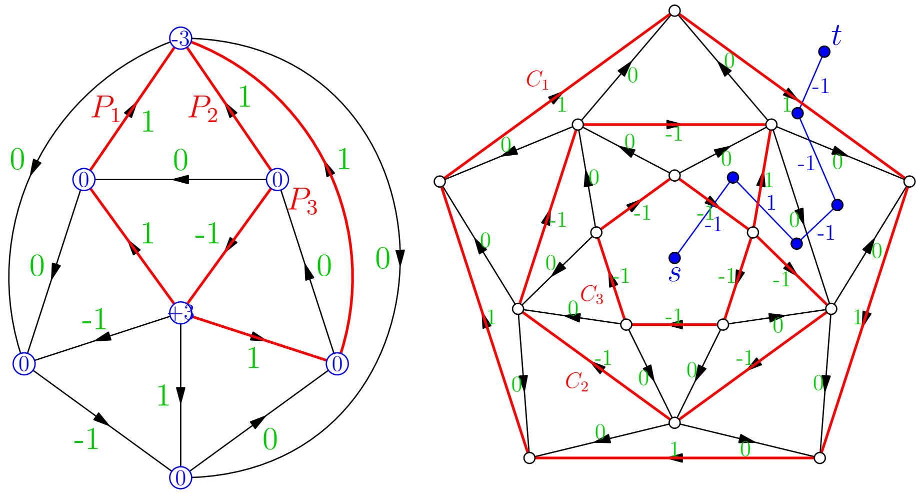
For a more general surface $\Sigma$ and $G$ embedded in $\Sigma$, a split along a contractible cycle $C$ is defined analogously, except that one of the parts is embedded in the sphere with a hole, and the other in $\Sigma$ with a hole.

Of particular interest to us is the torus. Let $G$ be a graph embedded in the torus and let $R$ be a non-contractible cycle in $G$. By *splitting* along $R$ we mean the following operation. Let $\lambda$ be the closed curve representing the embedding of $R$. We consider the surface obtained by deleting $\lambda$ from the torus, which is an open cylinder. We close both boundaries, and consider the graph $G'$ induced by the embedding. By this we mean that the cycle $R$ now exists in two copies in $G'$, each embedded along one of the surface boundaries, with natural correspondence between the two copies. Understanding the new cylindrical surface as the sphere with two holes, we call the faces corresponding to the holes *rings*.

The graph $G'$ is equivalent to a plane graph, however we keep awareness of its rings, since their presence is useful for working with several concepts that are not well defined for plane graphs with no rings and help us study properties of $G$ while inspecting properties of $G'$.

Similarly, a graph embedded into the Klein bottle can be split along a non-contractible orienting cycle into a plane graph with two rings. The difference between a plane graph obtained by splitting from the torus and from the Klein bottle is that the natural isomorphism of rings is oriented along the faces in the same direction if the graph was obtained from the Klein bottle, and in the opposite direction if it was obtained from the torus.

Clearly, any plane graph can be embedded into a sphere with two holes so that two of its faces become rings. These faces can be chosen arbitrarily and given the faces are of the same lengths, we may then un-split, obtaining either




A function $d$ (blue), a $d$-flow (green) and a corresponding $d$-linkage $\{P_1, P_2, P_3\}$ (red).

An $(s,t)$-circulation $\{C_1, C_2, C_3\}$ of size 3 (red) and a $\varnothing$-flow $h$ with $\int_Q h^\star = 1$ (green). The path $Q$ and the values of $\sigma(Q, e^\star)$ are in blue.

Figure 2.1: Flows, linkages and circulations.

a graph embedded in the torus or in the Klein bottle, possibly with multiedges or loops.

### 2.1.1 Flows, Circulations, Vorticity

For the purposes of this section, let us assume that all edges in given graphs have a fixed orientation, otherwise we fix an arbitrary orientation. The only purpose of this assumption is as a reference so that we can define directed values on edges (such as flows) using signed values.

Paths and walks in this section are always understood as ordered lists of edges, such that the consecutive edges share a vertex in the natural way. To give a full formal definition, each edge in the list is aligned to have a fixed order of vertices (independent of its orientation) so that the first vertex is only shared with the preceding edge, and the second vertex is shared with the following edge. A path or a walk is not required to respect the fixed orientations of edges, but through the order has a defined beginning- and end-vertex, and therefore a direction. Similarly, a cycle or a closed walk is understood as cyclically ordered list of edges (satisfying the usual requirements), hence there is always an associated direction.

Recall that under *edge function* and *vertex function* we understand functions assigning values from some set of values to (oriented) edges or vertices respectively. The sets of values we consider are always taken as (subsets of) Abelian groups, allowing commutative summation and subtraction.

For an edge function $f$, let *excess* $\delta_f(v)$ of vertex $v$ be defined as $\delta_f(v) = \sum_{uv \in E(G)} f(uv) - \sum_{vu \in E(G)} f(vu)$.

*Demand function* is a vertex function assigning whole numbers to vertices with zero sum, that is $d(V(G)) = \sum_{v \in V(G)} d(v) = 0$. We say that a demand function is *even* if for every vertex the demand value has the same parity as the degree of the vertex. We denote as $\varnothing$ the demand function that is zero everywhere.

For a demand function $d$, we define a *d-flow* as an edge function assigning elements from set $\{-1, 0, 1\}$ such that for every vertex $v$, $d(v) = \delta_d(v)$.

For a demand function $d$, we define a *d-linkage* as a set $\mathcal{P}$ of edge-disjoint paths satisfying that paths only begin in vertices with positive demand, end only in vertices with negative demand, and every vertex $u$ is the beginning- or end-vertex of exactly $|d(u)|$ paths in $\mathcal{P}$.

A *support set* $supp(f)$ of a $d$-flow is the set of all edges with non-zero flow value in $f$. Similarly, a *support set* of a $d$-linkage $\mathcal{P}$ is the set of edges in $\cup\mathcal{P}$.

**Observation 1.** *Let $G$ be a graph and $d$ a demand function on $G$. If there exists a $d$-flow $f$ on $G$, then there exists a $d$-linkage $\mathcal{P}$ such that $supp(\mathcal{P}) \subseteq supp(f)$. If there exists a $d$-linkage $\mathcal{P}$, then there exists a $d$-flow such that $supp(f) = supp(\mathcal{P})$.*

Let us define *circulation* $c$ on $G$ as an $\varnothing$-flow, a *cycle-set* $\mathcal{Q}$ as a set of edge-disjoint cycles in $G$ and $supp(c)$ and $supp(\mathcal{Q})$ analogously as before.

Clearly, a similar relation to that between $d$-flows and $d$-linkages exists between circulations and cycle-sets. Furthermore, the following relation ties all definitions together. Notice that the supports in the following observation form a disjoint union.

**Observation 2.** *Let $G$ be a graph and $d$ a demand function on $G$. A $d$-flow on $G$ exists, if and only if there exists a $d$-linkage $\mathcal{P}$ and a cycle-set $\mathcal{Q}$ such that $supp(f) = supp(\mathcal{P}) \dot\cup supp(\mathcal{Q})$ (possibly $\mathcal{Q} = \emptyset$).*

Let $\sigma$ denote *crossing function* defined as follows. For a walk $Q$ in $G$ and an (oriented) edge $e$ of $G$, $\sigma(Q, e)$ is the number of times $Q$ traverses $e$ in the direction of $e$ minus the number of times $Q$ traverses $e$ in the opposite direction. Note that if $Q$ is a path or a cycle, then $\sigma(Q, e) \in \{-1, 0, 1\}$ and $\sigma(Q, e) \neq 0$ if and only if $e$ is in $Q$.

For an edge function $p$ and a walk $Q$, we define the *gatherer* $\int_Q p$ of $p$ over $Q$ as $\int_Q p = \sum_{e \in E(G)} \sigma(Q, e)p(e)$. In essence, gatherer is the sum of values of $p$ on $Q$, but in a directed (and weighted) sense.

We say that an edge function $p$ is a *tension* of a vertex function $w$ if for every edge $e = uv$ we have $w(v) = w(u) + p(e)$. By extension, for any two vertices $u$ and $v$ we get $w(v) = w(u) + \int_Q p$ for any walk $Q$ from $u$ to $v$. In particular, a gatherer $\int_Q p$ over a closed walk $Q$ is always zero for any tension function $p$.

For a plane graph $G$, let us consider the dual $H$ of $G$. Recall that the fixed orientation of edges in $G$ induces an orientation of edges in $H$ such that each edge $e$ of $G$ is crossed (in the plane embedding) by its dual edge $e^\star$ of $H$ from left to right.

Confusingly, a dual of $H$ has the opposite orientations to $G$. In the notations used here it therefore matters whether $G$ is a dual of $H$ or $H$ is a dual of $G$. We will avoid repeated dualization by "undualizing" instead, that is, defining $G$ from given $H$ as the graph with dual $H$, whenever needed. However the change in directions appears a few times as a sign change. We will point out every occurrence of this.

For a walk $Q$ in $G$, let us exploit the notation and define $\sigma(Q, e^\star) = \sigma(Q, e)$, that is, the number of times $Q$ (or rather its embedding) crosses $e^\star$ from right to left minus the number of times it crosses in the opposite direction. A slightly more useful view to understand the value $\sigma(Q, e)$ is that it receives a positive contribution whenever $e$ goes along $Q$ and similarly $\sigma(Q, e^\star)$ receives positive contribution whenever $e$ crosses $Q$ from left to right. That is, the value is preserved by dualization of $e$, which effectively rotates $e$ clockwise into $e^\star$.

Let $f$ be a face of $G$ and $R$ a closed walk in $G$. Let $\omega_R(f)$ denote the *winding number* of $R$ around $f$. Informally, the winding number $\omega_R(f)$ denotes the number of times $R$ circles around $f$ in the clockwise direction. A formal definition of $\omega_R(f)$ can be formulated as follows: For any half-line $p$ beginning inside $f$ and intersecting $G$ only in edges, $\omega_R(f)$ is the number of times $R$ crosses $p$ from left to right minus the number of times it crosses $p$ from right to left. Let us remark that the value of $\omega_R(f)$ is independent of the choice of the half-line $p$.

**Observation 3.** *Let $G$ be a plane graph, $H$ its dual (with the natural embedding) and $R$ any closed walk in $G$. Then $\omega_R$ has the following properties:*

- *Let $f$ be the outer face of $G$, then $\omega_R(f) = 0$,*
- *Let $e^\star = (g^\star, h^\star) \in E(H)$, then $\omega_R(h^\star) = \omega_R(g^\star) + \sigma(R, e^\star)$,*

*Furthermore, a face function on $H$ is equal to the winding number (for fixed $R$) if and only if it satisfies both of these properties.*

The second property can be formulated as a tension. For a fixed $R$ we may consider the dual vertex function $\omega_R^\star$ on $H$. Clearly, by fixing the parameter $R$ in $\sigma(R, e^\star)$ we obtain a tension $\sigma_R(e^\star)$ of $\omega_R^\star$.

In the following key notation, we use the crossing function in the following way. Suppose $Q^\star$ is a walk in $G^\star$ and $e$ is edge of $G$, then by definition $\sigma(Q^\star, e) = -\sigma(Q^\star, e^\star)$. Note that the minus sign comes from considering a dual of $e^\star$, since $\sigma(Q^\star, e^\star)$ is well defined and $\sigma(Q^\star, (e^\star)^\star)$ has the same value by definition, but the edge $(e^\star)^\star$ has the opposite direction to $e$. The value of $\sigma(Q^\star, e)$ expresses how many times (the embedding of) $e$ crosses (the embedding of) $Q^\star$ from right to left.

Let $d$ be a demand function on $G$, and let $d^\star$ denote its dual face-function on $G^\star$. Consider a closed walk $R^\star$ in $G^\star$. Let us define *pressure* $\nabla(R^\star, d)$ of $d$ on $R^\star$ as $\nabla(R^\star, d) = \sum_{f \in F(G)} \omega_{R^\star}(f) d(f)$. Note that if $R$ is a cycle, then $\nabla(R^\star, d)$ is a simple sum of $d^\star$ over faces in the interior of $R$ (up to sign depending on the orientation of $R$).

The motivation behind pressure comes from the properties of flows. There exists a (multi-)set of edges $R$ in $G$ such that (up to ordering) $R^\star$ is the dual of $R$. The set $R$ acts as a cut in $G$ separating the interior and exterior of $R$. The pressure on $R$ describes exactly how many units of $d$-flow necessarily cross $R$ from inside out, as shown in the following lemma.

**Lemma 4.** *Let $G$ be a plane graph, $d$ a demand function on $G$ and $h$ a $d$-flow. For any closed walk $R^\star$ in $G^\star$*

$$\nabla(R^\star, d) = \int_{R^\star} h$$

*Let $s^\star, t^\star \in V(G^\star)$, and let $P_1, P_2$ be two $(s^\star, t^\star)$-walks in $G^\star$. If $R^\star$ is a closed walk obtained by the concatenation of $P_1$ and reverse of $P_2$, then*

$$\int_{P_1} h = \int_{P_2} h + \nabla(R^\star, d)$$

*Proof.* We explain the computation, and express it formally below. Recall the definition of pressure $\nabla(R^\star, d)$. It can be expressed as a sum over demands $d$ of all vertices of $G$ weighted by winding number of $R^\star$. Such sum is equal to a weighted sum over excesses, given a valid $d$-flow $h$ in $G$.

Expressing the excess as sum over edges and using Observation 3, we may reorder the sum as a sum over edges. Value of each edge is weighted by the difference of winding numbers of its endpoints, which is the crossing function. Clearly, crossing function is zero if edge is not from the walk $R^\star$. We obtain the expression of (negation of) $h$-gatherer over $R^\star$.

$$\begin{aligned}
\nabla(R^\star, d) &= \sum_{f \in F(G^\star)} \omega_{R^\star}(f) d(f) = \sum_{v \in V(G)} \omega_{R^\star}(v) \delta_h(v) \\
&= \sum_{v \in V(G)} \omega_{R^\star}(v) \Big( \sum_{uv \in E(G)} h(uv) - \sum_{vu \in E(G)} h(vu) \Big) \\
&= \sum_{uv \in E(G)} \Big( \omega_{R^\star}(v) - \omega_{R^\star}(u) \Big) h(uv) = \sum_{uv \in E(G)} -\sigma(R^\star, uv) h(uv) \\
&= \int_{R^\star} h
\end{aligned}$$

If $R^\star$ is the concatenation of $P_1$ and the reverse of $P_2$, we can split the final expression as $-\int_{R^\star} h = \int_{P_2} h - \int_{P_1} h$. By rearranging, we obtain the desired equality. $\square$

### 2.1.2 Existence of d-flows in the cylinder

In this section we explore the necessary and sufficient conditions for the existence of a particular kind of $d$-flows. Recall that any coloring of a plane graph induces a nowhere-zero flow on the dual. If the graph has a precolored connected subgraph, the precoloring can be interpreted as a preflow in the dual and the coloring can be obtained by completing the nowhere-zero flow (if possible). If the precoloring induces a disconnected subgraph, however, it is not enough to complete the preflow. Recall that in a nowhere-zero flow obtained from a $k$-coloring, every walk between two vertices of the same color is crossed by the same amount of flow from left to right as the amount of flow from right to left, up to modulo $k$. Analogous relation holds for all other pairs of vertices as well. In order for the completed nowhere-zero flow to give a consistent coloring across all connected components of the precoloring, such property of the flow must be enforced.

Our goal is to extend precoloring of a cylindrical graph, where the precoloring is given on the facial walks of its rings. We assume that the structure of the graph is such that the demand function $d$ we are trying to satisfy has a small number of non-zero entries. A potential nowhere-zero flow can then be decomposed into two edge-disjoint parts, a linkage satisfying the demand function and a circulation which covers the rest of the edges. To characterize the existence of a linkage, we use a standard max-flow min-cut argument. However, to ensure that the nowhere-zero $d$-flow provides a coloring consistent with the precoloring, we need to control the amount of flow circling around the cylinder. To this end we lay down some basic tools to quantify such property and ensure that we can obtain appropriately behaved circulations. We give constructive proofs characterizing when a nowhere-zero combination of a disjoint linkage and a sufficiently varied circulation exists.

Let $G$ be a graph embedded in the sphere and let $r_1, r_2$ be the rings of $G$ (given with order). For a cycle $C$ in $G$, let us consider the two parts into which $C$ cuts the sphere. We say that $C$ is *contractible* if both rings are contained in one part and otherwise, that is, if both sides of $C$ contain a ring, we say that $C$ is *non-contractible*. The contractibility notion is consistent with understanding $G$ as embedded in the sphere with rings realized as topological holes, or in an infinite cylinder.

In order to utilize tools from Section 2.1.1, we need to define the winding number in the sphere. Recall that the winding number extends the notion of interior and exterior of a closed curve, however interior is not well defined in the sphere. We use the Observation 3 which shows that given an embedding, it is enough to fix the outer face (with winding number 0) for the winding number to be well defined. We fix the second ring of $G$ ($r_2$) as the outer face. Equivalently, in a less formal approach, we may understand $G$ as embedded in the plane so that $r_2$ is the outer face.

Let $C$ be a cycle in $G$ and as before let us consider the two parts into which $C$ cuts the sphere embedding $G$. We call the part containing $r_2$ the *exterior* of $C$, denoted $ext(C)$, and the other part the *interior* of $C$. Observe that this is consistent with the above definition of the winding number.

Let $G$ be a plane graph with rings $r_1, r_2$. Let $c$ be a circulation in $G$ and $Q$ a path from $r_1$ to $r_2$ in $G^\star$. We say that $Q$ is *ring-connecting.* We define *vorticity* of $c$ as the value $\textstyle\int_Q c$. Note that the sign of the vorticity depends on the order of the rings, which is given.

**Observation 5.** *Vorticity of a circulation (in respect to ordered rings $r_1, r_2$) is well defined, that is, it does not depend in the choice of $Q$.*

*If there exists a circulation of vorticity $k > 0$ then there exist circulations on the same support set of all vorticities from the range $[k, -k]$ with the same parity as $k$.*

*Proof.* Let $G$ be a plane graph with rings $r_1, r_2$ and $c$ a circulation on $G$. Let $Q_1, Q_2$ be two ring-connecting paths in $G^\star$. Let $R^\star$ be the closed walk obtained as the concatenation of $Q_1$ and reverse of $Q_2$. By the properties of pressure shown in Lemma 4, $\textstyle\int_{Q_1} c = \textstyle\int_{Q_2} c + \nabla(R^\star, \varnothing)$, where $\nabla(R^\star, \varnothing) = 0$. We conclude that the value of vorticity of $c$ is independent of the choice of $Q$.

Given a circulation $c$ of vorticity $k$, reversing flow on every edge clearly produces a circulation with the opposite vorticity and the same support. Without loss of generality, suppose $k > 0$. We show that we can obtain circulation of vorticity $k - 2$, the claim then holds by induction.

By Observation 2 we may express $c$ as a collection of (oriented) edge-disjoint cycles, each carrying a flow of magnitude 1. By inspection of the definition of vorticity we may express the vorticity of $c$ as the sum of vorticities of the individual cycles. Since vorticity of a single cycle of unit flow is equivalent to the number of times it crosses $Q$ in one direction minus the times it crosses in the other direction, it is equivalent to the winding number of the cycle, in respect to $r_1$.

If there is a cycle $C$ with vorticity of magnitude greater than 1, it then wraps around $r_1$ multiple times and by planarity must intersect itself, contradicting the choice of the circulation decomposition into cycles rather than closed walks. We conclude that there are at least $k$ cycles of vorticity 1.

We define a new cycle set by reversing one of the cycles of vorticity 1 (in $c$). Clearly the support of the cycle set remains unchanged, however the vorticity of the associated flow has decreased by exactly 2. $\square$

For a graph $G$ with fixed rings $r_1, r_2$ and a demand function $d$ on $G$, we say that a $d$-flow $f$ is of *maximal vorticity* or *minimal vorticity*, if the value $\int_Q f$ is the maximal, respectively minimal value possible among all $d$-flows, given a fixed ring-connecting path $Q$.

While it is not true that the value $\int_Q f$ is independent of the choice of $Q$, we observe that maximum- and minimum-attaining flows are independent of the choice of $Q$.

**Observation 6.** *Let $f_1, f_2$ be flows in $H$ and $Q_1, Q_2$ ring connecting paths in $H^\star$, then $\int_{Q_1} f_1 - \int_{Q_1} f_2 = \int_{Q_2} f_1 - \int_{Q_2} f_2$.*

*Proof.* By Lemma 4 we have that $\int_{Q_1} f = \int_{Q_2} f + \nabla(R^\star, d)$ where $R^\star$ is a closed walk obtained as the concatenation of $Q_1$ and the reverse of $Q_2$. Since $\nabla(R^\star, d)$ is constant not depending on $f$ we observe that measurements of vorticities on $Q_1$ and $Q_2$ only differ by this constant for all $d$-flows. $\square$

We show that flows of maximal and minimal vorticities can be constructed and certified.

**Lemma 7.** *Let $H$ be a graph and $d$ an admissible demand function. Let $h_1, h_2$ be $d$-flows of maximal and minimal vorticity respectively, with inclusion-wise maximal support sets. Then there exist ring-connecting paths $Q_1$ and $Q_2$ such that $\int_{Q_1} h_1 = |E(Q_1)|$ and $\int_{Q_2} h_2 = -|E(Q_2)|$. Furthermore, $E(H) \setminus supp(h_1)$ and $E(H) \setminus supp(h_2)$ are forests, and in particular if $d$ is even, then $supp(h_1) = supp(h_2) = E(H)$.*

*Proof.* We prove both claims for $h_1$, the claims then follow for $h_2$ analogously.

Let $E_0 = E(H) \setminus \mathrm{supp}(h_1)$. For contradiction with the second point, let $E_0$ contain a cycle. Clearly, there exists a non-empty circulation $c$ such that $\mathrm{supp}(c) \subseteq E_0$. Let $Q$ be any ring-connecting path in $H$. We may choose $c$ so that $\int_Q c \geq 0$. By the choice of $c$, $h_1 + c$ is a $d$-flow with support set $\mathrm{supp}(h_1) \dot\cup \mathrm{supp}(c)$ and $\int_Q (h_1 + c) = \int_Q h_1 + \int_Q c \geq \int_Q h_1$, a contradiction with the definition of $h_1$. We conclude that $E(H) \setminus \mathrm{supp}(h_1)$ is a forest.

Suppose $d$ is even, then every vertex $v$ of $H$ is incident with number of edges in $\mathrm{supp}(h_1)$ of the same parity as $deg(v)$. In other words, $v$ is incident with even number of edges from $E(H) \setminus \mathrm{supp}(h_1)$. Since any forest on $H$ must have a leaf, we conclude that $E(H) \setminus \mathrm{supp}(h_1) = \emptyset$.

It remains to find a ring-connecting path $Q_1$ such that $\int_{Q_1} h_1 = |E(Q_1)|$. Let us consider $H^\star$ and define $W$ as the set of vertices of $H^\star$ reachable from $r_1^\star$ along edges $e$ such that the flow $h_1$ crosses $e$ from left to right (that is, the crossing function is positive). Clearly, if $r_2$ is reachable via some path $Q$, then $\int_Q h_1 = |Q|$.

Suppose for contradiction that $r_2 \notin W$. The set $W$ represents a connected subset of faces in $G$. Let $G'$ be obtained by deleting from $G$ all edges separating pairs of faces in $W$ and let $r$ be the face of $G'$ containing $r_1$. By definition, the facial walks of $r$ contain only edges with either no flow, or a unit of flow from right to left (as viewed from $r$). Let $c$ be the circulation in $G$ obtained by sending a unit flow from left to right along the edges of $r$ (clockwise as viewed from $r$). By definition, $h_1 + c$ is a $d$-flow. Additionally, $c$ has vorticity 1, and so vorticity of $h_1 + c$ is higher than vorticity of $h_1$, a contradiction. $\square$

It is easy to see that for any $d$-flow $h$, the existence of a ring-connecting path $Q$ such that $|\int_Q h| = |Q|$ implies that $h$ is either of maximal or minimal vorticity. Formally, suppose $\int_Q h = |Q|$ and for some other $d$-flow $h_0$ and some other $Q'$, $\int_{Q'} h_0 > \int_{Q'} h$. Then by Observation 6, $\int_Q h_0 > \int_Q h = |Q|$ which is not possible.

Our goal is to construct a linkage together with a disjoint circulation (or equivalently a cycle set) of high vorticity (in absolute value). Note that if we consider decomposition of a flow into a linkage and a cycle set, then a flow of maximal vorticity does not necessarily maximize the vorticity of its cycle set.

As an example, let us have $G$ such that there exist two ring-connecting paths $Q_1, Q_2$ in the dual, where $Q_1$ has length 1 and $Q_2$ has length 2. Suppose there is a single pair of vertices with non-zero demands $+1$ and $-1$, and they are separated by the cycle $Q_1 \dot\cup Q_2$. Let a flow $h$ of maximal vorticity satisfy $\int_{Q_1} h = 1$ and $\int_{Q_2} h = 0$, where $Q_1$ clearly certifies the maximality. The flow $h$ can either be composed of a linkage with a single path passing through $Q_1$ and an empty cycle set, or of a linkage with a single path passing through $Q_2$ and a cycle passing through both $Q_1$ and $Q_2$ of vorticity 1.

We say that a flow $f$ on graph $G$ is *nowhere-zero* if there is no edge $e$ in $G$ for which $f(e) = 0$. For a graph $H$, let $\|H\|$ denote the number of edges of $H$. Let $d$ be a vertex function on $G$, then we define *size of* $d$ as the $|d| = 1 + \sum_{v \in V(G)} |d(v)|$. We add the 1 so that $|d|$ is always positive.

**Lemma 8.** *Let $H$ be a connected plane graph, $r_1, r_2$ rings of $H$, and $d$ an even demand function on $H$. There exists an algorithm with time complexity*

$$O(|d| \cdot \|H\|)$$

*which finds nowhere-zero $d$-flows $h_1, h_2$ of maximal and minimal vorticity respectively, or decides that $H$ contains no $d$-flow.*

*Proof.* It suffices to find a $d$-flow $h_1$ of maximum vorticity, $h_2$ of minimum vorticity is obtained as the $d$-flow of maximum vorticity. We may assume that $H$ is embedded in the plane with outer face $r_2$.

We find a $d$-flow $h_0$ in $H$ or decide that no $d$-flow exists. This can be achieved in time $O(|d| \cdot \|H\|)$ using Ford-Fulkerson algorithm. Since $d$ is even, $H \setminus \mathrm{supp}(h)$ is Eulerian and can be partitioned into a cycle set in linear time, which can be lifted into a circulation $c$. Together $h_0 + c$ form a nowhere-zero $d$-flow $h$.

Let $\vec{H}$ be the graph obtained from $H$ by reorienting every edge along the flow $h$. Let $\vec{G}$ be a dual of $\vec{H}$, that is, every edge $e^\star$ in $\vec{G}$ now crosses its corresponding edge $e$ in $\vec{H}$ from right to left. We begin by initializing $s' = r_1$ and $\vec{B}$ as the subgraph of $\vec{H}$ drawn in the boundary of $r_1$.

We will delete parts of $\vec{H}$ (maintaining $\vec{G}$), extending the face $s'$ and maintaining the boundary $\vec{B}$ of face $s'$. During the process we possibly update $h$, increasing its vorticity. We continue until $s'$ merges with $r_2$ at which point there exists a ring-connecting path certifying that the current flow $h$ is of maximum vorticity by Lemma 7.

We iterate the following steps:

(i) While there exists an out-edge $e^\star$ from $s'$ to $f$

(i-a) If $e$ is bridge in $\vec{H}$, that is $f = s'$, delete $e$. Update the boundary $\vec{B}$ by deleting $e$.

(i-b) Otherwise, let $W$ be the facial walk of $f$, delete $e$. Update $\vec{B}$ by replacing $e$ with $W \setminus e$.

(ii) If $s'$ became the outer face of $\vec{H}$, then stop.

(iii) Boundary $\vec{B}$ of $s'$ is now a collection of cycles. Update $h$ by adding 2 units of flow against the direction of (edges in) $\vec{B}$. Update directions of all edges in $\vec{B}$ by reversing their direction in $\vec{H}$ (and $\vec{G}$).

Note that $\vec{B}$ becomes disconnected when $\vec{H}$ is disconnected in (i-a). While it would be more intuitive to immediately delete the component $C$ of $\vec{H}$ not incident with $r_2$, we let this removal play out in a lazy fashion to avoid the cost of identifying the correct component. Both components are deleted step by step in the same way by steps (i) and (iii).

The step (i) can be implemented efficiently by maintaining queue of edges satisfying the criteria, updated whenever $s'$ is updated or in step (iii). Note that edges in queue are a subset of edges in $\vec{B}$. Each edge added into queue for (i) can only leave the queue by being processed by (i) and consequently deleted. An edge can only be added into queue if it is part of $W$ in (i-b) or when it is reversed in (iii). Overall the total cost of maintaining the queue is $\mathcal{O}(\|H\|)$.

The step (i-a) has constant complexity. The step (i-b) is easily implemented with complexity proportional to $|W|$. No edge is assigned to $e$ more than once in (i), and no edge takes part in $W$ (in (i-b)) more than twice (each time an edge takes part in $W$, its incident face $f \neq s'$ is merged with $s'$). The complexity of the step (iii) is proportional to $|\vec{B}|$. The overall complexity of the algorithm is $\mathcal{O}(\|H\|)$. $\square$

For a closed walk $R^\star$ in $G^\star$ and a demand function $d$ on $G$ we define the *slack* of $R^\star$ as $\text{slack}(R) = |R| - |\nabla(R^\star, d)|$, where $|R|$ denotes the length of $R$ in the number of edges. Recall that $\nabla(R^\star, d)$ essentially quantifies the minimum amount of flow that must cross $R^\star$ from the inside out in any $d$-flow. Note that if $R^\star$ is a cycle, then $\nabla(R^\star, d)$ is equal to the sum over demand values of vertices of $G$ in the interior of $R^\star$, up to a sign which depends on the direction of $R$.

The well known max-flow min-cut theorem implies the following.

**Observation 9.** *Let $H$ be a plane graph and $d$ a demand function on $H$. Then $H$ allows a $d$-flow if and only if slack$(R^\star) \geq 0$ for every cycle $R^\star$ of $H^\star$.*

We say that a demand function on graph $H$ is *feasible* if $\text{slack}(R^\star) \geq 0$ for every cycle of $R^\star$ in $H^\star$.

Let $H$ be a connected plane graph, and $r_1, r_2$ rings of $H$, where $r_2$ is the outer face of $H$. In the context of graph (rather than its embedding) we use the following notational shortcuts. For a cycle $C^\star$ in $H^\star$ we denote as $int(C^\star)$ the set of vertices of $H$ embedded in the interior of $C^\star$, and we denote $int^\star(C^\star)$ as the set of faces of $H^\star$ embedded in the interior of $C^\star$, in other words, the faces dual to $int(C^\star)$. Analogously, let $ext(C^\star) = V(H) \setminus int(C^\star)$ and $ext(C^\star)^\star = F(H^\star) \setminus int^\star(C^\star)$.

For a demand function $d$, let us abuse the notation slightly and define $d(C^\star)$ as $d(C^\star) = \sum_{v \in int(C^\star)} d(v)$. Note that if $d$ is even, then $|C^\star| \equiv d(C^\star) \pmod 2$ for every $C^\star$.

Recall the definition of slack of a closed walk $R^\star$, $\text{slack}(R^\star) = |R^\star| - |\nabla(R^\star, d)|$ (given a demand function $d$). For a cycle $x$ we can simplify this as $\text{slack}(x) = |x| - |d(x)|$. We use this prescription to also extend the definition of slack to the case when $x$ is an edge of $H^\star$. We interpret $e$ as a 2-cycle with no actual interior face, extending the definitions above so that, $int^\star(e^\star) = int(e^\star) = \emptyset$, $d(e^\star) = 0$ and $|e| = 2$, and therefore $\text{slack}(e) = 2$ for any $e$. Though seemingly artificial, inclusion of single edges in this way is a natural consequence of duality between maximal flows and ring-connecting paths as we will now show.

Let $X$ be a set of cycles and edges of $H^\star$, we call such a set a *chain*.

For a chain $X$, we let $\text{slack}(X) = \sum_{x \in X} slack(x)$. For $e^\star \in E(H^\star)$ such that $e^\star$ is contained in $a$ cycles of $X$ and $b$ edges of $X$, we define the multiplicity of $e^\star$ in $X$ as $m(X, e^\star) = a + 2b$, consistently with the idea that individually included edges are 2-cycles.

Let $H$ be a graph with given pair of rings $r_1, r_2$. We say that a chain $X$ is *ring-connecting* if $X$ (viewed as union of all its edges) contains a path from $r_1^\star$ to $r_2^\star$.

We say that a chain $X$ is *laminar* if for any pair of cycles $C_1^\star, C_2^\star$ of the chain, the open disks in the plane bounded by these cycles are either disjoint or one is a subset of the other. In other words, no two cycles cross.

For $d$ feasible, let $\text{circ}(H, r_1, r_2, d)$ denote the maximum integer $k$ such that there exists a $d$-linkage $\mathcal{P}$ and a set of $k$ non-contractible (in respect to rings $r_1$ and $r_2$) cycles $\mathcal{C}$ such that $\text{supp}(\mathcal{P})$ and $\text{supp}(\mathcal{C})$ are disjoint. Equivalently, there exist a $d$-flow and a disjoint circulation of vorticity $k$.

We are now ready to prove the main result of this section, the min-max theorem for $\text{circ}(H, d)$. The argument used to prove the part (a) is based on the idea of Seymour [51] for 2-commodity flows.

**Theorem 10.** *Let $H$ be a connected plane graph with rings $r_1, r_2$ and $d$ a feasible demand function on $H$. Let $h_1, h_2$ be $d$-flows of maximum and minimum vorticity respectively, with $supp(h_1) = supp(h_2) = E(H)$. Then the following claims hold:*

*(a) For every ring-connecting path $Q$ in $H^\star$,*
*$circ(H, r_1, r_2, d) = \frac{1}{2}\Big( \int_Q h_1 - \int_Q h_2 \Big)$.*

*(b) $slack(X) \geq 2 circ(H, r_1, r_2, d)$ for every ring-connecting chain $X$ in $H^\star$.*

*(c) There exists a laminar ring-connecting chain $X$ in $H^\star$ such that*
*$slack(X) = 2 circ(H, r_1, r_2, d)$.*

*Furthermore there exists an algorithm with time complexity $O(|d| \cdot \|H\|)$ which given $H, r_1, r_2$ and $d$ returns both*

*(i) a $d$-linkage and a non-contractible cycle-set of size $circ(H, r_1, r_2, d)$ with disjoint supports, and*

*(ii) a laminar ring-connecting chain $X$ in $H^\star$ such that*
*$slack(X) = 2 circ(H, r_1, r_2, d)$*

*Proof.* Let $\mathcal{P}$ be a $d$-linkage and $\mathcal{C}$ a set of non-contractible cycles such that $\mathrm{supp}(\mathcal{P}) \cap \mathrm{supp}(\mathcal{C}) = \emptyset$ and the vorticity of $\mathcal{C}$ is maximal, that is, equal to $\mathrm{circ}(H, r_1, r_2, d)$. For any cycle $K^\star$ in $H^\star$, the set $K = \{e : e^\star \in K^\star\}$ forms an edge-cut in $H$ separating $int(K^\star)$ from $ext(K^\star)$. By definition, $\mathcal{P}$ contains at least $d(K^\star)$ paths with one end in $int(K^\star)$ and the other end in $ext(K^\star)$. Clearly, all but at most $\mathrm{slack}_d(K^\star)$ edges of $K$ are crossed by paths in $\mathcal{P}$.

Let us consider a ring-connecting chain $X$ in $H^\star$. Let $F$ be the multigraph obtained from $\cup X$ by giving each edge $e$ multiplicity $m(X, e)$. The reasoning here is that, by increasing multiplicities, we make the cycles in $X$ edge-disjoint and turn single edges into 2-cycles. According to the previous paragraph, all but at most $\sum_{x \in X} \mathrm{slack}(x) = \mathrm{slack}(X)$ edges of $F$ are are intersected (crossed) by paths in $\mathcal{P}$. All vertices of $F$ have even degree, and so each component of $F$ is 2-edge-connected, in particular, there exist two edge-disjoint $(r_1^\star, r_2^\star)$-paths in $F$. Each non-contractible cycle in $H$ intersects (crosses) at least one edge of each of these paths. Since the supports of $\mathcal{P}$ and $\mathcal{C}$ are disjoint, we conclude that $\mathrm{slack}(X) \geq 2|\mathcal{C}| = 2\mathrm{circ}(H, r_1, r_2, d)$. Therefore, (b) holds.

Let $Q$ be any ring-connecting path in $H^\star$. Using Observation 2, let $h_d$ be a $d$-flow interpretation of $\mathcal{P}$ and $h_c$ a circulation obtained from $\mathcal{C}$ by orienting all cycles so that $\int_Q h_c = |\mathcal{C}|$. The supports of $h_d$ and $h_c$ are disjoint, and thus $h_a = h_d + h_c$ and $h_b = h_d - h_c$ are $d$-flows in $H$. Since $h_1$ and $h_2$ have maximum and minimum possible vorticities, respectively, we have $\int_Q h_a \leq \int_Q h_1$ and $\int_Q h_b \geq \int_Q h_2$. Consequently,

$$\begin{aligned}\mathrm{circ}(H, r_1, r_2, d) = |\mathcal{C}| = \int_Q h_c = \int_Q \frac{1}{2}(h_a - h_b) \\ = \frac{1}{2}\left(\int_Q h_a - \int_Q h_b\right) \leq \frac{1}{2}\left(\int_Q h_1 - \int_Q h_2\right).\end{aligned}$$

On the other hand, by Lemma 7, since $d$ is even we can assume that $\mathrm{supp}(h_1) = \mathrm{supp}(h_2) = E(H)$. Let $h_+ = (h_1 + h_2)/2$ and $h_- = (h_1 - h_2)/2$.

Since $\mathrm{supp}(h_1) = \mathrm{supp}(h_2) = E(H)$, all values of $h_1$ and $h_2$ are odd, and thus all values of $h_+$ and $h_-$ are integers. Consequently, $h_+$ is a $d$-flow and $h_-$ is a circulation in $H$. Furthermore, observe that $\mathrm{supp}(h_+) \cap \mathrm{supp}(h_-) = \emptyset$. By Observation 2, we conclude that $H$ contains a $d$-linkage $\mathcal{P}'$ and a circulation $\mathcal{C}'$ with disjoint supports such that $|\mathcal{C}'| = \int_Q h_- = \frac{1}{2}\left(\int_Q h_1 - \int_Q h_2\right)$. Therefore, $\mathrm{circ}(H, r_1, r_2, d) \geq \frac{1}{2}\left(\int_Q h_1 - \int_Q h_2\right)$.

Combining the inequalities, we conclude that (a) holds. Note that $d$-flows of maximum and minimum vorticities with maximal supports can be found in time $O(|d| \cdot \|H\|)$ using the algorithm of Lemma 8, and they can be converted into a $d$-linkage and an circulation of size $\mathrm{circ}(H, r_1, r_2, d)$ in $H$ with disjoint supports in time $O(\|H\|)$ as described above.

Finally, let us prove the part (c). By Lemma 7, there exist ring-connecting paths $Q_1$ and $Q_2$ in $H^\star$ such that $\int_{Q_1} h_1 = |E(Q_1)|$ and $\int_{Q_2} h_2 = -|E(Q_2)|$. Let $R^\star$ be the closed walk obtained as the concatenation of $Q_1$ with the reversal of $Q_2$.

By Lemma 4, we have $\int_{Q_1} h_2 = \int_{Q_2} h_2 + \nabla(R^\star, d)$, and thus

$$\begin{aligned} 2 \cdot \mathrm{circ}(H, r_1, r_2, d) &= \int_{Q_1} h_1 - \int_{Q_1} h_2 && (2.1)\\ &= \int_{Q_1} h_1 - \int_{Q_2} h_2 - \nabla(R^\star, d) && (2.2)\\ &= |E(Q_1)| + |E(Q_2)| - \sum_{f \in F(H^\star)} \omega_{R^\star}(f) d(f). && (2.3)\end{aligned}$$

For an integer $i$, let $L_i$ (layers) be the set consisting of edges $e$ of $H^\star$ such that $e$ is incident with faces $f_1$ and $f_2$ satisfying $\omega_{R^\star}(f_1) \geq i$ and $\omega_{R^\star}(f_2) < i$. The motivation is the following. The winding function $\omega_{R^\star}$ is assigning various degrees of "insideness" of vertices of $H$ in respect to $R^\star$. One can view these as assigning heights to faces of $H^\star$, and $L_i$ then corresponds to the contour lines between areas (faces) where the height sharply crosses from below $i$ to at least $i$.

Let $A$ be the set of edges $a$ of $H^\star$ such that both $Q_1$ and $Q_2$ pass through $a$ in the same direction. We claim that the ring-connecting chain $X$ can be chosen to consist of the edges of $A$ and of the cycles into which the sets $L_i$ naturally decompose. Let us describe the construction precisely.

By Observation 3, for an edge $e^\star$ of $H^\star$ incident with faces $f_1$ and $f_2$ and letting $n = \max(\omega_{R^\star}(f_1), \omega_{R^\star}(f_2))$,

- if $e^\star \in (E(Q_1) \setminus E(Q_2)) \cup (E(Q_2) \setminus E(Q_1))$, then $e^\star$ belongs to exactly one of the sets $L_i$, namely to $L_n$,
- if $e^\star \in E(Q_1) \cap E(Q_2) \setminus A$, then $e^\star$ belongs exactly to two of the sets, namely to $L_n$ and $L_{n-1}$, and
- if $e^\star \in A$ or $e^\star \notin E(Q_1) \cup E(Q_2)$, then $e^\star$ does not belong to any of the sets.

It follows that

$$|E(Q_1)| + |E(Q_2)| = 2|A| + \sum_i |L_i|. \qquad (2.4)$$

Let $\overline{L_i}$ denote the subgraph of $H^\star$ with the edge set $L_i$ and the vertex set consisting of the vertices incident with the edges of $L_i$. Observe that if $f$ is a face of $\overline{L_i}$ and faces $f_1$ and $f_2$ of $H^\star$ satisfy $f_1, f_2 \subseteq f$, then either both $\omega_{R^\star}(f_1) \geq i$ and $\omega_{R^\star}(f_2) \geq i$, or both $\omega_{R^\star}(f_1) < i$ and $\omega_{R^\star}(f_2) < i$. Let $F_i^+$ and $F_i^-$ denote the sets of faces of $\overline{L_i}$ for that the former or the latter, respectively, holds. For $f \in F_i^+$, let $W_f$ denote the subgraph of $\overline{L_i}$ drawn in the boundary of $f$. Note that faces in $F_i^+$ only share edges with faces in $F_i^-$ and vice-versa. Consequently, $\overline{L_i} = \bigcup_{f \in F_i^+} W_f$, and the graphs $W_f$ for $f \in F_i^+$ are pairwise edge-disjoint and 2-edge-connected. For $f \in F_i^+$, let $K_f$ denote the set of 2-connected blocks of $W_f$; since $W_f$ is 2-edge-connected and all its edges are incident with $f$, $K_f$ is a set of cycles. For $C \in K_f$, let $\mathrm{out}_f(C) = \mathrm{ext}^\star(C)$ if $f$ is contained in the open disk of the plane bounded by $C$, and let $\mathrm{out}_f(C) = \mathrm{int}^\star(C)$ otherwise. Since $d(V(H)) = 0$, for each face $f \in F_i^+$ we have

$$\begin{aligned}\sum_{f' \in F(H^\star), f' \subseteq f} d^\star(f') &= -\sum_{f' \in F(H^\star), f' \not\subseteq f} d^\star(f') = \\ &= -\sum_{C \in K_f} \sum_{f'' \in \mathrm{out}_f(C)} d^\star(f'') \leq \sum_{C \in K_f} d(C). && (2.5)\end{aligned}$$

Let $K_i = \bigcup_{f \in F_i^+} K_f$ and $X = A \cup \bigcup_i K_i$. Clearly, $X$ is ring-connecting, since $\bigcup X = Q_1 \cup Q_2$. Observe that if $f \in F_i^+$ and $i > j$, then there exists a face $f' \in F_j^+$ such that $f \subseteq f'$, and thus the set $X$ is laminar. It remains to argue that $\mathrm{slack}_d(X) = 2 \cdot \mathrm{circ}(H, r_1, r_2, d)$.

Let $m$ be the minimum of $\{\omega_{R^\star}(f) : f \in F(H^\star)\}$. Note that for any $f' \in F(H^\star)$,

$$\omega_{R^\star}(f') = m + |\{i > m : f' \subseteq f \text{ for some } f \in F_i^+\}|. \tag{2.6}$$

By (2.6) and (2.5), and using the fact that $d(V(H)) = 0$, we have

$$\begin{aligned}\sum_{f' \in F(H^\star)} \omega_{R^\star}(f') d^\star(f') &= m \cdot d^\star(F(H^\star)) + \sum_{i>m} \sum_{f \in F_i^+} \sum_{f' \in F(H^\star), f' \subseteq f} d(f') \\ &\leq \sum_{i>m} \sum_{C \in K_i} d(C)\end{aligned}$$

and thus by (2.3),

$$2 \cdot \mathrm{circ}(H, r_1, r_2, d) \geq |E(Q_1)| + |E(Q_2)| - \sum_{i>m} \sum_{C \in K_i} d(C). \tag{2.7}$$

By (2.4), we have

$$\sum_{x \in X} |x| = 2|A| + \sum_i |L_i| = |E(Q_1)| + |E(Q_2)|,$$

and the definition of $X$ gives

$$\sum_{x \in X} d(x) = \sum_{i>m} \sum_{C \in K_i} d(C).$$

By (2.7), we conclude that

$$\begin{aligned}2 \cdot \mathrm{circ}(H, r_1, r_2, d) \geq |E(Q_1)| + |E(Q_2)| - \sum_{i>m} \sum_{C \in K_i} d(C) = \sum_{x \in X} |x| - \sum_{x \in X} d(x) = \\ = \mathrm{slack}_d(X)\end{aligned}$$

By (b), we conclude that $\mathrm{slack}_d(X) = 2 \cdot \mathrm{circ}(H, r_1, r_2, d)$, and thus (c) holds. Furthermore, observe that this construction of the set $X$ from the $d$-flows $h_1$ and $h_2$ can be performed in time $O(\|H\|)$. $\square$

**Corollary 11.** *There exists an algorithm with time complexity $O(|d| \cdot \|H\|)$ which given a connected plane graph $H$, with rings $r_1, r_2$, a path $Q$ from $r_1^\star$ to $r_2^\star$ in $H^\star$, an even demand function $d$, and an integer $m$ returns one of the following:*

- *A $d$-flow $h$ in $H$ such that supp$(h) = E(H)$ and $\int_Q h \equiv m \pmod 3$, or*
- *a cycle $C$ in $H^\star$ such that slack$(C) < 0$, or*
- *a laminar ring-connecting chain $X$ in $H^\star$ with slack$(X) \leq 2$, and $d$-flows $h_1$ and $h_2$ in $H$ such that $\int_Q h_2 = \int_Q h_1 +$ slack$(X)$ and $\int_Q h_1 \not\equiv m \not\equiv \int_Q h_2 \pmod 3$.*

*Proof.* If $d$ is not feasible, then a straightforward modification of any max-flow algorithm, in particular the Ford-Fulkerson algorithm, returns a cycle $C$ in $H^\star$ such that $\mathrm{slack}(C) < 0$. We can return such a cycle and stop.

Hence, suppose that $d$ is feasible. We use the algorithms from Lemma 8 and Theorem 10. Let $\mathcal{P}$ be a $d$-linkage and let $\mathcal{C}$ be an circulation of vorticity $\mathrm{circ}(H, r_1, r_2, d)$ in $H$ with $\mathrm{supp}(\mathcal{P}) \cap \mathrm{supp}(\mathcal{C}) = \emptyset$, and let $X$ be a laminar ring-connecting chain in $H^\star$ such that $\mathrm{slack}(X) = 2 \cdot \mathrm{circ}(H, r_1, r_2, d)$. Let $h_0$ be a $d$-flow interpretation of the linkage $\mathcal{P}$. Let $H'$ be the subgraph of $H$ with vertex set $V(H)$ and edge set $E(H) \setminus (\mathrm{supp}(\mathcal{P}) \cup \mathrm{supp}(\mathcal{C}))$.

Since $d$ is even, all vertices have even degree in $H'$, and by the maximality of $\mathcal{C}$, the graph $H'$ does not contain any non-contractible cycle. Hence, we can express $H'$ as an edge-disjoint union of contractible cycles, and by sending one unit of flow along each of them, we obtain a circulation $h'$ in $H$ such that $\mathrm{supp}(h_0) \cup \mathrm{supp}(h') \cup \mathrm{supp}(\mathcal{C}) = E(H)$, the supports are pairwise disjoint, and $\int_Q h'^\star = 0$.

Let $n = \min(\mathrm{circ}(H, s, t, d) + 1, 3)$, and for $i \in \{1, \dots, n\}$, let $c_i$ be a circulation with $\mathrm{supp}(c_i) = \mathrm{supp}(\mathcal{C})$ and $\int_Q c_i^\star = -|\mathcal{C}| + 2i - 2$, obtained by Observation 5. Let $h_i = h_0 + h' + c_i$, and note that $h_i$ is a $d$-flow with $\mathrm{supp}(h_i) = E(H)$ and $\int_Q h_i^\star = \int_Q h_1^\star + 2i - 2$. If there exists $i \in \{1, \dots, n\}$ such that $\int_Q h_i^\star \equiv m \pmod 3$, we return $h = h_i$ and stop.

Otherwise, we clearly have $n \leq 2$ and thus $\mathrm{circ}(H, r_1, r_2, d) \leq 1$ and $\mathrm{slack}(X) \in \{0, 2\}$. If $\mathrm{slack}(X) = 2$, then return $X$, $h_1$, and $h_2$. If $\mathrm{slack}(X) = 0$, then return $X$, $h_1$, and $h_1$. $\square$

Note that each of the last two outcomes of Corollary 11 certifies that the first outcome is impossible; the second one because no $d$-flow exists, the third one because $\int_Q h \in \left\{\int_Q h_1, \int_Q h_2\right\}$ for every $d$-flow $h$ in $H$ with $\mathrm{supp}(h) = E(H)$.

They way we typically use Corollary 11 is testing whether a nowhere-zero $d$-flow exists for some particular value $m$. In some applications at the end of Section 2.4 however, we wish to test for which values of $m$ from $\{0, 1, 2\}$ an appropriate $d$-flow exists. Note that just before the end, the algorithm constructs the flows $h_1, \dots, h_n$ of all possible $n$ distinct "$m$-types". Instead of running the algorithm separately for each value, it is a simple modification to output all feasible flow types.

### 2.1.3 Coloring Graphs in the Cylinder

In this section we explore the following 3-coloring problem. Let $G$ be a plane graph, $r_1, r_2$ its rings and let the boundaries of $r_1$ and $r_2$ be precolored. Is it possible to extend the precoloring into a 3-coloring of $G$?

We utilize the connection between colorings and nowhere-zero flows. First we design an algorithm such that it finds the desired coloring of $G$ from the problem above, if it exists. The algorithm is based on deciding existence of nowhere-zero flows in $G^\star$ and its running time is dependent on some structural properties of $G$, in particular its census (a multiset of sizes of faces other than 4). Then, we analyze the inner workings of the algorithm to derive sufficient structural conditions for the existence of the desired coloring. We use this characterization in Section 2.2 to then study necessary structural obstructions preventing a graph embedded in the torus to be 3-colorable.

Let $G$ be a connected plane graph and let $\vec{G}$ be an orientation of $G$. A (proper) 3-coloring $\psi : V(G) \to \{0, 1, 2\}$ of $G$ can be understood as a vertex function $\psi : V(G) \to \mathbb{Z}_3$ on $G$. Let $\delta_\psi$ be the tension function of $\psi$ on $\vec{G}$, specifically, with the values taken from $\mathbb{Z}_3$: for every edge $e = (u, v)$ of $\vec{G}$, $\delta_\psi$ is defined so that $\psi(v) \equiv \psi(u) + \delta_\psi(e) \pmod 3$. We interpret the possible values of $\delta_\psi$ as $\{-1, +1\}$ (note that $\delta_\psi$ is nowhere-zero).

We say that a demand function $d$ on $H$ is a *candidate demand function* if it is an even demand function, such that for every $v \in V(H)$, $|d(v)| \leq deg(v)$, and for every $v \in V(H)$, $d(v)$ is divisible by 3 (we denote the second property as $3|d$).

Let $\psi$ be a precoloring of two connected induced subgraphs $C_1, C_2$ of $G$. Let $h$ be an edge-function on $G$. We say that $h$ *tension-connects* $C_1$ and $C_2$ in respect to $\psi$ if for (any) path $Q$ in $G$ from a vertex $v_1 \in V(C_1)$ to a vertex $v_2 \in V(C_2)$, we have that $\psi(v_2) \equiv \psi(v_1) + \int_Q h \pmod 3$.

As discussed earlier, a tension of a coloring can be understood as a flow (in the dual). When, we want to obtain a coloring from a tension of a coloring, we are free to choose a color of any first vertex $v$ and then all other vertices obtain a color uniquely defined by the tension and the choice of the color for $v$. In the setting of precolorings, this is an issue when the precolored subgraph is disconnected; that is, one component of the precolored subgraph together with consistent tension $h$ produces a coloring which may or may not color the other components consistently with the precoloring. The property of tension-connection captures the property needed to avoid this problem, as we formalize in the following lemma.

We restate the connection between colorings of a plane graph and nowhere-zero flows in its dual, first observed by Tutte [34], in a precoloring extension settings as follows.

**Lemma 12.** *Let $G$ be an oriented connected plane graph, let $H$ be an oriented graph such that $G = H^\star$. Let $C_1$ and $C_2$ be connected induced subgraphs of $G$ and let $C = C_1 \cup C_2$.*

*A proper 3-coloring $\psi : V(C) \to \mathbb{Z}_3$ of $C$ extends to a 3-coloring of $G$ if and only if there exists a feasible candidate demand function d on $H$ and a nowhere-zero d-flow $h$ on $H$ such that the restriction of $h^\star$ to $E\Big(C\Big)$ is equal to the tension $\delta_\psi$ of $\psi$ and $h^\star$ tension-connects $C_1$ and $C_2$ in respect to $\psi$.*

*Furthermore, given such a d-flow $h$, we can obtain a 3-coloring of $G$ extending $\psi$ in time $O(\|G\|)$.*

Note that in the statement of the Lemma 12, as well as other lemmas further on, the oriented graph $H$ is chosen so that $G$ is the dual of $H$ rather than the other way around. Recall that edges in the dual of $G$ would be oriented in the opposite direction than in $H$. By choosing the relation between $G$ and $H$ as in the statement of the Lemma 12, we match the results from previous section dealing with existence of flows in $H$ while considering its dual $G$.

*Proof.* Consider a 3-coloring $\varphi : V(G) \to \mathbb{Z}_3$ of $G$ extending $\psi$. Let $h^\star$ be the tension of $\varphi$ on $G$ and $h$ the edge-function on $H$ to which $h^\star$ is dual, that is $h(e) = h^\star(e^\star)$ for every $e \in E\Big(H\Big)$. Clearly $h^\star$ restricted to $E(C)$ is equal to $\delta_\psi$ as $\varphi$ extends $\psi$.

Let us choose a demand function $d$ on $H$ such that for every $v \in V(H)$, the value $d(v)$ is the excess of $v$ for the edge function $h$. By choice, $\sum_{v \in V(H)} d(v) = 0$.

Since the values of $h$ are only $\{1, -1\}$, the parity of $d(v)$ is the same as the parity of $deg(v)$; in other words, $d$ is even. By definition of a $d$-flow, $h$ is a $d$-flow and by existence of the $d$-flow $h$, $d$ is clearly feasible.

Fix any $v \in V(H)$ and let $W$ be the facial walk of the face $f$ of $G$ such that $f^\star = v$. Recall that by definition, for any tension $t$ and a closed walk $Q$, it hold that $\int_Q t = 0$ in the group on which $t$ is defined. In particular, $\int_W h^\star \equiv 0 \pmod 3$. It follows that the excess of $v$ is divisible by 3, and therefore $3|d$. The demand function $d$ is therefore candidate and feasible.

From the assumption that $\varphi$ extends $\psi$, we have that tension of $\psi$ on $C$ is equal to the restriction of $h^\star$ to $E\left(\vec{C}\right)$. Finally, necessarily by definition of $h^\star$, we have $\varphi(v_2) \equiv \varphi(v_1) + \int_Q \delta_{\vec{G},\phi} = \varphi(v_1) + \int_Q h^\star \pmod 3$. Hence, $h^\star$ tension-connects $C_1$ and $C_2$ and the conclusions of the lemma hold.

Let us prove the converse implication. Let $d$ be a feasible candidate demand function. Let $\delta_\psi$ denote the tension of $\psi$ on $\vec{C}$, and let $h$ be a nowhere-zero $d$-flow in $H$ such that restriction of $h^\star$ to $E(\vec{C})$ is equal to $\delta_\psi$ and $\psi(v_2) \equiv \psi(v_1) + \int_Q h$ (mod 3).

Since $h$ is a $d$-flow, by Lemma 4 we have that for any closed walk $R$ in $G$, $\int_R h^\star = \nabla(R, d^\star) \equiv 0 \pmod 3$ where the last congruence follows from $3|d$.

Let us now define vertex function $\varphi$ on $G$, obtained from viewing $h^\star$ as a tension of $\varphi$, and show that $\varphi$ is a coloring as required. For each vertex $x \in V(G)$, let $Q_x$ be any walk from $v_1$ to $x$. Let $\varphi(x)$ be defined as $\varphi(x) \equiv \psi(v_1) + \int_{Q_x} h^\star$ (mod 3) for every $x \in V(G)$.

We observe that the definition of $\varphi(x)$ is independent of the choice of $Q_x$. Let $Q_x, Q'_x$ be two distinct $(v_1, x)$-walks, and let $R$ be the concatenation of $Q_x$ and the reverse of $Q'_x$. By Lemma 4, $\int_{Q_x} h^\star - \int_{Q'_x} h^\star = \int_R h^\star \equiv 0 \pmod 3$. By rearranging $\int_{Q_x} h^\star \equiv \int_{Q_x} h^\star \pmod 3$ and therefore the specific choice of $Q_x$ does not matter.

By similar reasoning, let $e = \{x, y\} \in E(G)$ and let $R$ be the concatenation of $Q_x$, $e$ and the reverse of $Q_y$. We obtain that $\int_{Q_x} h^\star + h^\star(e) - \int_{Q_y} h^\star \equiv 0$ (mod 3) and therefore $\varphi(y) \equiv \varphi(x) + h^\star(e)$. Since $h^\star(e) \in \{-1, +1\}$, we have that $\varphi(x) \neq \varphi(y)$ and $\varphi$ is a proper coloring.

Let $v$ be a vertex in $C$. If $v \in C_1$, we may assume that $Q_v$ is a walk in $C_1$. Clearly, $\psi(v) = \varphi(v)$ as $h^\star$ matches the tension of $\psi$ on $C$. If $v \in C_2$, we may assume that $Q_v$ is the concatenation of $Q$ and a walk from $v_2$ to $v$ in $C_2$. By the assumption on $Q$ and $h^\star$, we have that $\psi(v_2) = \varphi(v_2)$, and as before, $\psi(v) = \varphi(v)$. We conclude that $\varphi$ is extension of $\psi$.

Note that $\varphi$ can be constructed from $h$ in time $\mathcal{O}(\|G\|)$, using the fact that $\varphi(y) \equiv \varphi(x) + h^\star(e)$ for every edge $e = (x, y)$ of $G$, via a search-like propagation. □

By inspection of the proof, we may see that when $h$ is a $d$-flow, the property of tension-connecting $C_1$ and $C_2$ holds for every pair of vertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$ and any path $Q$ whenever it holds for any pair and a path between $C_1$ and $C_2$.

When applying Lemma 12, we usually try all candidate demand functions one by one. Once the function $d$ is fixed, we need to enforce the condition that the restriction of $h^\star$ to $E\left(\vec{C}\right)$ is equal to $\delta_{\vec{C},\psi}$. To this end, we use the following construction.

Let $G$ be a connected plane graph, let $H = G^\star$, let $\vec{H}$ be an arbitrary orientation of $H$, and let $\vec{G} = \vec{H}^\star$. Suppose we have a precoloring of two faces of $G$, $z_1^\star$ and $z_2^\star$. Let $C$ be the subgraph of $G$ consisting of the vertices and edges drawn in the boundaries of these faces. Let $\vec{C}$ be the orientation of $C$ induced by $\vec{G}$. The precoloring induces a preflow $\delta : E\left(\vec{C}\right) \to \mathbf{Z}$. Additionally, let us have a fixed demand function $d : V(H) \to \mathbf{Z}$.

The idea is to express the preflow $\delta$ in $H$ by adjusting the demand function. We delete the vertices $z_1, z_2$, and consequently also all edges with defined preflow. For all vertices incident with the deleted edges, we shift their demand by the contribution of the deleted edges as follows.

Let $(d/\delta) : V(H - \{z_1, z_2\}) \to \mathbf{Z}$ be the function defined by setting

$$(d/\delta)(v) = d(v) + \sum_{i \in \{1,2\}} \left( \sum_{e \in N^+_{\vec{H}}(z_i) \cap N^-_{\vec{H}}(v)} \delta(e^\star) - \sum_{e \in N^-_{\vec{H}}(z_i) \cap N^+_{\vec{H}}(v)} \delta(e^\star) \right)$$

for every $v \in V(H) \setminus \{z_1, z_2\}$.

Clearly, any valid $(d/\delta)$-flow on $H - \{z_1, z_2\}$ together with $\delta$ forms a valid $d$-flow on $H$. We formally restate this as the following observation.

**Observation 13.** *Let $G$ be a connected plane graph, let $H = G^\star$, let $\vec{H}$ be an arbitrary orientation of $H$, and let $\vec{G} = \vec{H}^\star$. Let $z_1$ and $z_2$ be vertices of $H$ and let $C$ be the subgraph of $G$ consisting of the vertices and edges drawn in the boundaries of the faces $z_1^\star$ and $z_2^\star$. Let $\vec{C}$ be the orientation of $C$ induced by $\vec{G}$. Let $d : V(H) \to \mathbf{Z}$ and $\delta : E\left(\vec{C}\right) \to \mathbf{Z}$ be functions such that*

$$d(z_i) = \sum_{e \in N^+_{\vec{H}}(z_i)} \delta(e^\star) - \sum_{e \in N^-_{\vec{H}}(z_i)} \delta(e^\star)$$

*for $i \in \{1, 2\}$.*

- *If $h'$ is a $(d/\delta)$-flow in $H - \{z_1, z_2\}$, then the function $h : E\left(\vec{H}\right) \to \mathbf{Z}$ defined by $h(e) = h'(e)$ for $e \in E\left(\vec{H} - \{z_1, z_2\}\right)$ and $h(e) = \delta(e^\star)$ for each edge $e$ incident with $z_1$ or $z_2$ is a $d$-flow in $H$.*
- *If $h$ is a $d$-flow in $H$ such that $h(e) = \delta(e^\star)$ for each edge $e$ incident with $z_1$ or $z_2$, then the restriction of $h$ to $E\left(\vec{H} - \{z_1, z_2\}\right)$ is a $(d/\delta)$-flow in $H - \{z_1, z_2\}$.*

Let us now introduce some notation used to express algorithmic complexity, and a simpler result dealing with one precolored face in plane graph.

For an integer $i$, let $r(i)$ denote the number of integers from $-i$ to $i$ with the same parity as $i$ and divisible by 3. For faces $f_1, \ldots, f_j$ of graph $G$, let $r_{f_1,\ldots,f_j}(G) = \prod_f r(|f|)$, where $f$ iterates over all faces of $G$ distinct from faces in $f_1, \ldots, f_j$. Note that the expression $r_{f_1,\ldots,f_j}(G)$ upper-bounds the number of candidate demand functions on $G^\star$, given that values for $d(f_1), \ldots, f(f_j)$ are already fixed.

For any plane graph $G$, let $q(G)$ be defined as $q(G) = 1 + \sum_{f:|f| \neq 4} |f|$, that is, a sum of lengths of all faces other than those of length four.

**Theorem 14** (Dvořák and Lidický [52, Lemma 4 and the discussion following it])**.** *There exists an algorithm which, given a simple connected plane graph $G$, a cycle $C_1$ bounding a face $f_1$ of $G$, and a 3-coloring $\psi$ of $C_1$, finds in time $O\Big(r_{f_1}(G)q(G)|G|\Big)$ a 3-coloring of $G$ extending $\psi$ or correctly decides that no such 3-coloring exists.*

We are now ready to derive an algorithm for 3-coloring graphs drawn in the cylinder with precolored boundary cycles.

**Theorem 15.** *There exists an algorithm which, given a simple connected plane graph $G$, cycles $C_1$ and $C_2$ bounding distinct faces $f_1$ and $f_2$ of $G$, and a 3-coloring $\psi$ of $C_1 \cup C_2$, finds in time $\mathcal{O}(r_{f_1,f_2}(G)q(G)|G|)$ a 3-coloring of $G$ extending $\psi$ or correctly decides that no such 3-coloring exists.*

*Proof.* If the cycles $C_1$ and $C_2$ are not disjoint, it suffices to split $G$ into subgraphs drawn in the closure of faces of the (connected) graph $C_1 \cup C_2$, and check whether $\psi$ extends to a 3-coloring of each of them. This can be done using the algorithm from Theorem 14. Note that the sum of lengths of the precolored faces is upper-bounded by the sum of lengths of $f_1$ and $f_2$, similarly face sizes and the values of $r$ and $q$ of the individual parts sum to a value upper-bounded by the respective values for $G$, up to a factor of 2. The complexity bound therefore holds. Hence, we can assume that the cycles $C_1$ and $C_2$ are vertex-disjoint.

Let $\vec{G}$ be an orientation of $G$ and let $\vec{H}$ be so that $\vec{H}^\star = \vec{G}$. Let $C = C_1 \cup C_2$ and let $\vec{C}$ be the orientation of $C$ induced by $\vec{G}$. Let $z_1$ and $z_2$ be the vertices of $H$ such that $z_i^\star$ is the face of $G$ bounded by $C_i$ for $i \in \{1,2\}$. We view the cycles $C_1$ and $C_2$ as directed so that the face $z_i^\star$ is to the right from $C_i$.

Let $Q$ be any path in $G$ from a vertex $v_1 \in V(C_1)$ to a vertex $v_2 \in V(C_2)$ intersecting $C_1 \cup C_2$ only in its endvertices, and let $m = (\psi(v_2) - \psi(v_1)) \bmod 3$.

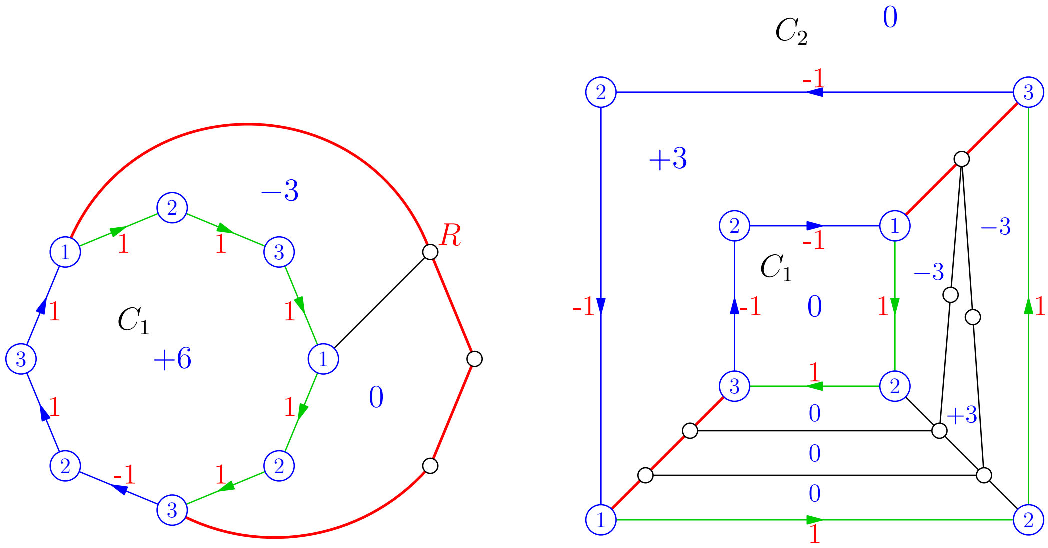
We iterate over all candidate demand functions $d$ on $H$, with values $d(v_i)$ fixed as $\int_{C_i} \delta_\psi$ for $i \in \{1,2\}$, that is, consistent with the precoloring. Clearly, $r_{f_1,f_2}(G)$ upper-bounds the number of possible choices for $d$.

By Lemma 12, it suffices to check whether for any such function $d$, there exists a nowhere-zero $d$-flow $h$ in $H$ such that the restriction of $h^\star$ to $E\Big(\vec{C}\Big)$ is equal to $\delta_\psi$ and $h^\star$ tension-connects $C_1$ and $C_2$ in respect to $\psi$. If such a $d$-flow exists, we can in time $O(|G|)$ turn $h$ into a 3-coloring extending $\psi$. Otherwise, Lemma 12 implies that no 3-coloring of $G$ extends $\psi$.

Let $g_1$ and $g_2$ be the faces of $H - \{z_1, z_2\}$ such that in $H$, the vertex $z_i$ is drawn in $g_i$ for $i \in \{1,2\}$ (since the cycles $C_1$ and $C_2$ are vertex-disjoint, we have $g_1 \neq g_2$). Note that the graph $G' = (H - \{z_1, z_2\})^\star$ contains a path $Q'$ from $g_1^\star$ to $g_2^\star$ with $E(Q') = E(Q)$. By Observation 13, existence of a nowhere-zero $d$-flow $h$ on $H$ is equivalent to existence of a $(d/\delta_{\vec{C},\psi})$-flow $h'$ in $H - \{z_1, z_2\}$ such that $\operatorname{supp}(h') = E(H - \{z_1, z_2\})$ and $\int_Q h'^\star \equiv m \pmod 3$. By Corollary 11, we can test whether such flow exists in time $O(|d| \cdot |G|) = O(q(G)|G|)$.

Since the test needs to be performed for at most $r_{f_1,f_2}(G)$ possible choices of $d$, we conclude the time complexity of the algorithm is $O(r_{f_1,f_2}(G)q(G)|G|)$. □

Using the ideas from the proof of Theorem 10, let us now explicitly formulate a sufficient condition for extendability of a 3-coloring of two cycles in a plane graph.




A generalized chord $R$ (in red, base in green) and a coloring $\psi$ and function $d$ (in blue, values of $\delta_{\vec{C},\psi}$ in red) such that $\text{slack}_{d^\star,\psi}(R) = 4 - |-3 + 5| = 2$.

A $(C_1, C_2)$-connector $Q$ (in red) and a coloring $\psi$ and function $d$ (in blue, values of $\delta_{\vec{C},\psi}$ in red) such that $\text{slack}_{d^\star,\psi}(Q) = 5 - |-3 + (2 + 2)| = 4$.

Figure 2.2: Constraints

We begin by defining a few structures to express some of the constraints that need to be satisfied for the feasibility of the flow we need.

Let $G$ be a connected simple plane graph and let $C_1$ and $C_2$ be vertex-disjoint cycles bounding its rings $r_1$ and $r_2$, where $r_2$ is the outer face of $G$; for $i \in \{1, 2\}$, we view the cycle $C_i$ as directed so that the face $r_i$ is to the right from $C_i$. Let $\vec{C}$ be any orientation of $C_1 \cup C_2$.

Let $\psi$ be a proper 3-coloring of $C_1 \cup C_2$ and let $d^\star : F(G) \to \mathbb{Z}$ be a candidate demand function such that $d^\star(f_i) = \int_{C_i} \delta_\psi$ for $i \in \{1, 2\}$.

If a path $R$ in $G$ has both ends in $C_i$ for some $i \in \{1, 2\}$ and is otherwise disjoint from $C_i$ and edge-disjoint from $C_{3-i}$, we say $R$ is a *generalized chord* of $C_i$, see the left part of Figure 2.2 for an example. Let $K_R$ be the unique contractible cycle in $C_i \cup R$, that is, not containing $r_1$ in its interior. Let $B$ be the path $K_R \cap C_i$ directed so that $r_i$ is to the right of $B$ (we say $B$ is the *base* of $R$). We define $\text{int}^\star(R) = \text{int}^\star(K_R)$.

Let us remark that since $r_2$ is the outer face of $G$, we have $r_1, r_2 \notin \text{int}^\star(R)$.

We define

$$\text{slack}_{d^\star,\psi}(R) = |E(R)| - \left| d^\star(\text{int}^\star(R)) + \int_B \delta_{\vec{C},\psi} \right|$$

In the language of flows, we see the generalized chord as a cut separating its interior from the rest of the graph. The slack of a generalized chord is the difference between the length of the cut, and the overall contributions from its interior and the preflow crossing the base.

A $(C_1, C_2)$-*connector* $Q$ is the union of two vertex-disjoint paths, both with one end in $C_1$, the other end in $C_2$, and otherwise disjoint from $C_1 \cup C_2$, see the right part of Figure 2.2 for an example. Let $K_Q$ be one of the two contractible (in respect to rings $r_1, r_2$) cycles in $C_1 \cup C_2 \cup Q$, and for $i \in \{1,2\}$, let $B_i$ be the path $K_Q \cap C_i$ directed so that $f_i$ is to the right of $B_i$.

Analogously to the generalized chord earlier, we define

$$\mathrm{slack}_{d^\star,\psi}(Q) = |E(Q)| - \left| d^\star(\mathrm{int}^\star(K_Q)) + \int_{B_1} \delta_{\vec{C},\psi} + \int_{B_2} \delta_{\vec{C},\psi} \right|$$

Observe that the value of $\mathrm{slack}_{d^\star,\psi}(Q)$ does not depend on which of the two cycles we choose as $K_Q$, since $d^\star(F(G)) = 0$ and $d^\star(f_i) = \int_{C_i} \delta_{\vec{C},\psi}$ for $i \in \{1,2\}$.

For a cycle $K$ in $G$ edge-disjoint from $C_1 \cup C_2$, we let $\mathrm{slack}_{d^\star,\psi}(K) = |K| - |d^\star(\mathrm{int}^\star(K))|$.

An edge $e$ of $G$ is *non-chord* if it is not the case that both vertices incident with $e$ are contained in the same cycle $C_i$, for $i \in \{1,2\}$. For a non-chord edge $e$, we let $\mathrm{slack}_{d^\star,\psi}(e) = 2$.

A *constraint* is a generalized chord, a $(C_1, C_2)$-connector, a cycle edge-disjoint from $C_1 \cup C_2$, or a non-chord edge. A set $X$ of constraints is $(C_1, C_2)$-*connecting* if $\bigcup X$ contains a path from $C_1$ to $C_2$. We define $\mathrm{slack}_{d^\star,\psi}(X) = \sum_{T \in X} \mathrm{slack}_{d^\star,\psi}(T)$.

We are finally ready to formulate the sufficient conditions.

**Lemma 16.** *Let $G$ be a connected simple plane graph and let $C_1$ and $C_2$ be vertex-disjoint cycles bounding its rings $r_1$ and $r_2$ of $G$. Let $\psi$ be a proper 3-coloring of $C_1 \cup C_2$ and let $d$ be a candidate demand function on $G^\star$ such that $d^\star(f_i) = \int_{C_i} \delta_\psi$ for $i \in \{1,2\}$. If $\mathrm{slack}_{d^\star,\psi}(T) \geq 0$ for every constraint $T$ and $\mathrm{slack}_{d^\star,\psi}(X) > 2$ for every $(C_1, C_2)$-connecting set $X$ of constraints, then $\psi$ extends to a 3-coloring of $G$.*

*Proof.* Without loss of generality, let $r_2$ be the outer face and for $i \in \{1,2\}$, we view the cycle $C_i$ as directed so that the face $r_i$ is to the right from $C_i$.

Let $\vec{G}$ be an orientation of $G$ and let $\vec{H}$ be so that $\vec{H}^\star = \vec{G}$. Let $Q$ be any path in $G$ from a vertex $v_1 \in V(C_1)$ to a vertex $v_2 \in V(C_2)$ intersecting $C_1 \cup C_2$ only in its endvertices, and let $m \equiv \psi(v_2) - \psi(v_1) \bmod 3$.

By Lemma 12, it suffices to show that there exists a nowhere-zero $d$-flow $h$ in $H$ such that the restriction of $h^\star$ to $E\left(\vec{C}\right)$ is equal to $\delta_\psi$ and $h^\star$ tension-connects $C_1$ and $C_2$, that is $\int_Q h^\star \equiv m \pmod 3$.

For $i \in \{1,2\}$, let $z_i = r_i^\star$, and let $g_1$ and $g_2$ be the faces of $H - \{z_1, z_2\}$ such that $z_i$ is drawn in $g_i$ for $i \in \{1,2\}$. Note that since the cycles $C_1$ and $C_2$ are vertex-disjoint, we have $g_1 \neq g_2$.

By Observation 13, it suffices to show there exists a nowhere-zero $(d/\delta_\psi)$-flow $h'$ in $H - \{z_1, z_2\}$ and $\int_Q h'^\star \equiv m \pmod 3$. By Corollary 11, it suffices to verify that

$$\mathrm{slack}_{d/\delta_\psi}(K) \geq 0 \tag{2.8}$$

for every cycle $K$ in $(H - \{z_1, z_2\})^\star$, and that

$$\mathrm{slack}_{d/\delta_\psi}(X) > 2 \tag{2.9}$$

for every $(g_1, g_2)$-connecting chain $X$ in $(H - \{z_1, z_2\})^\star$.

Consider any cycle $K$ in $(H - \{z_1, z_2\})^\star$, and let $T$ be the subgraph of $G$ with $E(T) = E(K)$ and $V(T)$ consisting of the vertices incident with these edges.

If $g_1^\star, g_2^\star \notin V(K)$, then $T$ is a cycle in $G$ vertex-disjoint from $C_1 \cup C_2$.

If $g_i^\star \in V(K)$ and $g_{3-i}^\star \notin V(K)$ for some $i \in \{1, 2\}$, then $T$ is either a cycle in $G$ intersecting $C_i$ in one vertex and disjoint from $C_{3-i}$ (and thus edge-disjoint from $C_1 \cup C_2$), or $T$ is a generalized chord of $C_i$ vertex-disjoint from $C_{3-i}$.

Finally, if $g_1^\star, g_2^\star \in V(K)$, then $T$ is a cycle intersecting each of $C_1$ and $C_2$ in one vertex, or a generalized chord of $C_i$ intersecting $C_{3-i}$ in one vertex for some $i \in \{1, 2\}$, or a $(C_1, C_2)$-connector.

In either case, $T$ is a constraint. Observe that by the definition of $d/\delta_{\vec{C},\psi}$, we have

$$\mathrm{slack}_{d^\star,\psi}(T) = \mathrm{slack}_{d/\delta_\psi}(K)$$

Furthermore, if $X$ is a $(g_1, g_2)$-connecting chain $X$ in $(H - \{z_1, z_2\})^\star$ and $X'$ is obtained from $X$ by transforming each cycle as described above and keeping the edges of $X$ that are non-chord in $G$, then $X'$ is $(C_1, C_2)$-connecting. Therefore, the inequalities (2.8) and (2.9) follow from the assumptions of this Lemma. □

Let us remark that while the condition that $\mathrm{slack}_{d^\star,\psi}(T) \geq 0$ for every constraint $T$ is also necessary for the extendability of $\psi$, indeed it is necessary for existence of any $d$-flow, the condition on set of constraints is only sufficient. A coloring can extend even if $\mathrm{slack}_{d^\star,\psi}(X) \leq 2$ for some $(C_1, C_2)$-connecting set $X$ of constraints. In such a case, we are not guaranteed that the condition of tension-connecting is satisfied (as the variety of flows may not be rich enough). This comes down to the issue that the preflow does not change if we permute the colors on one of the precolored components by changing each color by $+1$. For example, if $G$ contains an edge between $C_1$ and $C_2$, it is clearly not possible to guarantee consistent extension for every precoloring $\psi$ of $C_1$ and $C_2$, however some precoloring $\psi$ may extend.

## 2.2 Torus

In Section 2.1, we obtained a set of sufficient conditions for a graph in the cylinder to allow an extension of a 3-coloring of its rings. In this section we look more closely at the possible obstructions, in particular as far as the application to coloring in the torus is concerned.

We show that by an appropriate choice of the precoloring of the rings and handling of the $(\geq 5)$-faces, it is possible to systematically avoid existence of most obstructions. We use this to show that if the edge-width of a triangle-free graph embedded in the torus is at least 21, then the graph is 3-colorable. This result also follows from [30], where the authors show analogous result for general surface, however without an explicit bound on the edge-width. For our applications, we need to show that the bound is actually small.

This result, together with earlier results and the algorithm from the previous section, is then used to design a non-constructive 3-colorability test that runs in linear time.

### 2.2.1 Constraints and Their Interactions

Suppose for contradiction that $G$ is a 4-critical triangle-free graph drawn in the torus with edge-width at least 21. Let $C$ be a shortest non-contractible cycle in $G$. By splitting along $C$, we obtain a graph $G'$ with two rings $C_1, C_2$. We say that a graph with two rings is *cylindrical*. In this case, we also have a natural mapping $\lambda$ between the vertices (and edges) of the rings, we denote this relation as $(G', C_1, C_2, \lambda)$. We often understand $G'$ as a plane graph with the outer face bounded by $C_2$. We can now use Lemma 16 to deduce some structural properties of $G'$.

We use the following characterizations to bound slack of various constraints. Let $X$ be a set of faces, then by $S(X)$ we denote the multiset of lengths of the faces in $X$ other than 4. For a contractible cycle $C$ in $G$ (drawn in a surface other than the sphere), let $\mathrm{int}^\star(C)$ denote the set of faces of $G$ drawn in the open disc bounded by $C$. Recall that $S(G)$ is the census of $G$, the multiset of lengths of faces of $G$ other than 4. Note that $S(\mathrm{int}^\star(C))$ is the multiset of lengths of faces other than 4 drawn in the open disc bounded by $C$.

Let $G$ be a graph and $H$ its subgraph. We say that $G$ is $H$*-critical* if foe each edge $e$, such that $e \in E(G)$ and $e \notin E(H)$, there exists a proper coloring of $H$ that does not extend into a coloring of $G$, but does extend into a coloring $G - e$.

Let $k$ and $l$ be integers. Consider a plane graph $G$ of girth at least $k$, with outer face of length $l$ bound by a cycle $C$. Furthermore, let $G$ be $C$-critical. We denote $\mathcal{S}_{k,l}$ the set of all possible values of $S(\mathrm{int}^\star(C))$. Note that the elements represent all possible sets of lengths of all faces, excluding the outer $l$-face.

In the following two lemmas represent a concise collection of various results. Together these play a key role in every analysis of the structural properties of 4-critical graphs.

**Lemma 17.** *[53] Let $G$ e a plane graph of girth at least five, and let $C$ be a facial cycle in $H$ of length $k \leq 11$. If $H$ is $C$-critical, then*

*(a) $k \geq 8$, $V(G) = V(C)$ and $C$ is not induced, or*

*(b) $k \geq 9$, $H - V(C)$ is a tree with at most $k - 8$ vertices and every vertex of $V(G) - V(C)$ has degree at least 3 in $G$, or*

*(c) $k \geq 10$ and $G - V(C)$ is a connected graph with at most $k - 5$ vertices containing exactly one cycle, and the length of this cycle is 5. Furthermore, every tex of $V(G) - V(C)$ has degree at least 3 in $G$.*

**Lemma 18.** *The following relations hold*

- $\mathcal{S}_{4,3} = \mathcal{S}_{4,4} = \mathcal{S}_{4,5} = \emptyset$,
- $\mathcal{S}_{4,6} \subseteq \{\emptyset\}$,
- $\mathcal{S}_{4,7} \subseteq \{\{5\}\}$,
- $\mathcal{S}_{4,8} \subseteq \{\{6\}, \{5,5\}, \emptyset\}$,
- $\mathcal{S}_{4,9} \subseteq \{\{7\}, \{6,5\}, \{5,5,5\}, \{5\}\}$.
- $\mathcal{S}_{4,10} \subseteq \{\{8\}, \{7,5\}, \{6,6\}, \{6,5,5\}, \{6\}, \{5,5,5,5,5,5\}, \{5,5,5,5\}, \{5,5\}, \emptyset\}$

*In particular, let $G$ be a triangle-free graph with a 2-cell drawing in a surface other than the sphere and let $C$ be a contractible cycle in $G$ not bounding a face. If $G$ is 4-critical, then the following claims hold.*

- $|C| \geq 6$,
- *if* $S(int^{\star}(C)) \neq \emptyset$, *then* $|C| \geq 7$,
- *if* $|S(int^{\star}(C))| \geq 2$, *then* $|C| \geq 8$, *and*
- *if* $|S(int^{\star}(C))| \geq 3$, *then* $|C| \geq 9$.

*Proof.* We first prove that the second part of the statement is a consequence of the first part. Suppose that $G$ is 4-critical. Clearly, no proper 3-coloring of $C$ extends both to the interior and exterior of $C$. Consider removing any edge $e$ of $G$ from the interior of $C$, $G-e$ must allow a 3-coloring $\psi_e$ inducing a 3-coloring of $C$ that extends into the interior of $C$ in $G-e$ but does not extend in $G$. We conclude that the interior of $C$ is indeed $C$-critical. By inspecting the list of values in the first part of the Lemma, we observe that the second part is implied.

The first part of the Lemma is a collection of various results. The values of $\mathcal{S}_{4,l}$ for $l \leq 6$ are implied by [53] (Theorems 9 and 10). The original proofs are due to Gimbel and Thomassen [23], [54], [55], [27]. The other values of the statement can then be derived through refinement defined in [28] and from knowledge of the values of $\mathcal{S}_{5,l}$ for $l \leq 10$, obtained by results of Thomassen [22] and Walls [56].

For completeness, we give an example of derivation of the hardest set of values, that is, the value $\mathcal{S}_{4,10}$. For more detail on the underlying process, refer to [53] and [28], and for a more detailed derivation of $\mathcal{S}_{4,l}$ for $l \leq 9$, see [33].

A multiset $S_2$ is a *one-step refinement* of a multiset $S_1$ if for some $k \in S_1$ we have $S_2 = (S_1 \setminus \{k\}) \cup Z$ where $Z$ is a multiset from $\mathcal{S}_{4,k} \cup \mathcal{S}_{4,k+2}$. A multiset $S_2$ is a *refinement* of $S_1$ if $S_2$ can be obtained from $S_1$ by a (possibly empty) sequence of one-step refinements. By [28] (Lemma 5.2), every element of $\mathcal{S}_{4,k}$ is either $\{k-2\}$ or a refinement of some element from $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$.

Assuming we already know the values $\mathcal{S}_{4,l}$ for $l \leq 10$, we need the value of $\mathcal{S}_{5,10}$ to proceed with the refinement. We use the Lemma 17. We see that for every $G$ critical in respect to an outer cycle $C$ of length 10, the interior of $C$ either contains only a chord, or a single vertex of degree 3, or two connected vertices of degree 3, or five internal vertices of degree 3 forming a 5-cycle disjoint from $C$. By counting edges, we get that in the latter three cases the sum of lengths of internal faces is 16, 20 and 30 respectively, with the total number of faces being 3, 4 and 6 respectively. Clearly, in the latter two cases all of the faces are 5-faces, while in the first of the three we get exactly one 6-face and two 5-faces. We conclude that $\mathcal{S}_{5,10} \subseteq \{\{7,5\},\{6,6\},\{6,5,5\},\{5,5,5,5\},\{5,5,5,5,5,5\}\}$. Together with the set $\mathcal{S}_{4,8} \subseteq \{\{6\},\{5,5\},\emptyset\}$ we now consider all refinements. However, all one-step refinements produce other elements we already obtained. $\square$

Based on a fine structural analysis and aided by a simple computer-assisted search, the following is known about the census of triangle-free 4-critical graphs in the torus.

**Theorem 19.** *[33] Let $G$ be a 4-critical triangle-free graph with a 2-cell embedding in the torus. Then $S(F(G))$ is one of* $\{7,5\},\{6,5,5\},\{5,5,5,5\},\{5,5\}$ *or* $\emptyset$.

Furthermore, as implied by the results of [32], this result is tight, as each of the possible values of census is indeed realized, by infinitely many graphs (with the exception of $\emptyset$ which is realized by a single graph).

Let us consider the inner working of the algorithm from Section 2.1, namely Theorem 15. Suppose we cut a graph $G$ embedded in the torus along a non-contractible cycle $C$ into a graph $G'$ embedded in the cylinder. If we want to use Theorem 15 to find out whether $G$ is 3-colorable, we may iterate through all precolorings of $C$, inducing consistent precoloring of rings of $G'$, and the algorithm then iterates through candidate demand functions. If a single choice satisfies the sufficient conditions of Lemma 16, then the 3-coloring exists. Given a graph that is not 3-colorable, we are therefore free to choose any precoloring of $C$ together with any candidate demand function, and an unsatisfied constraint must exist.

In this section we consider the constraints from Lemma 16, in particular how their slack is influenced by the free choices indicated above. In general, the slack of constraints can be simply viewed as the relation between the length of its boundary and the amount of flow contributed by the sinks and sources, and the amount of flow contributed by the boundary of a ring. We show how to minimize the contribution of precoloring (determining the amount of flow contributed by the rings) against the slack of constraints.

Then we focus on the 4-critical graphs. According to Theorem 19 their census is limited, and so the overall contribution of flow from sources and sinks is limited as well. For instance, if the rings are far enough apart (given a good choice of precoloring), a connector constraint will always have a positive slack, given the limited contribution from sources and sinks. We show that the interaction between various constrains are limited enough to allow a careful choice of a demand function to minimize the contributions of sink and sources against the slack of all constraints. Given these choices, we show that the coloring algorithm cannot fail, on a graph of large edge-width, implying that such graph cannot actually be 4-critical after all.

Let $\vec{C}$ be an orientation of a cycle $C$ and let $\psi$ be a 3-coloring of $C$. In order to maximize $\mathrm{slack}_{d^\star,\psi}$ for all constraints, it is convenient if $\left|\int_Q \delta_{\vec{C},\psi}\right|$ is relatively small for all subpaths $Q$ of $C$. We say that $\psi$ is *tame* if $\left|\int_Q \delta_{\vec{C},\psi}\right| \le 2$ for every subpath $Q$ of $C$ of length at most 5. Since $\int_Q \delta_{\vec{C},\psi}$ and $|E(Q)|$ have the same parity, if $|E(Q)| \in \{1,3,5\}$, then $\left|\int_Q \delta_{\vec{C},\psi}\right| = 1$. Any longer subpath $Q$ of $C$ can be partitioned into paths of length 5 and one path of length $|E(Q)| \bmod 5$, giving us the following bound.

**Observation 20.** *Let $\psi : V(C) \to \{0,1,2\}$ be a coloring of a cycle $C$, let $\vec{C}$ be an orientation of $C$, and let $Q$ be a subpath of $C$. If $\psi$ is tame, then*

$$\left|\int_Q \delta_{\vec{C},\psi}\right| \le \lfloor |E(Q)|/5 \rfloor + m(|E(Q)| \bmod 5),$$

*where $m(0) = 0$, $m(1) = m(3) = 1$, and $m(2) = m(4) = 2$.*

Let us focus mainly on the generalized chord constraints. Let $(G_0, C_1, C_2, \lambda)$ be obtained from a 4-critical triangle-free graph in the torus by cutting along a non-contractible cycle. Let $U$ be a (possibly empty) set of odd-length faces of $G_0$, let $k$ and $t$ be positive integers, and let $B$ be a subpath of $C_i$ for some $i \in \{1,2\}$. We say that $U$ is $(t,k)$*-tied to (the subpath $B$ of) $C_i$* if $|E(B)| = t$ and $C_i$ has a generalized chord $R$ of length $k$ with base $B$ such that $U$ is exactly the set of odd-length faces contained in $\mathrm{int}^\star(R)$.

For a positive integer $n$, we say that $U$ is *strongly* $(\leq n)$*-tied to* $C_i$ if $U$ is $(k,k)$-tied to $C_i$ for some $k \leq n$, and $U$ is $n$*-loose with respect to* $C_i$ otherwise. A face $f$ of $G_0$ is $k$*-near* to $C_i$ if $f \in \mathrm{int}^\star(Q)$ for some generalized chord $Q$ of $C_i$ of length at most $k$.

**Lemma 21.** *Let $G$ be a 4-critical triangle-free graph with a 2-cell drawing in the torus of edge-width at least 15, and let $C$ be a shortest non-contractible cycle in $G$. Let $(G_0, C_1, C_2, \lambda)$ be obtained from $G$ by splitting along $C$. Fix $i \in \{1,2\}$ and let $U$ be a set of odd-length faces of $G_0$ that is $(t,k)$-tied to a subpath $Q$ of $C_i$. Then the following claims hold.*

*(a)* $t \leq k$

*(b) If $|U| = 2$ and $t < k \leq 6$, then $(t,k) \in \{(3,5),(2,6),(4,6)\}$.*

*(c) If $|U| = 2$, $t < k \leq 6$, $U$ is also $(t',k')$-tied to another subpath $Q'$ of $C_i$, $t' < k' \leq 6$, and $U$ is 7-loose with respect to $C_i$, then $Q \subseteq Q'$ or $Q' \subseteq Q$, or $U$ is $(3,5)$-tied to either $Q \cup Q'$ or $Q \cap Q'$ (and in particular the union or intersection is a path of length three).*

*(d) If $|U| = 2$, $k = t \leq 7$, $U'$ is a set of odd-length faces of $G_0$ such that $|U'| = 2$, and $|U \cap U'| = 1$, then $U'$ is 7-loose with respect to $C_i$.*

*(e) If $|U| \geq 3$, then $U$ is 4-loose.*

*Proof.* Let $R$ be a generalized chord of $C_i$ of length $k$ with base $Q$ such that $U$ is the set of odd-length faces of $\mathrm{int}^\star(R)$.

Let $C'$ be the closed walk in $G$ obtained from $C$ by replacing $Q$ by $R$. Note that $C'$ is homotopically equivalent to $C$, and thus $C'$ is non-contractible. Since $C$ is a shortest non-contractible cycle in $G$, we have $|C| - t + k = |C'| \geq |C|$, and thus $t \leq k$. Consequently, (a) holds.

Using (a) together with Lemma 18, we have that $t+k \geq 9$ and $t \leq k$ implying that $k \geq 5$ and therefore (e) holds.

In the cases (b), (c), and (d), $\mathrm{int}^\star(R)$ contains exactly two odd-length faces, and thus $k+t$ is even. Furthermore, by Lemma 18, $k+t \geq 8$. Hence, if $t < k \leq 6$, then $(t,k) \in \{(3,5),(2,6),(4,6)\}$. Therefore, (b) holds.

For the cases (c) and (d), let $R'$ be a generalized chord of $C_i$ of length $k'$ with base $Q'$ of length $t'$, with $U'$ being the set of odd-length faces of $\mathrm{int}^\star(R')$, where

- in the case (c) $U = U'$, and
- in the case (d) we for contradiction assume $t' = k' \leq 7$.

Let $d = k - t$ and $d' = k' - t'$, so that $d = d' = 0$ in case (d) and $d, d' \in \{2,4\}$ in case (c) by (b). Let $f_1$ and $f_2$ be the faces of $G_0$ bounded by $C_1$ and $C_2$. Let $\Delta$ and $\Delta'$ be the disks in the plane bounded by $R \cup Q$ and $R' \cup Q'$, respectively. Consider the plane graph $G' = C_i \cup R \cup R'$. Let $D_0$ denote the set of faces of $G'$ not contained in $\Delta \cup \Delta'$, and let $D_2$ denote the set of faces of $G'$ contained in $\Delta \cap \Delta'$; let $D = D_0 \cup D_2$. Note that if an edge $e \in E(R \cup R')$ is in the boundaries of two faces of $D$, then $e \in E(R \cap R')$. Furthermore, an edge $e$ of $C_i$ is incident with a face of $D$ distinct from $f_i$ if and only if $e$ is contained in either both or neither of $Q$ and $Q'$.

Hence,

$$\sum_{f\in D}|f| \le k+k'+2|C|-(t-|E(Q\cap Q')|)-(t'-|E(Q\cap Q')|)$$
$$\le 2|C|+d+d'+2|E(Q\cap Q')|. \tag{2.10}$$

Let $f_0$ denote the face of $G'$ containing $f_{3-i}$; clearly, $f_0\in D_0$, and since $C$ is a shortest non-contractible cycle in $G$, we have $|f_0|\ge|C|\ge 15>k+k'$. Hence, the boundary of $f_0$ intersects $C_i$ in at least one edge.

Let $\ell_2=\sum_{f\in D_2}|f|$. If $E(Q\cap Q')$ is non-empty, then the face of $G'$ distinct from $f_i$ whose boundary contains $E(Q\cap Q')$ belongs to $D_2$ and $\ell_2\ge 2|E(Q\cap Q')|$ by (a). If $E(Q\cap Q')=\emptyset$, then $\ell_2\ge 2|E(Q\cap Q')|$ trivially. Furthermore, the odd-length faces of $G$ drawn in $\Delta\cap\Delta'$ are exactly those belonging to $U\cap U'$, and thus $\ell_2$ and $|U\cap U'|$ have the same parity. Hence, $\ell_2$ is odd in the case (d), implying $\ell_2>2|E(Q\cap Q')|$, and since $d=d'=0$, (2.10) gives

$$2|C|+2|E(Q\cap Q')|\ge\sum_{f\in D}|f|\ge|f_i|+|f_0|+\ell_2>2|C|+2|E(Q\cap Q')|,$$

which is a contradiction. Therefore, (d) holds.

From now on, we assume the case (c). Then two odd-length faces of $G$ are drawn in $\Delta\cap\Delta'$, and Lemma 18 implies $\ell_2\ge 8$. We can assume that $Q\not\subseteq Q'$ and $Q'\not\subseteq Q$, and thus $|E(Q\cap Q')|\le\min(t,t')-1\le 3$.

Let $\ell_0=\sum_{f\in D_0\setminus\{f_i,f_0\}}|f|$. By (2.10), we have

$$2|C|+\ell_0+8\le|f_i|+|f_0|+\ell_0+\ell_2=\sum_{f\in D}|f|\le 2|C|+d+d'+2|E(Q\cap Q')|. \tag{2.11}$$

If $D_0\ne\{f_i,f_0\}$, then $\ell_0\ge 4$, and (2.11) implies $d+d'+2|E(Q\cap Q')|\ge 12$. Since $|E(Q\cap Q')|\le 3$, it follows that $\max(d,d')\ge 3$, and thus $\min(t,t')=2$. But then $|E(Q\cap Q')|\le\min(t,t')-1=1$, and the same argument gives $\max(d,d')\ge 5$, which is a contradiction. We conclude that $D_0=\{f_i,f_0\}$, and consequently $V(Q\cap Q')\ne\emptyset$, as otherwise $C_i$ has two non-empty subpaths edge-disjoint from $Q\cup Q'$, each incident with a distinct face of $D_0\setminus\{f_i\}$.

Since the boundary of $f_0$ contains an edge of $C_i$, we conclude both $Q\cup Q'$ and $Q\cap Q'$ are non-empty connected subpaths of $C_i$ (with $Q\cap Q'$ possibly consisting of a single vertex). Let $T$ denote the part of the boundary of $f_0$ edge-disjoint from $C_i$. Since $\Delta$ and $\Delta'$ are not disjoint, observe that $T$ is a path intersecting $C_i$ only in its endpoints; hence, $T$ is a generalized chord of $C_i$ with base $Q\cup Q'$, and $\mathrm{int}^\star(T)=U$.

We claim that $|E(Q\cup Q')|\le 7$. Indeed, since $t,t'\le 4$, we could have $|E(Q\cup Q')|>7$ only if $t=t'=4$ and $E(Q)\cap E(Q')=\emptyset$; but then $d,d'=2$, contradicting (2.11). Since $U$ is 7-loose with respect to $C_i$, it follows that $|E(T)|>|E(Q\cup Q')|$. Furthermore, since $U$ consists of the odd-length faces in $\mathrm{int}^\star(T)$ and $|U|$ is even, $|E(T)|$ and $|E(Q\cup Q')|$ have the same parity, and thus $|E(T)|\ge|E(Q\cup Q')|+2$.

Note that $|f_0|=|C|+|E(T)|-|E(Q_1\cup Q_2)|$, and thus as in (2.11), we have

$$2|C|+10\le 2|C|+|E(T)|-|E(Q_1\cup Q_2)|+8\le|f_i|+|f_0|+\ell_2$$
$$\le\sum_{f\in D}|f|\le 2|C|+d+d'+2|E(Q\cap Q')|. \tag{2.12}$$

Since $d, d' \leq 4$, it follows that $|E(Q \cap Q')| \geq 1$.

If $|E(Q \cap Q')| = 1$, then (2.12) shows that $d = d' = 4$ (and thus $t = t' = 2$ and $|E(Q \cup Q')| = 3$) and $|E(T)| = |E(Q \cup Q')| + 2 = 5$, implying that $U$ is $(3, 5)$-tied to $Q \cup Q'$. If $|E(Q \cap Q')| \geq 2$, then $t, t' \geq 3$, and thus $d = d' = 2$. Hence, (2.12) shows that $|E(Q \cap Q')| \geq 3$. Since $t, t' \leq 4$, we have $|E(Q \cap Q')| = 3$, and thus all the inequalities in (2.12) are tight; in particular $\ell_2 = 8$. Considering the face of $G'$ distinct from $f_i$ whose boundary contains $Q \cap Q'$ and whose length is, we $\ell_2$ conclude that $U$ is $(3, 5)$-tied to $Q \cap Q'$. □

Analogously, we can bound the amount of interaction between bound sets of faces.

**Lemma 22.** *Let $G$ be a 4-critical triangle-free graph with a 2-cell drawing in the torus, and let $C$ be a shortest non-contractible cycle in $G$. Let $(G_0, C_1, C_2, \lambda)$ be obtained from $G$ by splitting along $C$, let $f_1, \ldots, f_4$ be distinct odd-length faces of $G$, and fix $i \in \{1, 2\}$.*

*Suppose that $\{f_1, f_2\}$ is $(t_1, k_1)$-tied to a subpath $Q_1$ of $C_i$ and $t_1 < k_1$ and analogously $\{f_3, f_4\}$ is $(t_2, k_2)$-tied to a subpath $Q_2$ of $C_i$ and $t_2 < k_2$. Then $|E(Q_1 \cap Q_2)| \leq (t_1 + t_2 + k_1 + k_2)/2 - 8 \leq k_1 + k_2 - 10$.*

*Proof.* For $j \in \{1, 2\}$, let $R_j$ denote a generalized chord of $C_i$ of length $k_j$ with base $Q_j$ and with $\{f_{2j-1}, f_{2j}\}$ being exactly the odd-length faces in $\mathrm{int}^\star(R_j)$. Let $\Delta_j$ denote the open disk bounded by $Q_j \cup R_j$, let $D_j$ denote the set of faces of $G' = C_i \cup R_1 \cup R_2$ drawn in $\Delta_j \setminus \Delta_{3-j}$, and let $\ell_j$ be the sum of the lengths of these faces. Since $f_{2j-1}$ and $f_{2j}$ are contained in $\Delta_j \setminus \Delta_{3-j}$, Lemma 18 implies $\ell_j \geq 8$.

Note that if an edge $e$ is in boundaries of two faces of $D_1 \cup D_2$, then $e \in E(R_1 \cap R_2)$. Furthermore, an edge $e$ of $C_i$ is incident with a face of $D_1 \cup D_2$ if and only if $e$ is contained in exactly one of $Q_1$ and $Q_2$.

Hence,

$$16 \leq \ell_1 + \ell_2 \leq k_1 + k_2 + |E(Q_1)| + |E(Q_2)| - 2|E(Q_1 \cap Q_2)|.$$

Note that $t_j = |E(Q_j)|$ and $k_j$ have the same parity, since $\mathrm{int}^\star(R_j)$ contains exactly two odd-length faces, and thus $t_j \leq k_j - 2$. This implies the inequality from the statement of the lemma. □

Let $G$ be a 4-critical triangle-free graph with a 2-cell drawing in the torus, and let $C$ be a non-contractible cycle in $G$. Let $(G_0, C_1, C_2, \gamma)$ be obtained from $G$ by cutting along $C$, and let $f_1$ and $f_2$ be the faces of $G_0$ bounded by $C_1$ and $C_2$.

Our goal now is to design a candidate demand function on the dual of $G_0$ so that generalized chords gain enough slack. We do this by minimizing the absolute value of the sources and sinks and assigning opposite signed values to pairs of faces tied together by a generalized chord, making their contributions cancel out.

More precisely, we say that $d^\star$ is a *standard assignment of sources and sinks* if $d^\star(F(G_0)) = 0$, $d^\star(f) = 0$ for every even-length face $f \in F(G_0)$, $d^\star(f) \in \{-3, 3\}$ for every odd-length face $f \in F(G_0)$, $d^\star(f_1) \in \{0, 3\}$ and $d^\star(f_2) = -d^\star(f_1) \in \{0, -3\}$, and $d^\star(U) = 0$ for every two-element set $U$ of odd faces strongly $(\leq 7)$-tied to $C_1$ or $C_2$.

Note also that a standard assignment is even, $3|d^\star$ and $|d^\star(f)| \leq |f|$ for every $f \in F(G_0)$, and therefore is a candidate demand function. Let us remark that the last condition concerning tied pairs ensures that the main contribution against positive slack of short generalized chords comes from the precoloring of the rings (which we can choose), not from the sources and sinks. It is however not clear that this condition can always be met, which is the purpose of the following lemma.

**Lemma 23.** *Let $G$ be a 4-critical triangle-free graph with a 2-cell drawing in the torus of edge-width at least 15, and let $C$ be a shortest non-contractible cycle in $G$. Let $(G_0, C_1, C_2, \gamma)$ be obtained from $G$ by cutting along $C$. Then there exists a standard assignment of sources and sinks $d^\star : F(G_0) \to \mathbb{Z}$.*

*Proof.* Let $f_1$ and $f_2$ be the faces of $G_0$ bounded by $C_1$ and $C_2$. The values of $d^\star$ are exactly determined by the definition of a standard assignment except on odd faces distinct from $f_1$ and $f_2$, where the sign of the value is not determined. We need to choose the values on these odd faces so that $d^\star(F(G_0)) = 0$ and the last condition from the definition holds.

Note that $G_0$ does not contain a path $P$ of length less than $|C|/2$ between $C_1$ and $C_2$, as otherwise one of the three non-contractible cycles in $C \cup P$ in $G$ is shorter than $C$.

First we show that each face $f$ is 7-near to at most one of $C_1$ and $C_2$. Suppose not, that is, for $i \in \{1, 2\}$ there exists generalized chord $R_i$ of $C_i$ such that $f \in int^\star(C_1)$. The union $R_1 \cup R_2$ contains two edge-disjoint paths connecting $C_1$ and $C_2$. Since each path is of length at least $|C|/2$, we get that $|E(R_1)|+|E(R_2)| \geq |C|$. By assumption on $C$, $|C| \geq 15$, which is a contradiction with $f$ being 7-near to both $C_1$ and $C_2$.

Consider all pairs of odd-length faces $U = \{f, f'\}$ such that $U$ is strongly $(\leq 7)$-tied to $C_1$ and similarly for $C_2$. To satisfy the last condition of the standard assignment, faces in each pair must be given opposite values.

By Lemma 21(d) and the argument in the previous paragraph, there are no two distinct pairs $U$ and $U'$ such that $U \cap U' \neq \emptyset$. We conclude that each odd-length face appears in at most one constraint, inducing a partial pairing of odd-length faces. We complete the pairing, and in each pair choose arbitrarily which of the paired faces is assigned 3 and $-3$. Naturally, since all non-zero values are paired, the sum of all values is 0 as requested by the definition of a standard assignment. $\square$

### 2.2.2 Satisfying Constraints

In this section, we identify the type of constraints that are potentially not satisfied by the choices defined in the previous section.

Let $G$ be a 4-critical triangle-free graph with a 2-cell drawing in the torus, and let $C$ be a non-contractible cycle in $G$. Let $(G_0, C_1, C_2, \gamma)$ be obtained from $G$ by splitting $G$ along $C$, so that $G_0$ is the obtained graph, $C_1$ and $C_2$ the cycles bounding the rings of $G_0$ and $\gamma$ a natural projection of $\gamma : V(G_0) \cup E(G_0) \to V(G) \cup E(G)$. Let $d^\star$ be a standard assignment of sources and sinks. Let $f_1$ and $f_2$ be the two rings of $G_0$ bounded by $C_1$ and $C_2$. Let $\vec{C}$ be a cyclic orientation of $C$ chosen so that the paths in $C_1$ for which $f_1$ is to their right are mapped by

$\gamma$ to paths along $\vec{C}$ (and thus, the paths in $C_2$ for which $f_2$ is to their right are mapped by $\gamma$ to paths in the opposite direction to $\vec{C}$).

The $(G, C, d^\star)$*-request* is the system of all pairs $(Q, s)$, where $Q$ is a subpath of $C$ directed along $\vec{C}$ and $s$ is a non-zero integer, such that one of the following holds:

(Ra) $G_0$ has a two-element set $U$ of odd faces such that $d^\star(U) \neq 0$, for some $i \in \{1, 2\}$, $U$ is $(3, 5)$-tied to a subpath $Q'$ of $C_i$, $\gamma(Q')$ is (up to reversal) equal to $Q$, and $s = (-1)^i d^\star(U)$; or

(Rb) $G_0$ has a two-element set $U$ of odd faces such that $d^\star(U) \neq 0$, for some $i \in \{1, 2\}$, $U$ is $(2, 6)$- or $(4, 6)$-tied to a subpath $Q'$ of $C_i$, $\gamma(Q')$ is (up to reversal) equal to $Q$, and $s = (-1)^i d^\star(U)$; or

(Rc) $G_0$ has a 5-face $f$ such that for some $i \in \{1, 2\}$, the boundary of $f$ intersects $C_i$ in a path $Q'$ of length two, $\gamma(Q')$ is (up to reversal) equal to $Q$, and $s = (-1)^i d^\star(f)$.

The $(G, C, d^\star)$-request $\mathcal{R}$ is *satisfied* by a 3-coloring $\psi$ of $C$ if $\int_C \delta_{\vec{C},\psi} = d^\star(f_1)$ and for every $(Q, s) \in \mathcal{R}$,

$$s \cdot \int_Q \delta_{\vec{C},\psi} \geq 0.$$

Note this means that the amount of flow sent across $Q'$ according to $\psi$ "compensates" for the flow originating in $U$ or $f$ according to $d^\star$. It turns out this suffices to ensure extendability of $\psi$.

**Lemma 24.** *Let $G$ be a 4-critical triangle-free graph with a 2-cell drawing in the torus of edge-width at least 21, let $C$ be a shortest non-contractible cycle in $G$, and let $\vec{C}$ be a cyclic orientation of $C$. Let $(G_0, C_1, C_2, \gamma)$ be obtained from $G$ by cutting along $C$, let $d^\star$ be a standard assignment of sources and sinks, and let $\psi_0$ be a 3-coloring of $C$. If $\psi_0$ is tame, then $\psi_0$ does not satisfy the $(G, C, d^\star)$-request $\mathcal{R}$.*

*Proof.* Suppose for a contradiction that $\psi_0$ satisfies $\mathcal{R}$. Let $f_1$ and $f_2$ be the rings of $G_0$ bounded by $C_1$ and $C_2$. By Theorem 19, $G$ has at most four odd-length faces, and $|d^\star(U)| \leq 6$ for any $U \subseteq F(G_0) \setminus \{f_1, f_2\}$, implying that $|d^\star(U)| \leq 9$ for any $U \subseteq F(G_0)$.

Let $\psi$ be the 3-coloring of $C_1 \cup C_2$ corresponding to $\psi_0$, and let $\delta = \delta_{C_1 \cup C_2, \psi}$ be the tension induced by $\psi$. Let us discuss possible constraints.

- Let $K$ be a non-contractible cycle in $G_0$ edge-disjoint from $C_1 \cup C_2$. Since $K$ is non-contractible, $\gamma(K)$ is non-contractible, and thus $|K| \geq 21$ and $\mathrm{slack}_{d^\star,\psi}(K) = |K| - |d^\star(\mathrm{int}^\star(K))| \geq 12$.
- Let $K$ be a contractible cycle in $G_0$ edge-disjoint from $C_1 \cup C_2$. Since $|K| \leq 7$, then by Lemma 18 $\mathrm{int}^\star(K)$ contains at most one odd-length face, and

$$\mathrm{slack}_{d^\star,\psi}(K) = |K| - |d^\star(\mathrm{int}^\star(K))| = \begin{cases} |K| & \text{if } |K| \text{ is even} \\ |K| - 3 & \text{if } |K| \text{ is odd.} \end{cases}$$

If $|K| \geq 8$, then $\mathrm{slack}_{d^\star,\psi}(K) = |K| - |d^\star(\mathrm{int}^\star(K))| \geq |K| - 6 \geq 2$.

We observe that if $\mathrm{slack}_{d^\star,\psi}(K) < 4$ then $\mathrm{slack}_{d^\star,\psi}(K) = 2$ and either $|K| = 5$ and $\mathrm{int}^\star(K) = \{5\}$, or $|K| = 8$ and $\mathrm{int}^\star(K)$ contains two odd-length faces.

- Let $R$ be a $(C_1, C_2)$-connector consisting of paths $P_1$ and $P_2$. Let $Q_1$ and $Q_2$ be subpaths of $C_1$ and $C_2$ directed so that $f_1$ and $f_2$ are to the right of them, and such that $K = Q_1 \cup Q_2 \cup P_1 \cup P_2$ is a contractible cycle.

  Note that there are two possible choices for $Q_1$ and $Q_2$, and we choose one where $|E(Q_1)| + |E(Q_2)| \leq |C|$. For $i \in \{1, 2\}$, let $a_i = \int_{Q_i} \delta$. Since $C$ is a shortest non-contractible cycle in $G$, we have $|E(P_1)|, |E(P_2)| \geq |C|/2$. We have $\mathrm{slack}_{d^\star,\psi}(R) \geq |E(R)| - |a_1| - |a_2| - |d^\star(\mathrm{int}^\star(K))| \geq |C| - |a_1| - |a_2| - 6$. Since $\psi$ is tame, Observation 20 gives $|a_1| + |a_2| \leq |E(Q_1)|/5 + |E(Q_2)|/5 + 4 \leq |C|/5 + 4$, and thus $\mathrm{slack}_{d^\star,\psi}(R) \geq \frac{4}{5}|C| - 10 > 4$.

- Let $R$ be a generalized chord of $C_i$ for some $i \in \{1, 2\}$. Let $Q$ be the base of $R$, and let $k = |E(R)|$, $t = |E(Q)|$ and $a = \int_Q \delta$. Since $C$ is a shortest non-contractible cycle, we have $k \geq t$ and $k \geq 2$. Recall that $\mathrm{slack}_{d^\star,\psi}(R) = k - |a + d^\star(\mathrm{int}^\star(R))|$.

  - If $d^\star(\mathrm{int}^\star(R)) = 0$, then $k$ and $t$ have the same parity. By Observation 20, we have $\mathrm{slack}_{d^\star,\psi}(R) \geq 0$ if $k = 2$, $\mathrm{slack}_{d^\star,\psi}(R) \geq 2$ if $k \in \{3, 4\}$, and $\mathrm{slack}_{d^\star,\psi}(R) \geq 4$ if $k \geq 5$.
  - If $|d^\star(\mathrm{int}^\star(R))| = 3$, then by parity we have $k \geq t + 1$, and since $G$ is triangle-free, $k \geq 3$. If $k = 3$, then $R \cup Q$ is a 5-cycle, and by Lemma 18 $R \cup Q$ bounds a 5-face $f$. Since $t = 2$, $(\gamma(Q), -d^\star(f)) \in \mathcal{R}$, and since $\psi$ satisfies $\mathcal{R}$, $a$ and $d^\star(\mathrm{int}^\star(R))$ do not have the same sign. Consequently, $|a + d^\star(\mathrm{int}^\star(R))| \leq 3$ and $\mathrm{slack}_{d^\star,\psi}(R) \geq 0$.

    Using Observation 20, we obtain that $\mathrm{slack}_{d^\star,\psi}(R) \geq 0$ if $k \in \{4, 5\}$, $\mathrm{slack}_{d^\star,\psi}(R) \geq 2$ if $k \in \{6, 7, 8\}$, and $\mathrm{slack}_{d^\star,\psi}(R) \geq 4$ if $k \geq 9$.
  - If $|d^\star(\mathrm{int}^\star(R))| = 6$, then $\mathrm{int}^\star(R)$ contains exactly two odd-length faces $g_1$ and $g_2$ such that $d^\star(g_1) = d^\star(g_2) \in \{-3, 3\}$. Since $d^\star$ is a standard assignment, if $k \leq 7$ then $k \geq t + 2$.

    If $k \leq 5$, then by Lemma 21(b) we have $k = 5$ and $t = 3$. Hence, $(\gamma(Q), -d^\star(\mathrm{int}^\star(R))) \in \mathcal{R}$, and since $\psi$ satisfies $\mathcal{R}$, $a$ and $d^\star(\mathrm{int}^\star(R))$ have opposite signs.

    Consequently $|a + d^\star(\mathrm{int}^\star(R))| \leq 5$ and $\mathrm{slack}_{d^\star,\psi}(R) \geq 0$. If $k = 6$, then an analogous argument gives $|a + d^\star(\mathrm{int}^\star(R))| \leq 6$ and $\mathrm{slack}_{d^\star,\psi}(R) \geq 0$. Once again, by Observation 20, we have $\mathrm{slack}_{d^\star,\psi}(R) \geq 0$ if $k \in \{7, 8, 9\}$, $\mathrm{slack}_{d^\star,\psi}(R) \geq 2$ if $k \in \{10, 11, 12\}$, and $\mathrm{slack}_{d^\star,\psi}(R) \geq 4$ if $k \geq 13$.

Hence, $\mathrm{slack}_{d^\star,\psi}(R) \geq 0$ for every constraint $R$. Since $G$ is 4-critical, $\psi_0$ does not extend to a 3-coloring of $G$, and $\psi$ does not extend to a 3-coloring of $G_0$. By Lemma 16 we conclude that there exists a $(C_1, C_2)$-connecting set $X$ of constraints such that $\mathrm{slack}_{d^\star,\psi}(X) \leq 2$. In particular, we have $\mathrm{slack}_{d^\star,\psi}(R) \leq 2$ for every $R \in X$.

According to the preceding analysis, we conclude that $X$ contains for $i \in \{1, 2\}$ either a non-chord edge $R_i$ with one end in $C_i$ or a generalized chord $R_i$ of $C_i$, and in case $R_1$ and $R_2$ are generalized chords with $\mathrm{slack}_{d^\star,\psi}(R_1) = \mathrm{slack}_{d^\star,\psi}(R_2) = 0$, $X$ can additionally contain a non-chord edge $R_3$ or a contractible cycle $R_3$ with $\mathrm{slack}_{d^\star,\psi}(R_3) = 2$.

Note that $G_0$ contains a path from $C_1$ to $C_2$ of length at most $\sum_{R\in X} \lfloor |R|/2 \rfloor$, and by a straightforward case analysis using the description of constraints $R$ with $\mathrm{slack}_{d^\star,\psi}(R) \le 2$ we obtained above, this path has length at most 10. However, since $C$ is a shortest non-contractible cycle in $G$, any such path must have length at least $|C|/2 > 10$, which is a contradiction. □

**Lemma 25.** *If $G$ is a 4-critical triangle-free graph drawn in the torus, then the edge-width of the drawing is at most* 20.

*Proof.* Suppose for a contradiction that $G$ is a 4-critical triangle-free graph with a 2-cell drawing in the torus of edge-width at least 21. We prove the statement by constructing a 3-coloring satisfying the request of Lemma 24.

Let $C$ be a shortest non-contractible cycle in $G$. Let $(G_0, C_1, C_2, \gamma)$ be obtained from $G$ by cutting along $C$. Let $\vec{C}$ be a cyclic orientation of $C$ chosen so that the paths in $C_1$ for which $f_1$ is to their right are mapped by $\gamma$ to paths along $\vec{C}$.

We say that an odd-length face $f \in F(G_0) \setminus \{f_1, f_2\}$ is *isolated* if no two-element set containing $f$ is strongly $(\le 7)$-tied to $C_1$ or $C_2$, and that $f$ is *aligned* if for some $i \in \{1, 2\}$, $f$ is 7-near to $C_i$ and $d^\star(f) = (-1)^i \cdot 3$. Let $d^\star$ be a standard assignment of sources and sinks such that the number of aligned isolated faces is maximum. Let $\mathcal{R}$ be the $(G, C, d^\star)$-request.

If $|C|$ is odd, let $\psi_0$ be a 3-coloring of $C$ where vertices in order have colors

$$\mathbf{0}, \mathbf{1}, \mathbf{2}, 1, 2, \mathbf{1}, \mathbf{2}, \mathbf{0}, 2, 0, \mathbf{2}, \mathbf{0}, \mathbf{1} \tag{2.13}$$

followed by $0, 1$ repeated $(|C| - 13)/2$ times; if $|C|$ is even, let $\psi_0$ be a 3-coloring of $C$ where vertices in order have colors

$$\mathbf{0}, \mathbf{1}, \mathbf{2}, 1, \mathbf{2}, \mathbf{1}, \mathbf{0}, 1 \tag{2.14}$$

followed by $0, 1$ repeated $(|C| - 8)/2$ times. The boldface emphasizes the places where $\delta_{\vec{C},\psi_0}(e_1) = \delta_{\vec{C},\psi_0}(e_2)$ for two consecutive edges $e_1$ and $e_2$ of $C$; clearly, $\psi_0$ is tame and $\int_C \delta_{\vec{C},\psi_0} = d^\star(f_1)$. For $k \in \{1, \ldots, |C|-1\}$, let $\psi_k$ denote the coloring of $C$ obtained by rotating $\psi_0$ on $C$ by $k$ vertices. We say the index $k$ is *killed* by $(Q, s) \in \mathcal{R}$ if $s \cdot \int_Q \delta_{\vec{C},\psi_k} < 0$. By Lemma 24, $\mathcal{R}$ is not satisfied, and thus every index $k \in \{0, \ldots, |C| - 1\}$ is killed by some element of $\mathcal{R}$.

Let $U^+$ and $U^-$ be the set of faces of $F(G_0) \setminus \{f_1, f_2\}$ to that $d^\star$ assigns the value 3 and $-3$, respectively. Recall that by Theorem 19, $G$ has at most 4 odd-length faces, and thus $|U^+| = |U^-| \le 2$. For $i \in \{1, 2\}$, let $U_i^+$ denote those of the faces in $U^+$ that are 7-near to $C_i$; $U_i^-$ is defined analogously. Let us now discuss the elements of $\mathcal{R}$ arising from $U^+$. Since $C$ is a shortest non-contractible cycle and $|C| \ge 21$, the sets $U_1^+$ and $U_2^+$ are disjoint.

- If $|U_1^+| \le 1$ and $|U_2^+| \le 1$, then $U^+$ only contributes to $\mathcal{R}$ by (Rc): $U_1^+$ may contribute $(Q_1, -3)$ and $U_2^+$ may contribute $(Q_2, 3)$ for some subpaths $Q_1, Q_2 \subset C$ of length two. An inspection of the coloring $\psi_0$ shows that in total the contributed elements kill at most 3 indices if $|C|$ is odd and at most two indices if $|C|$ is even.

- Suppose $|U_1^+| = 2$ (and thus $U_2^+ = \emptyset$). Note that $U_1^+$ is 7-loose with respect to $C_1$, since $d^\star$ is a standard assignment of sources and sinks.
  - If $U_1^+$ is $(3,5)$-tied to a subpath $Q$ of $C_1$, then $U^+$ contributes to $\mathcal{R}$ the element $(Q, -6)$, and by Lemma 21(c) can additionally contribute only elements $(Q', s)$ where $s \in \{-3, -6\}$ and $Q'$ is either a length-two subpath or a length-four superpath of $Q$; and observe that each index killed by $(Q', s)$ is also killed by $(Q, s)$. Consequently, the elements contributed by $U^+$ kill $(|C| + 9)/2$ indices if $|C|$ is odd and $|C|/2$ indices if $|C|$ is even.
  - If $U_1^+$ is not $(3,5)$-tied to $C_1$, but is $(4,6)$-tied to a subpath $Q$ of $C_1$, then $U^+$ contributes to $\mathcal{R}$ the element $(Q, -6)$, and by Lemma 21(b) and (c) can additionally contribute only elements $(Q', s)$ where $s \in \{-3, -6\}$ and $Q'$ is a length-two subpath of $Q$. Note that each index killed by $(Q, -6)$ is also killed by $(Q', -3)$ for some length-two subpath of $Q$, and by considering three consecutive length-two subpaths of $C$, we conclude the elements contributed by $U^+$ kill at most 9 indices if $|C|$ is odd and at most 3 indices if $|C|$ is even.
  - If $U_1^+$ is neither $(3,5)$-tied nor $(4,6)$-tied to $C_1$, then by Lemma 21(b) and (c) contributes to $\mathcal{R}$ either one element $(Q_1, -6)$ for a length-two subpath $Q_1$, or (by (Rc)) two elements $(Q_1, -3)$ and $(Q_2, -3)$ for length-two subpaths $Q_1$ and $Q_2$. Consequently, the elements contributed by $U^+$ kill at most 6 indices if $|C|$ is odd and at most 2 indices if $|C|$ is even.
- The case that $|U_2^+| = 2$ is symmetric, but the signs are switched; hence, in the three considered subcases, the elements contributed by $U^+$ kill
  - $(|C| - 9)/2$ indices if $|C|$ is odd and $|C|/2$ indices if $|C|$ is even,
  - no indices if $|C|$ is odd and at most 3 indices if $|C|$ is even, and
  - no indices if $|C|$ is odd and at most 2 indices if $|C|$ is even, respectively.

The situation for $U^-$ is symmetric, up to switching of signs. Let us first consider the case that $|C|$ is even. Since all indices are killed and $|C|/2 > 3$, this implies each of $U^+$ and $U^-$ is $(3,5)$ tied to one of $C_1$ and $C_2$, and by the preceding analysis, each index is killed by exactly one of elements $(Q^+, s^+)$ and $(Q^-, s^-)$ of $\mathcal{R}$, where $Q^+$ and $Q^-$ are length-three subpaths of $C$ and $s^+, s^- \in \{-6, 6\}$. Without loss of generality, we can assume that the index 0 is killed by $(Q^+, s^+)$ and in $\psi_0$, the path $Q^+$ covers the first four colors of (2.14); all other cases are symmetric. Then also 1, $|C| - 1$, and $|C| - 2$ are killed by $(Q^+, s^+)$, and thus none of them is killed by $(Q^-, s^-)$. This is only possible if $Q^-$ covers the last four colors of (2.14). However, then 2 is killed by neither of the elements, which is a contradiction.

Therefore, $|C|$ is odd. We can by symmetry assume that at least $|C|/2 > 9$ indices are killed by elements contributed by $U^+$, and thus $|U^+| = 2$ and $U^+$ is $(3,5)$-tied to $C_1$. If $|U_2^-| = 2$, then since $d^\star$ is a standard assignment of sources and sinks, all elements of $U^+ \cup U^-$ are isolated but not aligned, and there exists a standard assignment of sources and sinks with more isolated aligned faces, obtained by assigning $-3$ to faces in $U^+$ and $+3$ to faces in $U^-$, contradicting the choice of $d^\star$. Hence, $|U_2^-| \leq 1$. Since all indices are killed, the elements contributed by $U^-$ kill at least $(|C|-9)/2 > 3$ indices. It follows that $|U_1^-| = 2$ and $U^-$ is $(3,5)$-tied to $C_1$. By the preceding analysis, each index is killed by exactly one of elements $(Q^+, -6)$ and $(Q^-, 6)$ of $\mathcal{R}$, where $Q^+$ and $Q^-$ are length-three subpaths of $C$; and by Lemma 22, $Q^+$ and $Q^-$ are edge-disjoint. Without loss of generality, we can assume that in $\psi_0$, the path $Q^+$ starts two vertices before the first color of (2.13), and thus $(Q^+, -6)$ kills 0, 1, 2, and 3. Hence, $(Q^-, 6)$ does not kill 0, 1, 2, and 3, which is only possible if $Q^-$ is shifted by 5 or 10 vertices to the right from $Q^+$ in (2.13). However, then $|C|-2$ is not killed by either of the elements, which is a contradiction. □

## 2.3 Templates

Before combining the results of Sections 2.1 and 2.2 into algorithmic applications in Section 2.4, we take a detour into a very different approach. We will present analysis of the structure of triangle-free 4-critical graphs embedded in the torus based on exhaustive search performed by a computer. The results presented played a significant role in inspiring the analysis in Section 2.2. In fact, the computer assisted proof shows that even stronger results should be obtainable by a more detailed treatment of the constraints, however, the tools of our choice seem to fall short of providing theoretical basis for replicating the full extent of these results purely theoretically.

The content of this section is a shorter version of [32], focusing more on the features common with the results from Sections 2.2 and 2.4. The main result is an almost-exact description of 4-critical triangle-free toroidal graphs, providing a precise characterization of 3-colorability of triangle-free graphs drawn in the torus, and in principle also a linear-time decision algorithm to test 3-colorability. There are infinitely many such 4-critical graphs, however there are only a few faces in these graphs other than 4-faces. A typical 4-critical graph is therefore a huge near-quadrangulation with at most four faces of length 5, 6 or 7 floating somewhere among the other 4-faces. We call these faces *long* faces.

Let us consider the behavior of nowhere-zero flows in the dual of such graph. In the spirit of representing the nowhere-zero flow as linkages and circulations, since 4-faces do not act as sources or sinks, they essentially only conduct the linkage connecting the long faces. It comes then as no surprise that the exact structure of the quadrangulation stops playing any role in colorability far away from the long faces. We therefore aim at abstracting away the quadrangulations to the extent that does not influence the colorability in the hope of describing the infinite class in a finite set of structures that can then be studied.

Thomassen [23] proved that every graph embedded in the torus without contractible ($\leq 4$)-cycles (but possibly with non-contractible triangles or 4-cycles) is 3-colorable. Moreover, every contractible 4-cycle in an embedded 4-critical graph is know to bound a face. Consequently, every 4-critical triangle-free graph $G$ drawn in the torus has a 4-face. A natural way of dealing with 4-faces is to identify opposite vertices of one of them, effectively removing it from the graph.

### 2.3.1 Collapsing 4-faces and Reduction Operation

Consider a graph $G$ embedded in a surface and a 4-face $f$ of $G$ bounded by the cycle $v_1v_2v_3v_4$. Let $G'$ be the graph drawn in the torus obtained from $G$ by identifying $v_2$ with $v_4$ and suppressing the parallel edges from the resulting vertex $v$ to $v_1$ and $v_3$. Informally, we obtain an embedding of $G'$ naturally by sliding the unified verticies to the same location within $f$.

Let $P$ be the path $v_1vv_3$ in $G'$. We say that $G'$ is obtained from $G$ by *collapsing $f$ to $P$*. Let us remark that there are two ways to collapse a 4-face, to path $v_1vv_3$ or $v_2vv_4$, we say that these collapses are performed in different directions.

**Observation 26.** *Let $G$ be a graph drawn in a surface, $f$ its 4-face and $G_1, G_2$ be graphs obtained from $G$ by collapsing $f$ in . Then the following hold:*

- *If $G$ is not 3-colorable, then neither $G_1$ nor $G_2$ is 3-colorable.*
- *If $G$ is 3-colorable, then at least one of $G_1, G_2$ is 3-colorable.*
- *Any coloring of $G_1$ and $G_2$ extends to a coloring of $G$.*

*Proof.* If we have a 3-coloring of $G_1$ or $G_2$, we may naturally modify it to 3-color $G$ as well, by coloring the two extra vertices in $G$ by the color of their unifications of $G_i$. This implies the first and the last point. On the other hand, if we have a 3-coloring of $G$, then it must assign the same color to at least one pair of opposite vertices of $f$. Collapsing $f$ in the direction that unifies these vertices produces a graph that is 3-colored by a natural modification of the coloring of $G$. This implies the latter point. □

This operation is of particular interest for the study of triangle-free 4-critical graphs, because it reduces size of the graph. In particular, when applied to a 4-critical graph $G$, the resulting graph $G'$ contains a (smaller) 4-critical subgraph $H$, although possibly not a triangle-free one. If $G'$ is also triangle-free, we say that $H$ is a *reduction* of $G$.

In [33], we proved the following technical lemma relating $H$ to a subgraph of $G$ (there are several cases depending on whether the edges $vv_1$ and $vv_3$ and the vertices $v_1$ and $v_3$ belong to $H$ or not), see Figure 2.3 for an illustration. The Lemma serves mainly as a prescription of how to inverse the reduction operation.

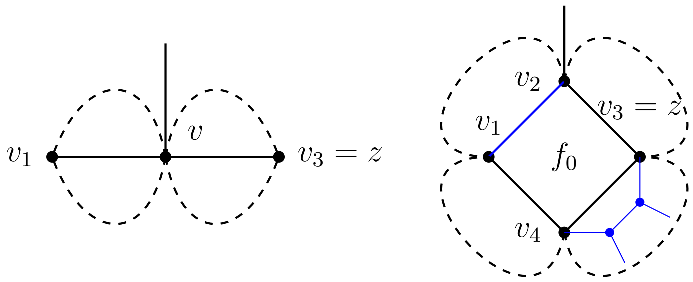
**Lemma 27** ( [33] Lemma 17)**.** *Let $G$ be a 4-critical triangle-free graph drawn in the torus and let $H$ be a 4-critical subgraph of a graph obtained from $G$ by collapsing a 4-face. If $H$ is triangle-free, then there exists*

- *a subgraph $G_1$ of $G$ whose drawing in the torus is 2-cell, and*
- *a path $v_2zv_4$ contained in the boundary of a face $f_0$ of $G_1$, such that $f_0$ is not a face of $G$,*

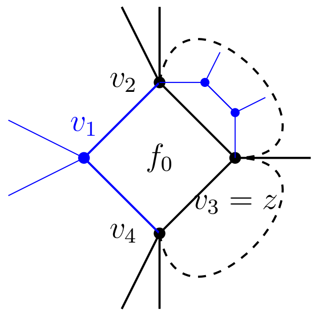
*such that one of the following claims holds.*

(i) *$H$ is obtained from $G_1$ by identifying $v_2$ with $v_4$ to a new vertex $v$ within $f_0$ and suppresing the resulting 2-face $vz$, or*

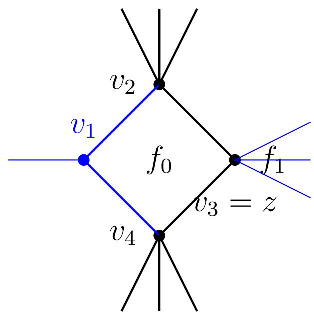
(ii) *$H$ is obtained from $G_1$ by identifying $v_2$ with $v_4$ to a new vertex $v$ within $f_0$ and deleting both resulting edges between $v$ and $z$, or*




(i) $vz \in E(H)$ for some $z \in \{v_1, v_3\}$




(ii) $vv_1, vv_3 \notin E(H)$, $z \in V(H)$ for some $z \in \{v_1, v_3\}$




(iii) $v_1, v_3 \notin V(H)$

Figure 2.3: A subgraph $G_1$ of $G$ (on the right side) corresponding to a reduction $H$ of $G$ (on the left side). Blue edges and vertices indicate parts of $G$ not belonging to $G_1$.

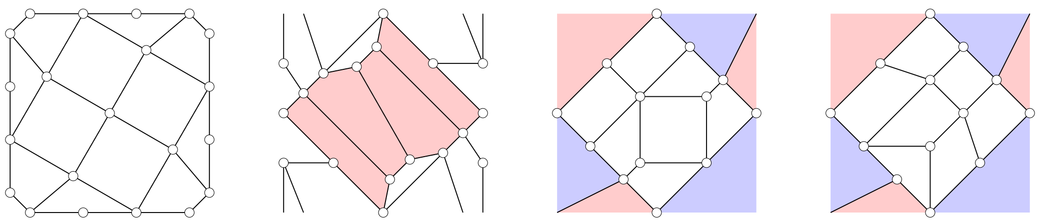
(iii) *$z$ has degree two and it is incident with two distinct faces $f_0$ and $f_1$ in $G_1$, $f_1$ is not a face of $G$, and $H$ is obtained from $G_1$ by contracting both edges incident with $z$.*

Note that no parallel edges except for those explicitly mentioned in the statement of Lemma 27 are created by the identification of $v_2$ with $v_4$ (in particular, when the collapsed 4-face of $G$ is bounded by a cycle $v_1v_2v_3v_4$ and $z = v_3$, at most one of the edges $v_1v_2$ and $v_1v_4$ belongs to $G_1$; this is also the reason why we can assume $f_0$ is not a face of $G$).

Lemma 27 is useful in conjunction with the basic properties of critical graphs, stated in Lemma 18. The graph $G$ can be obtained from its reduction $H$ by first finding its subgraph $G_1$ with properties described in Lemma 27, then filling some of the faces of $G_1$ by graphs drawn in a disk and critical with respect to the boundary cycle.

The reduction operation forms a natural hierarchy within the class of triangle-free 4-critical graphs and a natural framework to either reduce its graphs of any size down to small instances, and to generate the whole infinite class from a finite seed. We say that $G$ is *irreducible* if collapsing of any 4-face in any direction produces a triangle. In another paper, we have identified all irreducible graphs.

**Theorem 28** (Dvořák and Pekárek [33])**.** *There are only four non-homeomorphic irreducible graphs drawn in the torus: $I_4, I_5, I_7^a, I_7^b$, as depicted in Figure 2.4.*



(a) $I_4$ (b) $I_5$ (c) $I_7^a$ (d) $I_7^b$

Figure 2.4: Irreducible 4-critical graphs drawn on torus. Red and blue faces are 5- and 7-faces, respectively.

As an example why this is useful, using the census information about critical graphs in disks from Lemma 18, one can inductively prove Theorem 19 by considering how census may change when the reduction is reversed. See [33] for details.

For the purposes of this paper, we need to describe the inverse of the reduction operation in more detail, in the setting of templates. For this purpose, we describe several operations describing the necessary steps in a more abstract setting.

A *template* $T$ consists of a graph $G_T$ 2-cell embedded in the torus and a function $\theta_T$ assigning to each face of $G_T$ a multiset of integers greater or equal to five such that $\sum \theta_T(f) \equiv |f| \pmod 2$ for every face $f \in F(G_T)$. We say a graph $H$ 2-cell embedded in the torus *is represented* by the template $T$ if there exists a homeomorphism $\kappa$ of the torus mapping $G_T$ to a subgraph of $H$, such that for each face $f$ of $G_T$, $\theta_T(f)$ is equal to the census of the set of faces of $H$ drawn in $\kappa(f)$.

We say a face $f \in F(G_T)$ is *proper* if either $|f| = 4$ and $\theta_T(f) = \emptyset$, or $\theta_T(f) = \{|f|\}$. By Lemma 18 we see that that if $G$ is a triangle-free 4-critical graph represented by template $T$ via a homeomorphism $\kappa$ and $f$ is a face of $G_T$ that is proper, then $\kappa(f)$ is a face of $G$. On the other hand, each non-proper face $f$ of $G_T$ bounded by the facial walk $C$ corresponds to a $\kappa(C)$-critical subgraph of $G$ of census $\theta_T(f)$.

Let $T$ be a template such that every face $f$ with $\theta_T(f) \neq \emptyset$ is proper, we say that $T$ is *direct*. Note that direct template reflects exact positions of all ($\geq 5$)-faces of any represented graph, and allows the represented graphs to differ only in quadrangulations of some areas of the torus.

A template $T$ is *relevant* if

- the graph $G_T$ is triangle-free,
- $\bigcup_{f\in F(G_T)} \theta_T(f) \in \{\emptyset, \{5,5\}, \{5,5,5,5\}, \{5,5,6\}, \{5,7\}\}$, and
- for every face $f$, $\theta_T(f) \in \mathcal{S}_{4,|f|} \cup \{\{|f|\}\}$ if $|f| \neq 4$ and $\theta_T(f) = \emptyset$ if $|f| = 4$.

By Theorem 19, if a triangle-free 4-critical graph is represented by a template $T$, then $T$ is relevant.

### 2.3.2 3-colorability of Templates

Consider a graph $G$ with a proper coloring $\varphi$ by the elements of $\mathbb{Z}_3$, and let $\omega_\varphi : E(G) \to \{-1, +1\}$ be the tension of $\varphi$. Let $W$ be a closed walk, we use the notation $\omega_\varphi(W) = \int_W t_\varphi$. For a 2-cell face $f$, if $W$ is the closed walk bounding $f$ in the clockwise direction, then we define $\omega_\varphi(f) = \int_W \omega_\varphi$.

Note that if $W$ is a cycle, then $\varphi$ can be extended in a natural way to a continuous mapping from the drawing of the cycle to a triangle in the plane, representing the three colors of $\varphi$, and in this representation $\omega_\varphi(W)$ is equal to three times the winding number of the corresponding closed curve around the interior of the triangle; thus, we will call $\omega_\varphi(W)$ the *winding number* of $\varphi$ on $W$ (ignoring the factor of three for convenience).

Recall the following properties of tension functions:

**Observation 29.** *Let $G$ be a graph 2-cell embedded in an orientable surface, let $\varphi$ be a 3-coloring of $G$, let $H$ be a subgraph of $G$, $W$ a clockwise facial walk of a face $f$ of $H$, and $m = |W|$. Then*

- $\omega_\varphi(f) = \sum_{g\in F(H), g\subseteq f} \omega_\varphi(g)$
- $3|\omega_\varphi(W)$, $\omega_\varphi(W) \equiv m \pmod 2$, *and* $|\omega_\varphi(W)| \leq m$
- *In particular, if $W$ has length 4, then $\omega_\varphi(W) = 0$.*
- $\sum_{f\in F(G)} \omega_\varphi(f) = 0$

We need the following important result. For a set $F$ of faces and an integer $k$, a *winding number assignment summing to $k$* is a function $n : F \to \mathbb{Z}$ such that $\sum_{f\in F} n(f) = k$ and for every $f \in F$, $n(f)$ is divisible by 3, $n(f)$ has the same parity as $|f|$, and $|n(f)| \leq |f|$. This notion of a winding number of a coloring around a face is essentially dual to the notion of a candidate demand function.

**Lemma 30** ([52, a reformulation of Lemma 5])**.** *Let $G$ be a graph with a 2-cell drawing in an orientable surface, let $H$ be a subgraph of $G$, and let $f$ be a 2-cell face of $H$. Let $\varphi$ be a 3-coloring of $H$ and let $F$ be the set of faces of $G$ contained in $f$. The coloring $\varphi$ does not extend to a 3-coloring of the subgraph of $G$ drawn in the closure of $f$ if and only if for every winding number assignment $n$ for $F$ summing to $\omega_\varphi(f)$, either*

(i) *there exists a path $P$ in $G$ drawn in $f$ and intersecting the boundary of $f$ exactly in its endpoints, and denoting by $Q$ a part of the clockwise boundary walk of $f$ between the endpoints of $P$ and by $F' \subseteq F$ the set of faces contained in the part of $f$ bounded by $Q + P$, we have*

$$\left|\omega_\varphi(Q) - \sum_{f \in F'} n(f)\right| > |E(P)|;$$

*or,*

(ii) *there exists a cycle $C$ in $G$ drawn in $f$ and intersecting the boundary of $f$ in at most one point, and denoting by $F' \subseteq F$ the set of faces contained in the part of $f$ bounded by $C$, we have we have*

$$\left|\sum_{f \in F'} n(f)\right| > |C|.$$

For a multiset $I$ of positive integers, let us analogously define a *winding number assignment summing to $k$* as a function $n : I \to \mathbb{Z}$ such that $\sum_{i \in I} n(i) = k$ and for every $i \in I$, $n(i)$ is divisible by 3, $n(i)$ has the same parity as $i$, and $|n(i)| \leq i$. Let $\omega(I)$ be the set of all integers $k$ such that there exists a winding number assignment for $I$ summing to $k$. A *proper 3-coloring* of a template $T$ is a proper 3-coloring $\varphi$ of $G_T$ such that $\omega_\varphi(f) \in \omega(\theta_T(f))$ for every face $f \in F(G_T)$. A justification for this definition is the following fact.

**Lemma 31.** *Let $T$ be a template and let $\varphi$ be a proper 3-coloring of $G_T$. Then $\varphi$ is a proper 3-coloring of $T$ if and only if there exists a graph $G$ represented by $T$ (via a homeomorphism $\kappa$) such that $\varphi \circ \kappa^{-1}$ extends to a proper 3-coloring of $G$.*

*Proof.* Suppose that there exists a graph $G$ represented by $T$ with a 3-coloring $\psi$ extending $\varphi \circ \kappa^{-1}$, i.e., $\varphi(v) = \psi(\kappa(v))$ for every $v \in V(G_T)$. Let $s = 1$ if $\kappa$ preserves orientation and $s = -1$ if $\kappa$ reverses the orientation. For any face $f \in F(G_T)$, let $W_f$ denote the closed walk of $G$ bounding $\kappa(f)$ in the clockwise direction and let $F_f(G)$ denote the set of faces of $G$ contained in $\kappa(f)$. By Observation 29, we have

$$\omega_\varphi(f) = s \cdot \omega_\psi(\kappa(W_f)) = s \cdot \sum_{g \in F_f(G)} \omega_\psi(g).$$

Since $\theta_T(f)$ is the census of $F_f(G)$, Observation 29 implies $\omega_\varphi(f) \in \omega(\theta_T(f))$. Hence, $\varphi$ is a proper 3-coloring of $T$.

The converse is proved by filling in the faces of $G_T$ by a suitably generic subgraphs (avoiding short paths between points on the boundary and short separating cycles) with faces of appropriate census, so that Lemma 30 implies $\varphi$ extends to a 3-coloring of these subgraphs. As we will not need this implication, the details are left to the reader. $\square$

We say that a template $T$ is 3-*colorable* if it has a proper 3-coloring. Note that not all realizations of a 3-colorable template are 3-colorable, but the converse is true, as is easy to see from Lemma 31.

**Corollary 32.** *If $G$ is represented by a template $T$ and $T$ is not 3-colorable, then $G$ is not 3-colorable.*

Testing whether a template is 3-colorable is of course NP-hard, but for reasonably small templates, a brute-force algorithm enumerating all proper 3-colorings of $G_T$ and testing the winding number conditions for each is fast enough for our purposes.

Throughout the rest of this section, let $T$ be a template.

### 2.3.3 Operations on Templates

#### Hiding, Revealing, Subtemplates

Let $e$ be an edge of $G_T$, where either $e$ is incident with two distinct faces or with a vertex $v$ of degree one. Then $T \diamond e$ denotes a template obtained from $T$ as follows: in the former case $G_{T\diamond e} = G_T - e$, in the latter case $G_{T\diamond e} = G_T - v$. Let $f_e$ denote the face of $G_T - e$ in which $e$ used to be drawn, and let $X$ be the set of (at most two) faces of $G_T$ incident with $e$. The function $\theta_{T\diamond e}$ matches $\theta_T$ on $F(G_T) \setminus X$, and $\theta_{T\diamond e}(f_e) = \bigcup \theta_T(X)$. We say that $T \diamond e$ is obtained from $T$ by *hiding* the edge $e$. Conversely, we say that $T$ is obtained from $T \diamond e$ by *revealing the edge $e$*. Note that revealing an edge may add new vertex of degree one into the template.

A template $T'$ is a *subtemplate* of $T$ if a template homeomorphic to $T'$ is obtained from $T$ by repeatedly hiding edges. Equivalently, there exists a homeomorphism $\kappa$ of the torus mapping $G_{T'}$ to a subgraph of $G_T$, such that for each face $f'$ of $G_{T'}$, we have

$$\theta_{T'}(f') = \bigcup_{f \in F(G_T), f \subseteq \kappa(f')} \theta_T(f).$$

Note that if a graph is represented by $T$, then it is also represented by $T'$.

#### Splitting

Let $I$ be a multiset of integers. We say a multiset $A$ is obtained from $I$ by *splitting* if $A$ is obtained from $I$ by

- removing an element of value 6, or
- replacing an element of value $i \geq 7$ by an element of value $i - 2$, or
- replacing an element of value $i \geq 8$ by two elements of values $i_1$ and $i_2$ such that $i_1, i_2 \geq 5$ and $i_1 + i_2 = i + 2$.

For a face $f \in F(G_T)$ with $\theta_T(f) \neq \emptyset$ and $\max \theta_T(f) \geq 6$, we say a template $T'$ is obtained from $T$ by *splitting inside* $f$ if $G_{T'} = G_T$, $\theta_{T'}(g) = \theta_T(g)$ for $g \in F(G_T) \setminus \{f\}$, and $\theta_{T'}(f)$ is obtained from $\theta_T(f)$ by splitting. Note that if $H$ is a triangle-free graph represented by $T$ via a homeomorphism $\kappa$ and we add a chord to a face $h$ of $H$ (splitting it into two faces) so that the resulting graph $H'$ is triangle-free, then $H'$ is represented by a template obtained from $T$ by splitting inside $f$, where $f$ is the face of $G_T$ such that $h \subseteq \kappa(f)$.

### Filling

We say a multiset $A$ is a *filling* of a multiset $I$ if $A$ is obtained from $I$ by replacing each element $i \in I$ by the elements of a multiset belonging to $\{\{i\}\} \cup \mathcal{S}_{4,i}$. We say that a template $T'$ is obtained from $T$ by *filling* if $G_{T'} = G_T$ and $\theta_{T'}(f)$ is a filling of $\theta_T(f)$ for $f \in F(G_T)$. By the definition of the sets $\mathcal{S}_{4,i}$, if $H$ is a graph represented by $T$ and $H'$ is a 4-critical triangle-free supergraph of $H$ in the torus, then $H'$ is represented by a filling of $T$.

### Boosting

We say a multiset $A$ is obtained from a multiset $I$ by *boosting* if either $A = I$ or $A$ is obtained from $I$ by replacing an element of value $i$ by the elements of some multiset from $\mathcal{S}_{4,i+2}$.

This operation interprets mainly the case of (iii) of Lemma 27. In particular, reverting a collapse of a 4-face can increase a length of at most two faces by 2. When these faces are represented as faces of a template, but abstracted via the $\theta$ function, the corresponding values are boosted in the reverse reduction.

### Partial Amplification

Note that a vertex $v$ of a graph $G$ can appear in the boundary of a face $f$ several times. The following definition is used to indicate a particular incidence of $f$ with $v$. We fix an open neighborhood $\delta$ of $v$ small enough so that no other vertex appears in $\delta$, and for each edge $e$ intersecting $\delta$, $e$ is incident with $v$ and $e \cap \delta$ is an initial segment of $e$ starting in $v$. An *angle* of $f$ at $v$ is an arcwise-connected subset $a$ of $\delta$ after removing the drawing of $G$ such that $a \subset f$.

A *partial amplification* of $T$ is a template obtained in one of the following ways:

(i-a) For some face $f \in F(G_T)$, we change $\theta_T(f)$ to a multiset obtained from it by boosting.

(i-b) For a vertex $v$ of $G_T$, we first either choose an edge $e$ incident with $v$, or reveal an edge $e$ incident with $v$. Then we choose an angle $a$ of a face $f$ at $v$, add an edge $e'$ parallel to $e$ so that $e$ and $e'$ bound a 2-face $f'$, and split the vertex $v$ into two vertices so that $f'$ merges with the angle $a$. Finally, we change $\theta_T(f)$ to a multiset obtained from it by boosting.

(ii-a) For some face $f \in F(G_T)$, we split a face inside $f$ and then change $\theta_T(f)$ to a multiset obtained from it by boosting.

(ii-b) For a vertex $v$ of $G_T$ and incident face $f$, we split a face inside $f$, then reveal an edge $e$ incident with $v$ and drawn in $f$. Then we choose an angle $a$ of a face $f'$ at $v$, add an edge $e'$ parallel to $e$ so that $e$ and $e'$ bound a 2-face $f''$, and split the vertex $v$ into two vertices so that $f''$ merges with the angle $a$. Finally, we change $\theta_T(f')$ to a multiset obtained from it by boosting.

(iii-a) For some face $f \in F(G_T)$, we change $\theta_T(f)$ to a multiset obtained from it by boosting twice.

(iii-b) For a vertex $v$ of $G_T$ and distinct angles $a_1$ and $a_2$ of (not necessarily distinct) faces $g_1$ and $g_2$ at $v$, we split $v$ into two vertices $v_2$ and $v_4$ and add a new vertex $z$ and a path $v_2zv_4$ so that the angles $a_1$ and $a_2$ now extend along this path. Then we change $\theta_T(g_1)$ and $\theta_T(g_2)$ to multisets obtained from them by boosting (boosting twice when $g_1 = g_2$).

Comparing this definition with Lemma 27 (the "a" cases corresponding to the situation where the vertex $v$ discussed in the Lemma does not belong to $G_T$) and using the interpretations of the operations of revealing an edge, splitting and boosting we introduced in this section, we conclude the following lemma holds.

**Lemma 33.** *Let $G$ be a 4-critical triangle-free graph drawn in the torus and let $H$ be a 4-critical subgraph of a graph obtained from $G$ by collapsing a 4-face; suppose $H$ is triangle-free and let $G_1 \subseteq G$, $f_0$ and possibly $f_1$ be as described in Lemma 27. Let $G_2$ be the subgraph of $G$ obtained from $G_1$ by, for $i \in \{0, 1\}$, adding the vertices and edges of $G$ drawn in $f_i$. If $H$ is represented by a template $T$, then $G_2$ is represented by a partial amplification of $T$.*

*Proof.* We give the argument in the case (ii) from the statement of Lemma 27, the arguments in the remaining two cases are similar.

Let $H'$ be the graph obtained from $G_1$ by identifying $v_2$ with $v_4$ to a new vertex $v$ within $f_0$ and let $g$ be its 2-face bounded by the edges from $v$ to $z$. Let $g_0$ be the face of $H'$ corresponding to $f_0$ and let $a_0$ be the angle of $g_0$ such that $g$ merges with $a_0$ when we split $v$ back to $v_2$ and $v_4$.

Without loss of generality, assume the homeomorphism showing that $H$ is represented by $T$ is the identity. Let $f$ be the face of $H$ in which the edges between $v$ and $z$ are drawn in $H'$. A template $T'$ representing $H'$ is obtained from $T$ by splitting inside $f$ and in case that $v \in V(G_T)$, additionally revealing the edges $e$ and $e'$ between $v$ and $z$.

If $v \in V(G_T)$, then let $f'$ be the face of $G_{T'}$ and $a$ its angle containing the angle $a_0$. As described in more detail after the definition of the boosting operation, a template representing $G_2$ is obtained from $T'$ by splitting the vertex $v$ into two vertices $v_2$ and $v_4$ so that the 2-face bounded by $e$ and $e'$ merges with the angle $a$, and changing $\theta_{T'}(f')$ to a multiset obtained from it by boosting. This matches the case (ii-b) from the definition of partial amplification. If $v \notin V(G_T)$, and thus $v$ is drawn inside $f$, we achieve the same effect just by boosting the multiset $\theta_{T'}(f')$, matching the case (ii-a). □

## Amplification

An *amplification* of $T$ is a filling of a partial amplification of $T$. Since the graph $G_2$ in Lemma 33 is a subgraph of $G$, a template representing $G$ is obtained from one representing $G_2$ by filling. An example of the amplification operation is given on the right in Figure 2.5.

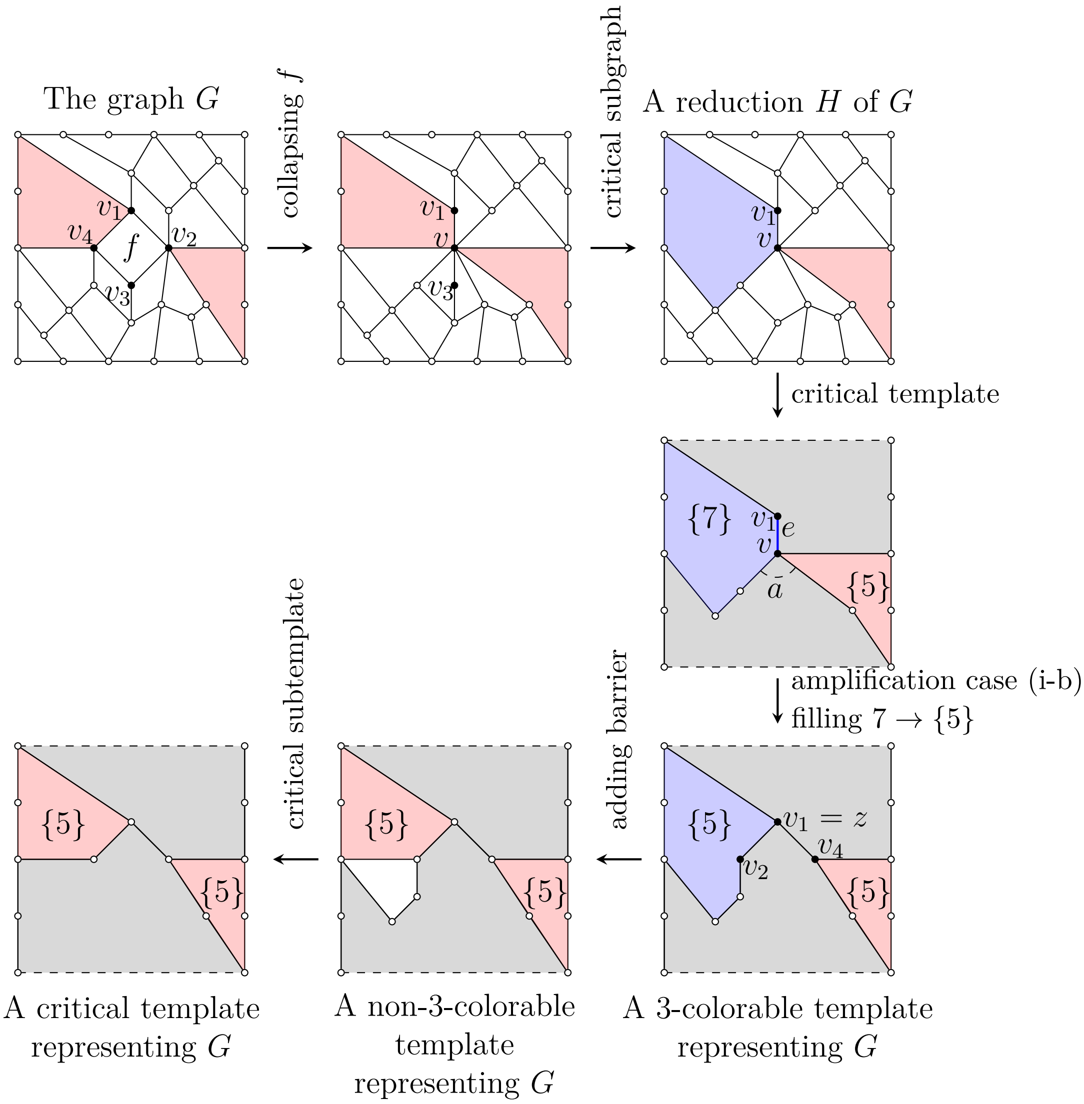



Figure 2.5: The process of deriving a critical template for a 4-critical triangle-free graph $G$ from a template for its reduction $H$.

**Corollary 34.** *Let $G$ be a 4-critical triangle-free graph drawn in the torus and let $H$ be a 4-critical subgraph of a graph obtained from $G$ by collapsing a 4-face. If $H$ is triangle-free and $H$ is represented by a template $T$, then $G$ is represented by an amplification of $T$.*

However, note that even if $T$ is not 3-colorable, some of its amplifications can be 3-colorable. Next, we deal with this issue.

### 2.3.4 Making a Template Non-3-colorable

The following definitions are motivated by Lemma 30(i), essentially restating what an obstruction to extendability of a 3-coloring may look like from the point of view of templates. Consider a template $T$ and let $f$ be a face of $G_T$. A *strut* of $f$ is a directed path $P$ whose endpoints $s$ and $t$ are vertices of $G_T$ in the boundary of $f$, and the rest of $P$ is drawn inside $f$. For a strut $P$, let $R(P)$ denote the subwalk of the clockwise boundary walk of $f$ starting in $t$ and ending in $s$, and let $r(P)$ denote the part of $f$ bounded by the cycle formed by the concatenation of $P$ and $R(P)$. Consider a proper 3-coloring $\varphi$ of $T$ and a winding number assignment $n$ for $\theta_T(f)$ summing to $\omega_T(\varphi)$. An *$f$-barrier* for $(\varphi, n)$ is a pair $(P, I)$, where $P$ is a strut of $f$ and $I$ is a multisubset of $\theta_T(f)$ such that

$$\left|\omega_\varphi(R(P)) - \sum_{i\in I} n(i)\right| > |E(P)|.$$

A set $B$ of $f$-barriers is *drawing-consistent* if the intersection of the drawings of any two of the struts is a union of vertices and edges; in such a case, let $\bigcup B$ denote the graph consisting of the union of the struts. We say that $B$ *blocks* $\varphi$ if for every winding number assignment $n$ for $\theta_T(f)$ summing to $\omega_T(\varphi)$, $B$ contains an $f$-barrier for $(\varphi, n)$.

Given a system $\mathcal{B} = \{B_f : f \in F(G_T)\}$, where $B_f$ is a drawing-consistent set of $f$-barriers for each face $f \in F(G_T)$, a *realization* of $\mathcal{B}$ is a template $T'$ such that

$$G_{T'} = G_T \cup \bigcup_{f\in F(G_T)} \bigcup B_f,$$

$T$ is a subtemplate of $T'$, i.e., every face $f \in F(G_T)$ satisfies

$$\theta_T(f) = \bigcup_{h\in F(G'_T), h\subseteq f} \theta_{T'}(h),$$

and for every $f \in F(G_T)$ and $(P, I) \in B_f$,

$$I = \bigcup_{h\in F(G_{T'}), h\subseteq r(P)} \theta_{T'}(h).$$

The last condition expresses that in the realization, the values from $I$ are exactly those assigned to the faces of the realization contained in $r(P)$. For this reason, a system $\mathcal{B}$ does not necessarily have a realization even when it is drawing-consistent, since it may not be possible to choose $\theta_{T'}$ so that the last condition holds. On the other hand, it may also be possible to choose $\theta_{T'}$ (and thus a realization) in several different ways. If $\mathcal{B}$ has a realization, we say that it is *consistent.* For a proper 3-coloring $\varphi$ of $T$, we say that $\mathcal{B}$ *blocks* $\varphi$ if there exists $f \in F(G_T)$ such that $B_f$ blocks $\varphi$. The following claim is essentially clear from the definitions and Lemma 30.

**Lemma 35.** *Let $T$ be a relevant template and let $\mathcal{B} = \{B_f : f \in F(G_T)\}$ be a consistent system of sets of barriers. Let $T'$ be a realization of $\mathcal{B}$. If $\mathcal{B}$ blocks every proper 3-coloring of $T$, then $T'$ is not 3-colorable.*

*Proof.* Suppose for a contradiction $T'$ has a proper 3-coloring $\varphi'$, and let $\varphi$ be the restriction of $\varphi'$ to $V(G_T)$. Consider any face $f$ of $G_T$, and let $F_f = \{g \in F(G_{T'}) : g \subseteq f\}$. Let $n_f : \bigcup_{g \in F_f} \theta_{T'}(g) \to \mathbb{Z}$ be a function whose restriction to $\theta_{T'}(g)$ is a winding number assignment for $\theta_{T'}(g)$ summing to $\omega_{\varphi'}(g)$ for every $g \in F_f$; such a function $n_f$ exists, since $\varphi'$ is a proper 3-coloring of $T'$.

Since $T'$ is a realization of $T$, Observation 29 implies

$$\begin{aligned}\omega_\varphi(f) = \omega_{\varphi'}(f) &= \sum_{g \in F_f} \omega_{\varphi'}(g)\\ &= \sum_{g \in F_f} \sum_{i \in \theta_{T'}(g)} n_f(i) = \sum_{i \in \theta_T(f)} n_f(i),\end{aligned}$$

and thus $n_f$ is a winding number assignment for $\theta_T(f)$ summing to $\omega_\varphi(f)$. Since this holds for every $f \in F(G_T)$, we conclude that $\varphi$ is a proper 3-coloring of $T$.

Therefore, $\mathcal{B}$ blocks $\varphi$, and thus for some $f \in F(G_T)$, $B_f$ contains an $f$-barrier $(P, I)$ for $(\varphi, n_f)$. Since $\varphi'$ is a proper 3-coloring of $T'$ and by Observation 29, we have

$$\begin{aligned}\omega_{\varphi'}(P + R(P)) &= \sum_{g \in F(G_{T'}), g \subseteq r(P)} \omega_{\varphi'}(g)\\ &= \sum_{g \in F(G_{T'}), g \subseteq r(P)} \sum_{i \in \theta_{T'}(g)} n_f(i)\\ &= \sum_{i \in I} n_f(i)\end{aligned}$$

Since $|\omega_{\varphi'}(u,v)| \le 1$ for any adjacent $u, v \in V(G_{T'})$, we have $|\omega_{\varphi'}(P)| \le |E(P)|$.

Consequently

$$\begin{aligned}\left|\omega_\varphi(R(P)) - \sum_{i \in I} n(i)\right| &= \left|\omega_{\varphi'}(R(P)) - \sum_{i \in I} n(i)\right|\\ &\le \left|\omega_{\varphi'}(P + R(P)) - \sum_{i \in I} n(i)\right| + |E(P)| = |E(P)|,\end{aligned}$$

which is a contradiction, since $(P, I)$ is an $f$-barrier for $(\varphi, n_f)$. $\square$

More interestingly, a converse holds as well.

**Lemma 36.** *Let $T$ be a relevant template. If a 4-critical triangle-free graph $G$ is represented by $T$, then there exists a consistent system $\mathcal{B}$ of sets of barriers which blocks every proper 3-coloring of $T$ such that $G$ is represented by a realization of $\mathcal{B}$.*

*Proof.* Without loss of generality, we can assume $T$ represents $G$ via the identity homeomorphism, and thus $G_T \subseteq G$. For a face $f \in F(G_T)$, let $F_f$ denote the set of faces of $G$ contained in $f$, and let us fix a bijection $\gamma_f$ mapping each face $g \in F_f$ of length at least 5 to an element of $\theta_T(f)$ of value $|g|$.

Consider a proper 3-coloring $\varphi$ of $G_T$. Since $G$ is not 3-colorable, there exists $f \in F(G_T)$ such that $\varphi$ does not extend to a 3-coloring of the subgraph of $G$

drawn in $f$. Consider any winding number assignment $n$ for $\theta_T(f)$ summing to $\omega_\varphi(f)$. For $g \in F_f$, let $n'(g) = n(\gamma_f(g))$ if $|g| > 4$ and $n'(g) = 0$ if $|g| = 4$; then $n'$ is a winding number assignment for $F_f$ summing to $\omega_\varphi(f)$. We now apply Lemma 30 to $\varphi$ and $n'$.

Suppose first that (ii) holds; let $C$ and $F'$ be as in the statement, and let $I = \gamma_f(F')$, so that $\left|\sum_{i\in I} n(i)\right| > |C|$. Since $G$ is 4-critical and triangle-free, $I \in \mathcal{S}_{4,|C|}$. If $I = \{i\}$, then since $n$ is a winding number assignment, we have $|C| = |n(i)| \leq i = \max I$, contradicting Lemma 18. Consequently $|I| \geq 2$, and thus Lemma 18 implies $|C| \geq 8$. Since $|n(5)| \leq 3$ and $\left|\sum_{i\in I} n(i)\right| > |C| \geq 8$, we have $I \neq \{5, 5\}$, and thus Lemma 18 implies $|C| \geq 9$. Since $|n(6)| \leq 6$ and $|n(7)| \leq 3$, the same argument now gives $I \neq \{5, 6\}$, $I \neq \{5, 7\}$, and $I \neq \{5, 5, 5\}$. Let $J = \bigcup_{h\in F(G_T)} \theta_T(h)$; since $T$ is relevant and $I \subseteq J$, we conclude that $J$ is either $\{5, 5, 5, 5\}$ or $\{5, 5, 6\}$ and $I = J$. Therefore,

$$\sum_{i\in I} n(i) = \sum_{h\in F(G_T)} \sum_{i\in\theta_T(h)} n(i) = \sum_{h\in F(G_T)} \omega_\varphi(h) = 0$$

by Observation 29. Therefore, $\left|\sum_{i\in I} n(i)\right| = 0 < |C|$, which is a contradiction.

Therefore, (i) holds; let $P$ be as in the statement (with $F' = \{g \in F(G) : g \subseteq r(P)\}$) and let $I = \gamma_f(F')$. Then $(P, I)$ an $f$-barrier for $(\varphi, n)$. Collecting all such barriers for all proper 3-colorings $\varphi$ of $T$ and for all choices of $n$ gives us a consistent system $\mathcal{B}$ of sets of barriers which blocks every proper 3-coloring of $T$, with a realization representing $G$. □

A system $\mathcal{B}$ of sets of barriers is *$T$-minimal* if $\mathcal{B}$ blocks all proper 3-colorings of $T$, but removing any barrier from any of the sets results in a sytem that no longer blocks all proper 3-colorings of $T$. Note that a $T$-minimal system has bounded size, and thus there are (up to homeomorphism) only finitely many $T$-minimal systems and their realizations. Let us remark that a realization of a $T$-minimal set of barriers may still contain a proper non-3-colorable subtemplate.

A template $T'$ is *critical* if $T'$ is not 3-colorable, but all proper subtemplates of $T'$ are 3-colorable. Observe that if a template is 3-colorable, then all its subtemplates are also 3-colorable. Hence, a non-3-colorable template $T''$ has a critical subtemplate, which can be obtained from $T''$ by repeatedly hiding edges whose removal does not cause the template to become 3-colorable.

We can now combine all the results. We say that a template $T_3$ is *grown from a template* $T$ if there exists an amplification $T_1$ of $T$, a $T_1$-minimal consistent system $\mathcal{B}$ of sets of barriers, and a realization $T_2$ of $\mathcal{B}$ such that $T_3$ is a relevant critical subtemplate of $T_2$. See the bottom part of Figure 2.5 for an illustration.

**Theorem 37.** *Let $G$ be a 4-critical triangle-free graph drawn in the torus and let $H$ be a 4-critical subgraph of a graph obtained from $G$ by collapsing a 4-face. If $H$ is triangle-free and $H$ is represented by a template $T$, then $G$ is represented by a template grown from $T$.*

*Proof.* By Corollary 34, we can choose an amplification $T_1$ of $T$ representing $G$. Since $G$ is 4-critical and triangle-free, Theorem 19 implies that the template $T_1$ is relevant. By Lemma 36, there exists a $T_1$-minimal consistent system $\mathcal{B}$ of sets of barriers such that $G$ is represented by a realization $T_2$ of $\mathcal{B}$. By Lemma 35, $T_2$ is not 3-colorable, and thus it has a critical subtemplate $T_3$, which also represents $G$. □

### 2.3.5 A Complete Description of 4-critical Δ-free Graphs Embedded in the Torus

For a set $\mathcal{T}$ of templates and a graph $G$, $G$ is *represented* by $\mathcal{T}$ if there exists $T \in \mathcal{T}$ such that $G$ is represented by $T$. A set $\mathcal{T}$ of templates $\mathcal{T}$ is *closed under growing* if for every $T \in \mathcal{T}$, all templates grown from $T$ belong to $\mathcal{T}$. A set $\mathcal{T}$ of templates is *total* if $\mathcal{T}$ is a set of relevant critical templates such that the graphs $I_4$, $I_5$, $I_7^a$, and $I_7^b$ are represented by $\mathcal{T}$ and $\mathcal{T}$ is closed under growing.

**Corollary 38.** *If $\mathcal{T}$ is a total set of templates, then every 4-critical triangle-free graph drawn in the torus is represented by $\mathcal{T}$, and no graph represented by $\mathcal{T}$ is 3-colorable.*

*Proof.* For the first claim, the proof is illustrated in Figure 2.5. Suppose for a contradiction $G$ is a 4-critical triangle-free graph drawn in the torus and not represented by $\mathcal{T}$ with the smallest number of vertices. Since all irreducible graphs are represented by $\mathcal{T}$, $G$ contains a 4-face that can be collapsed without creating a triangle; let $H$ be the corresponding reduction of $G$. Since $H$ is 4-critical, triangle-free, and $|V(H)| < |V(G)|$, the minimality of $G$ implies $H$ is represented by a template $T \in \mathcal{T}$. By Theorem 37, $G$ is represented by a template grown from $T$. This is a contradiction, since $\mathcal{T}$ is closed under growing.

The second claim follows from Corollary 32, since all templates in $\mathcal{T}$ are critical, and thus not 3-colorable. □

It remains to show that a finite total set of templates exists. This claim is not at all evident. While it is clear that an infinite set representing all 4-critical graphs exists, a finiteness may be doubted. As motivated earlier, there are plenty of informal reasons to assert the existence of a finite set with confidence. All the represented graphs have limited census, and in the spirit of bariers, such graphs are colorable once there are no tight obstructions separating their long faces. On the other hand, quadrangulations far away from these faces become structurally irrelevant.

We do not give a firm theoretical basis to confirm the existence of a finite total set, but we explicitly constructed the set $\mathcal{T}$ using a computer search. For more details and the total set, see [32]. In fact, we give the following stronger result.

**Claim 39.** *There exists a total set of direct templates of size* 186.

The fact that we can get away with only having direct templates came as a bit of a surprise to us. Note that the operations of filling, boosting, or hiding edges may (and typically do) turn a direct template into a non-direct one. It is fortuitous that adding struts when turning a template into a non-3-colorable one counteracts these effects. On an intuitive side, this gives more evidence to how crucial the close surroundings of each ($\geq 5$)-face is.

By studying the total set, we may also observe further properties. For instance, if we focus on the subgraphs of the template graphs induced by the edges of their long faces, the subgraphs are always connected and non-planar. We observe that all templates have edge-width at most four, with a single exception. As a consequence, we get the following.

**Lemma 40** (computer assisted)**.** *Let $G$ be a triangle-free graph drawn in the torus.*

- *If $G$ has edge-width at least six, then $G$ is 3-colorable.*
- *If $G$ has edge-width at most five, then $G$ is 3-colorable if and only if it does not contain $I_4$ as a subgraph, in which case it has edge-width exactly five.*

### 2.3.6 Algorithmic Application Remarks

From Claim 39, we have a way to test whether a triangle-free toroidal graph $G$ is 3-colorable, by checking whether it contains a subgraph represented by an element $T \in \mathcal{T}$. Performing this test efficiently is not entirely trivial, though.

We may use a standard result:

**Lemma 41** (Eppstein [57],[58])**.** *Let $H$ be a fixed graph and $\Sigma$ a surface. There exists an algorithm which for an input graph $G$ embedded in $\Sigma$ tests whether $H \subseteq G$ in time $\mathcal{O}(||G||)$.*

We give a very brief sketch of the proof. One of the standard approaches to solving a generally hard problem in restricted graph classes is to use limited tree-width together with dynamic programming approach. Given fixed $H$, the relation $H \subseteq G$ can be expressed as a logical formula, which is testable in linear time as long as the tree-width of $G$ is limited, by Theorem of Courceille [59]. A result of Eppstein [57] shows that a tree-width of a limited-distance neighborhood of any vertex $v$ in a graph $G$ embedded in the fixed surface $\Sigma$ has limited tree-width. We can clearly combine these two results to obtain a quadratic algorithm, first guessing mapping of one vertex from $H$ to $G$, and then testing whether $H$ is a subgraph of a limited neighborhood of the mapped vertex. By a more sophisticated approach, similar construction can be made to work in linear time [58].

Of course, we are still faced with the issue that for a template $T = (G_T, \theta_T)$, it is not enough to test whether $G_T \subseteq G$. We may use the same method as Lemma 46 in 2.4 to remove short separating cycles from $G$ or conclude that $G$ cannot be a supergraph of a 4-critical graph according to its census. Then, if $G_T \subseteq G$ we use the fact that $T$ is direct, implying that we only need to test whether some faces of $G_T$ map to quarangulated subgraphs of $G$. If not, then there are more ($\geq 5$)-faces in $G$ then allowed by $T$ and therefore $G$ is not represented by $T$, even if a different mapping of $G_T$ into $G$ exists.

We may also use a more basic approach, avoiding the enormous constants involved in the machinery for the subgraph test. First, we eliminate all contractible separating cycles. Then we guess mapping of long faces of $G_T$ to $G$. As mentioned earlier, the long faces of $G_T$ form a connected subgraph, and so up to a choice of mapping of one edge (in a directed sense), the mapping is unique. It then remains to find mapping of the remaining vertices in $G_T$. There are typically only a few vertices and only rarely some of them do not neighbor the long faces directly. In a typical case this boils down to testing whether there exists a path across of length at most 3 connecting some pair of already mapped vertices. With sophisticated effort, this approach could also be made to finish the test in linear time. However, there turns out to be a simpler solution, which we present in Section 2.4.

So far, we discussed testing whether a graph s represented by a template, essentially testing whether it is 3-colorable. In theory, it is possible to use templates to obtain a 3-coloring as well, using methods similar to those in Section 2.4, basically repeatedly reducing $G$ via collapses of 4-faces to smaller graph while maintaining 3-colorability. However, as we discuss in Section 2.4, this is not straightforward in a triangle-free setting on any surface other than the sphere.

## 2.4 Coloring Algorithms

In this section we combine the Theorem 15, with the results from Sections 2.2 and 2.3 to test and obtain 3-colorings of triangle-free graphs embedded in the torus.

The overall approach is the following. Theorem 15 allows us to efficiently decide 3-colorability for graphs that have properties similar to those observed in the 4-critical graphs, that is, limited census and at least one short non-contractible cycle. We first deal with parts of graph that are not parts of potential 4-critical subgraphs, if possible, until the remaining graph is similar to a 4-critical graph and we can decide colorability efficiently.

To actually obtain a 3-coloring, we employ a strategy based on collapse of 4-faces to reduce (virtually) any 3-colorable graph on the input to a smaller 3-colorable graph and deal with structural issues arising from 4-face collapses, namely non-contractible triangles. To do that, we exploit the fact that Theorem 15 operates on graphs embedded in the cylinder, without the necessity of its rings or their precolorings to match. We can therefore divide the graph embedded in the torus into multiple pieces embedded in cylinders and work with each piece separately.

### 2.4.1 Testing of 3-colorability

Let $G$ be a given graph, triangle-free and with 2-cell embedding in the torus. We may assume without loss of generality that the input graph is of minimum degree 3. If that is not the case, any vertex of degree at most 2 may be deleted at any point without affecting colorability of the graph.

Suppose $G$ is not 3-colorable and therefore contains a 4-critical subgraph $H$. Let us compare $H$ and $G$. Starting with $H$, we may construct $G$ by adding elements into some of its faces. Note that representativity of $H$ is at least 2, as implied by templates or from a more direct approach [33, 32], and therefore each face of $H$ is a face of $G$ or its boundary is a separating cycle in $G$. By Theorem 19, all faces of $H$ are of length at most 7 and Lemma 18 characterizes what structure may be contained inside a separating cycle in $H$. We observe that if we delete the interiors of all separating ($\leq 7$)-cycles that do not contain one of the structures specified by Lemma 18, the obtained graph $H'$ still contains a 4-critical subgraph $\bar{H}$, possibly different than $H$. Furthermore, if we find a coloring of $H'$, we are guaranteed that a proper extension to $G$ exists. The only difference between $H'$ and $\bar{H}$ is that a 6- or 7- face of $\bar{H}$ may be a separating cycle in $H'$, quadrangulated in the former case, and near-quadrangulated up to a single 5-face in the latter case.

We say that the multiset of integers is *modest* if it is one of the values from Theorem 19 or one of their subsets. That is one of

$$\{7,5\},\{7\},\{6,5,5\},\{6,5\},\{6\},\{5,5,5,5\},\{5,5,5\},\{5,5\},\{5\},\emptyset$$

We say that a graph is modest if its census is modest.

Note that in the reasoning above, both $\bar{H}$ and $H'$ are modest. The former by Theorem 19, and the later by considering the difference between $\bar{H}$ and $H'$ allowed by Lemma 18. Furthermore, by Corollary 25, the representativity of $\bar{H}$ is at most 20, and so is the representativity of $H'$ and $G$.

The high-level strategy for testing 3-colorability is as follows. Given $G$, we construct a subgraph $H'$ which is close enough to $\bar{H}$ so that it is modest and of low edge-width, unless $G$ is 3-colorable (implying that $\bar{H}$ does not exist) and crucially $H'$ is 3-colorable if and only if $G$ is 3-colorable. We then use results from Section 2.1 to decide whether $H'$ is 3-colorable, where the parameters influencing the time complexity are now limited.

Let $G$ be a graph, $H$ a subgraph of $G$ and $f$ a face of $H$ that is a 2-cell face. We say that $f$ is a *cell* (of $G$). We say that the (clockwise) facial walk of $f$ (in $H$) *bounds* the cell $f$. Note that by orientation, the bounding walk defines $f$ uniquely and therefore we do not need to strictly distinguish between cell and its bounding walk. We say that a cell is *separating* if it contains any edges or vertices of $G$ in its interior.

It is easy to see that if $G$ is a critical graph and $f$ its separating cell bounded by a closed walk $C$, then the graph embedded in $f$ is $C$-critical, the same way as this is true for separating cycles. Furthermore, even though the boundary of a cell is not necessarily a cycle, when considering $C$-criticality of the interior, we can split some vertices of $C$ to make it into a cycle bounding a disk in which the interior of the cell is naturally embedded. Therefore, if a triangle-free 4-critical graph contains a cell bounded by a closed walk $C$ of length $l$, then the structure of the interior of the cell corresponds to a planar graph critical in respect to its outer face of length $l$.

To detect and remove separating cells efficiently, we use a slight modification of the following notion introduced by Dvořák, Král' and Thomas [31]. A face $f$ of length $l$ is *$k$-free* if it is not contained inside a separating cell of length at most $k$. We say that a face of length $l$ is *free* when it is $l$-free. We extend the same terminology to general parts of the graph (such as an edge or a pair of faces), that is, a part of a graph is $k$-free if it is not contained (all individual parts simultaneously) in the interior of a separating cell of length at most $k$.

In order to test freeness, we need a standard technical method based on the concept of universal cover. Let us give an informal description of the construction. Given an embedding of a graph $G$, we split the torus along a non-contractible cycle $C_1$, obtaining a cylinder. Let $P$ be a shortest path connecting the two copies of $C_1$ arising by splitting and let $C_2$ be a non-contractible cycle of $G$ obtained as $P$

together with arc of $C_1$ connecting the endpoints of $P$. We cut the cylinder again along $P$ and obtain a patch-like piece of surface with opposite sides corresponding to copies of the path along which the cuts were made.

Let us denote the resulting graph $T$. We say that the graph $T$ together with its embedding is a *tile*. If the endpoints of $P$ represent the same vertex of $C$ we say that the tile is square, and interpret the tile as embedded in a unit square with the opposite sides corresponding to $C_1$ or $P$ embedded identically along the edges of the square. Otherwise, we say that $T$ is hexagonal tile. We interpret $T$ as embedded into a unit hexagon, with the opposite sides representing the copies of $C_1 \cap C_2$, $P$ and $C_1 \setminus C_2$, with identical embeddings along the edges of the hexagon.

We say that two hexagons of a hexagonal grid, or two squares of a square grid are neighboring (at distance 1) if they share a vertex. A graph $H$ is a (hexagonal or square) *grid* of size $k$ tiled by $G$ with a *center tile* $C$, if $H$ is obtained from tiling all hexagons or squares at distance at most $k$ from some initial hexagon or square $h$ using the tile $T$ obtained from $G$. By this we mean embedding a copy of the tile $T$ into each cell of the grid and unifying the vertices and edges with overlapping embeddings (on the edges and vertices of the grid). The center tile $C$ is then the copy of $T$ embedded into the initial hexagon $h$.

The grid $H$ has natural non-injective projection into the graph $G$, as each element of $H$ is essentially a copy of an element of $G$.

In the following observation, we use a grid of infinite size to concisely describe properties of a large enough grid, given that we do not run into its outer boundary. We then use this observation to specify what size of grid is large enough for applications so that its boundary is never an issue.

**Observation 42.** *Let $G$ be a graph embedded in the torus that is not planar. Let $H$ be a grid of infinite size tiled by $G$. Let $W$ be a walk in $G$ from vertex $v$ to vertex $w$ (possibly $v = w$). For any preimage $v' \in V(H)$ of $v \in V(G)$, there exists a walk in $W'$ in $H$ such that all of the following holds:*

- *$W'$ begins in $v'$, projects to $W$ and $|W'| = |W|$*
- *If $W$ is a path or a cell, then $W'$ is a path or a cell respectively.*
- *$W'$ is contained in tiles (including their borders) at distance at most $|W|$ from any tile that is incident with $v'$. Furthermore, if $W$ bounds a cell in $G$, then $W'$ is contained in tiles at distance at most $\lfloor |W|/2 \rfloor$.*

*Proof.* The first point can be seen by considering the inverse of projection of $H$ into $G$. We obtain $W'$ by starting in the vertex $v'$ and following edges that project to edges of $W$, in the order as they appear on $W$. Note that these are always unique. The walk $W'$ contains preimages of all edges and vertices of $W$. Furthermore $W'$ contains the same number of preimages of each edge $e$ as is the number of times $e$ appears in $W$, and similarly for vertices. Therefore, if $W$ is a path, then so is $W_{v'}$ (note that the opposite implication does not hold since the projection is not injective).

Suppose now that $W$ is a cell. Since the projection preserves incidences of edges with faces, we observe that all faces on the left side of $W$, when walking along $W$, have their respective preimages on the left side of $W'$. Therefore, $W'$ does not cross itself, and because it is also closed, it bounds a cell.

The last point follows from the fact that by the definition of a tile $T$ obtained from $G$, the boundary of (the embedding of) $T$ is exactly the embedding of the cycle bounding the outer face of $T$. Let $W$ be a walk in $H$. Let us consider moving along the walk and for each visited vertex $v$ measure the distance $t$ of a tile containing $v$ from the initial tile. Additionally, if $v$ is contained in multiple tiles, we take the lowest distance value of all tiles containing $v$.

The first vertex of $W'$ is incident with the initial tile, so its distance is 0. For every edge $e = (uv)$ of $W$, there exists a tile containing $e$, and so both endpoints of an edge are incident with the same tile. On the other hand, each vertex may be incident only with mutually neighboring tiles. We conclude that every step changes the distance by at most 1. If $W'$ is closed and of length $k$, the maximum distance reached is at most $\lfloor k/2 \rfloor$. □

For the purposes of determining the algorithm complexity we also need the following

**Observation 43.** *A tile $T$ obtained from $G$ can be constructed in time $\mathcal{O}(|G|)$.*

*Proof.* For example, we may run a BFS search from any arbitrary vertex of $G^\star$. We obtain a (spanning) BFS tree $S$ of faces, together with its embedding in the torus. We construct the tile $T$ so that $S$ is a spanning tree of the $T^\star$(with the exception of the outer face of $T$). For simplicity, we may create a separate copy of the facial walk for every vertex of $S$ and unify the edges represented by edges of $S$. By walking around the outer boundary of the obtained structure we exhaustively unify together pairs of copies of the same edge neighboring around a vertex. We obtain a graph $T$ where only the outer face contains duplicates of elements of $G$. In particular, since BFS touches every edge outside of the BFS tree exactly twice, each edge of $G$ appears exactly twice on the outer face boundary, or exactly once otherwise.

We argue that $T$ is indeed a tile. We may project the facial walk of the outer face back into $G$, where it forms a closed walk $W$ and the embedding of $W$ has a single 2-cell face (containing the embedding of $S$). Using Euler's formula for the torus, we get that $e(W) = v(W) + f(W) = v(W) + 1$. We conclude that all vertices are of degree 2 up to either one vertex of degree 4 or two vertices of degree 3. We see that $W$ is the union of two non-contractible cycles, either intersecting in a single vertex of degree 4, yielding a square tile, or intersecting in a subpath connecting the two vertices of degree 3, yielding a hexagonal tile. □

**Lemma 44.** *There exists an algorithm which given a triangle-free graph $G$ embedded in the torus removes interiors of all separating 4-cycles in $G$ in time $\mathcal{O}(|G|)$.*

*Proof.* The following construction an adaptation of a construction from a bachelor thesis by Urmanov [60]. We give a sketch of the algorithm. First, we need to orient $G$ so that the maximum out-degree is bounded. Since triangle-free graphs embedded in the torus (or the plane) have average degree at most 4, we can always obtain out-degree at most 4 by a greedy approach: iteratively find a vertex $v$ of degree at most 4, orient all edges of $v$ out of $v$, continue with $G - \{v\}$. This process can be implemented to run in time $\mathcal{O}(|G|)$.

Observe that in $\vec{G}$, oriented with out-degrees at most $\Delta$, the number of oriented walks of length $\ell$ from any vertex is bounded by $\Delta^\ell$.

We consider all possible orientations of a 4-cycle $C$ and describe how to iterate over each type of orientations in $G$. We say that a vertex $v$ of $C$ is a *source* or *sink*, if both edges of $C$ are oriented away from $v$ or into $v$ respectively. If $C$ has no source, then $C$ is one of the oriented walks of length 4 beginning in any vertex of $C$. If $C$ has a single source $v$, then it is the union of two oriented walks beginning in $v$ with the sum of length 4. In both cases, there is a limited number of options to iterate through.

Finally, if $C$ has two sources, we proceed a bit more carefully. Let us fix a vertex $w$ and suppose $w$ is one of the two sinks of $C$. For each edge $(v_i, w)$ we iterate through all out-neighbors $x$ of $v_i$. We store at the vertex $x$ the information that $v_i$ is a potential corner of a 4-cycle, together with $x$ (and the fixed vertex $w$). At any point, any pair of corners $v_i, v_j$ stored at $x$ forms a 4-cycle $w, v_i, x, v_j$. Once we finish iteration through all edges $(v_i, w)$, we delete all stored corners. We repeat this process for all choices of $w$. Clearly, over all choices of $w$ we iterate through each edge $(v_i, w)$ only once, for each there are at most $\Delta$ choices of $x$ and so overall we store at most $\mathcal{O}(||G||)$ corners. It may of course happen that a single vertex $x$ would store many $v_i$'s. If we process all new pairs of corners when storing a new corner, we avoid this issue. It can be shown that any embedding of $K_{2,9}$ has a non-facial contractible 4-cycle, and therefore there can never be more than 8 corners stored at a single vertex without one corner being in the interior of a 4-cycle defined by a pair of the other corners.

For each 4-cycle we need to determine whether it is contractible and which elements of $G$ lie in its interior. We construct a grid of size 4 tiled by $G$. For each 4-cycle we find a preimage $W$ which is a walk of length 4 in $H$, beginning in the center tile. If the walk is not closed, we conclude that $C$ is non-contractible, by Lemma 42. Otherwise it forms a 4-cycle in $H$ with a uniquely determined interior.

We may determine which side of $W$ is the interior by the following construction. We fix an arbitrary spanning tree $\vec{T}$ of $H^\star$ rooted in the outer face of $H$ and oriented towards the root. For each edge of $\vec{T}$ we precompute the number of vertices in its subtree. We observe that if we sum the values of edges of $\vec{T}$ crossing $C$ oriented out from the interior of $C$ and subtract values of edges oriented into the interior, we obtain the number of faces in the interior of $W$. If we do the same for the exterior, we get a negative value. Once we determined the interior, we may simply run search of the interior of $W$, project all elements into $G$ where we delete them together with all their preimages in $H$. Note that we do not need to update the tree $\vec{T}$ as we delete elements of $G$, since the edges crossing a given 4-cycle still correctly count faces in the original grid $H$ and therefore distinguish correctly the interior from the exterior. $\square$

**Lemma 45.** *There exists an algorithm which for any triangle-free graph $G$ with a 2-cell embedding in the torus, $F$ a set of faces of $G$, and $k$ an integer such that $k \leq 9$, satisfying one of the following conditions:*

- $|F| = \{f\}$, $k \leq |f| + 1$
- $|F| \leq 3$, *every $f \in F$ is 4-free*

*decides whether $F$ is $k$-free. If $F$ is not $k$-free, the algorithm returns a certifying separating cell $C$ such that $|C| \leq k$ and $C$ itself is $|C|$-free. The algorithm runs in time $\mathcal{O}_k(|G|)$.*

*Proof.* The main idea is to test freeness using a max-flow algorithm, an adaptation of approach appearing across literature. We omit some of the more technical details. We proceed as follows.

In general, we represent $G$ as a tile and use it to tile a portion of the plane. By Observation 42, a grid of size at most $\lfloor \frac{k+1}{2} \rfloor$ is always enough to contain any closed walk of length at most $k$ incident with the central tile.

We first describe the construction for the case when $|F| = \{f\}$ and $k < |f|$. We then present modifications to solve the remaining cases. Let $H$ be the plane graph obtained as a grid of size $\lfloor \frac{k+1}{2} \rfloor$ tiled by $G$. We use $H^\star$ to define the flow network. For the outer face $g$ of $H$, we declare $g^\star$ to be the source and set capacities of all edges to 1. We create a new sink vertex, and a new junction vertex. We connect the junction vertex to the sink via edge of capacity $k+1$ and connect the preimage of $f$ in the central tile to the junction vertex. We can now run a max-flow algorithm, for instance the Ford-Fulkerson algorithm, to obtain integer-valued flow, in time $\mathcal{O}(m \cdot ||H^\star||)$ where $m$ is the size of a maximum flow, and $m \leq k+1$ by the construction of the network.

Suppose that a separating cell $C$ of length $m$, such that $m \leq k$, exists in $G$. Then we can find preimage $C'$ of $C$ in $H$ such that it separates $f$ in the central tile from $g$. Note that while $C'$ is incident with the central tile, it is not necessarily contained in the central tile. We observe that the maximum possible flow in the flow network is at most $m$, as $C'^\star$ is a cut of the network. We conclude that if the maximum flow is of size $k+1$, then $f$ is $k$-free.

Conversely, suppose that the maximum flow is of size $m$, such that $m \leq k$. Let us construct cut $S$ as the closest cut to $g^\star$ as follows. Consider the set $A$ of vertices of the network reachable from $g^\star$ along non-saturated edges. We take the set of (saturated) edges with exactly one end in this set as a cut $S$ and denote $K$ the set of edges of $H$ dual to edges in $S$. We claim that $K$ is a closed walk bounding a cell in $H$. Consider the components of $H$ after deleting $S$. Clearly, $A$ forms a component and all other components neighbor only with $A$, by definition of $S$. If there was a component $D$ not containing a sink, then there must be flow both entering and exiting $D$ across $S$, but then vertices in $D$ are reachable along the edge with flow from $D$ into $A$. Therefore $V(H^\star) \setminus A$ form a single component, which is exactly the interior of the cell in $H$ bound by $K$.

Project the closed walk $K$ into $G$ and denote it $C$. If the projection of $K$ intersects itself, $C$ may have multiple faces, but only one of them contains $f$. Let $Q$ be the closed walk bounding this face. We argue that $|Q| \leq |K|$. Clearly, only the edges of $K$ form the boundary of $Q$, however, due to wrapping around the torus, it may happen that both sides of an edge belong to the boundary of $Q$. Consider the projection of the cell $\kappa$ (understood set of its interior faces) bounded by $K$ into $G$. For each face defined by the embedding of $C$ count how many times it is covered by the projection of $\kappa$. Let $e$ be an edge of $Q$ such that exactly one edge of $K$ projects to $e$, then we observe that the two faces of $C$ incident with $e$ are covered by $\kappa$ distinct number of times and therefore are not the same face. We conclude that if $e$ appears twice in $Q$, then it also has at least two preimages in $K$, which implies that indeed $|Q| \leq |K|$. If $Q \subset K$, we get a contradiction with the minimality of $S$, and therefore $Q = K$.

We obtain a cell bound by $K$ of length at most $m$ which is separating and contains $f$ in its interior, as required.

Note in the case $|F| = 1$, by minimality of $S$, $K$ is in fact a cycle. Also, note that if $k \leq 7$, then the non-existence of triangles implies that $C$ cannot intersect itself in $G$ and therefore $C = Q$ automatically. In the cases of $k \in \{8, 9\}$ we may get a cell wrapped around the torus and kissing itself in a single vertex. This may even happen twice, once for each direction it may wrap around the torus.

To solve the case when $|F| = \{f\}$ and $k \geq |f|$, we need to deal with the problem that the facial walk of $f$ is itself a small cut. We observe that if $f$ is contained in a cell $C$, then the interior of $C$ contains $f$ together with at least one of its neighbors. We iterate through each neighbor $f_i$ of $f$ in the central tile of $H$. Consider removing the edge between $f$ and $f_i$ and applying the previous case on $f'$. Since $|f_i| \geq 4$, we get that $|f'| \geq |f| + 2$ solving the problem with application to $f$ directly. Note that $f'$ may neighbor with itself, however even if a subset of the facial walk of $f'$ forms a small cut, we still get a separating cell $C$ in $G$ other than the facial walk of $f$. We conclude that the previous correctness analysis analogously holds when applying to $f'$. In implementation however, we prefer to perform an equivalent adjustment, where we connect both $f$ and $f_i$ to the junction vertex instead of removing their shared edge. Not removing the edge does not disturb the tile and preserves the properties of Observation 42 ensuring that the size of the grid is sufficient.

If we obtain a separating cell $C$ for any $i$, we conclude that $f$ is not $k$-free and output $C$. If all tests fail to find a separating cell, then $f$ is $k$-free. We now argue that it is in fact sufficient to run at most $k+1$ tests even if $f$ has more than $k + 1$ neighbors. Suppose that $f$ is not $k$-free, there exists a cell $C$ separating $f$, but a test with some choice $f_i$ fails to find $C$. This is only possible if $f_i$ is not in the interior of $C$, and therefore the edge $e_i$ between $f$ and $f_i$ is an edge of $C$. In each test which does not produce a separating cell we conclude that $e_i \in C$ for a different edge $e_i$. After $k + 1$ tests we conclude that $|C| > k$, a contradiction.

For the case when we test $F$ of size at least 2, we need to deal with the following problem. Let $F = \{f_1, f_2, \dots\}$. Suppose $C$ is a separating cell in $G$ with all elements of $F$ in its interior. We may find a preimage $C'$ of $C$ such that it separates a preimage of $f_1$ in the central tile from the outer face $g$. Then $C'$ may separate from $g$ different preimages of $f_2, \dots$ than those in the central tile. We therefore iterate over all possible choices of tiles into which we can project the faces $f_2, \dots$, and for each choice we proceed analogously to the previous case, connecting the junction vertex to each selected projection.

We argue that if the flow is at most $k$, we again obtain a separating cell. It may conceivably happen that we obtain a cut $S$ such that the corresponding set $K$ of edges in $H$ induces multiple cells separating distinct subsets of $F$ from $g$, since we now have multiple sinks in $H$ (connected to a common artificial sink). However, since each of the elements in $f$ is at least 4-free, each of the cells must have boundary of length at least 5. By minimality, no edge is in the boundary of at least two cells, and therefore $|S| \geq 10$, which is a contradiction with the assumption that $k \leq 9$. We therefore obtain a cell $C$ as in the first case.

Again, if any of the projection choices yields a separating cell, we return this cell, and if none choice yields a separating cell, then $F$ is $k$-free.

Let us analyze the complexity of the algorithm. In all cases, the leading factors are the runs of max-flow algorithm. Considering $k$ as a parameter, the flow network is of size $\mathcal{O}(k^2|G|)$, and so the max-flow algorithm runs in time $\mathcal{O}(k^3\|G\|)$ since the maximum flow is upper-bounded by $k+1$. In the cases when $|F| = 1$ we run at most $k+1$ instances of the max-flow algorithm. In the case when $1 < |F| \leq 3$, we run at most $(2k+1)^{2(|F|-1)}$ instances of the max-flow algorithm. Naturally, it is possible to greatly reduce these factors, especially whenever $G$ has non-trivial edge-width and so the bounds in Observation 42 can be reduced. Overall, since $k$ is upper-bounded by a constant, we get that the time complexity is at worst $\mathcal{O}(k^7|G|) = \mathcal{O}_k(|G|)$. □

As a side note, we do not actually need the cases when $k \in \{8, 9\}$ or $|F| = 3$ for testing of 3-colorability, however they will be useful when re-purposing the same algorithm to obtain a 3-coloring later on. Also note that the cell given on the output uniquely determines its interior by orientation.

**Lemma 46.** *There exists an algorithm that, given a triangle-free graph embedded in the torus terminates in time $\mathcal{O}(|G|)$ and either*

- *produces subgraph $H$ of $G$ such that $S(H)$ is modest and $H$ is 3-colorable if and only if $G$ is 3-colorable; or*
- *determines that $G$ is 3-colorable.*

*Proof.* Suppose $G$ is a triangle-free graph embedded in the torus. Suppose that $G$ is not 3-colorable, and therefore contains a 4-critical subgraph $H$. The graph $G$ can be obtained from $H$ by filling some faces of $H$ with planar graphs. In other words, every face of $G$ that is $k$-free either contributes to $S(H)$ or is contained inside a separating cell of length more than $k$ which forms a face of $H$. Suppose $f$ is a face of $G$ but not a face of $H$. Since $S(H)$ is modest, and in particular the largest possible face of $H$ is a 7-face, $f$ is not 7-free. Conversely, any 7-free face of $G$ contributes to $S(H)$.

The algorithm builds a set $F$ of faces with high freeness. In doing so, the algorithm iteratively removes interiors of short separating cells, following the results of Lemma 18, which guarantees that the (non-)colorability is maintained. By the reasoning above, if lengths of faces in $F$ exceed the limits of modesty, a 4-critical subgraph $H$ cannot exists and $G$ is correctly decided to be 3-colorable. Otherwise the remaining graph $H$ is modest.

In particular, following Lemma 18, the faces in $F$ should maintain the following properties:

- Every 5-face is 6-free
- Every pair of 5-faces is 7-free
- Every ($\geq 6$)-face is 7-free

Before running the procedure, we remove interiors of separating 4-cycles, if any exist, using the algorithm from Lemma 44. We start with the initial graph $G$ and an empty set $F$. Iteratively we pick an unprocessed ($\geq 5$)-face $f$ and use the algorithm from Lemma 45 to test whether it is $min\{|f|+1, 7\}$-free. If a separating cell is found, we delete its interior and add the new resulting face $f_s$ into $F$, otherwise we add $f$ into $F$. If the face being added is a $(\geq 8)$-face (and therefore at least 7-free), modesty of $F$ is violated and we terminate instead.

After adding a new face into $F$, we need to maintain the above listed properties. If the added face was $f$, then it already satisfies the conditions on individual faces as it is $min\{|f|+1, 7\}$-free. If however the added face $f_s$ arose from a separating cell, it is only free. Note that in this case, $|f_s| \geq 5$, since no separating 4-cycles remain. We need to further refine the face to ensure that it is 6- or 7-free to satisfy the conditions. We repeat the test above, for $f_s$.

If the added face was a 5-face, then we test it in combination with all 5-faces already in $F$ for 7-freeness. As before, we use the algorithm from Lemma 45 and if a separating cell of length 7 is found, we delete its interior and add the new face to $F$. Note that all 5-faces in $F$ are already individually 6-free, and so a separating cell of length 6 requiring further refinement may not be produced at this point.

The set $F$ now satisfies all of the listed conditions. If it is not modest, we terminate concluding that $G$ is 3-colorable. Otherwise we continue processing faces of the remainder of the graph until all $(\geq 5)$-faces are in $F$ and we return the remainder of the graph.

Let us consider the complexity. Each iteration adds at least one new face into $F$, while possibly deleting other faces. This may happen when deleting interiors of separating cells that contain some faces already in $F$. Note that all faces in $F$ are at least free, in particular whenever a subset of faces is deleted from $F$, a new face that is strictly longer than each deleted face is added into $F$. This also implies that 7-free faces of $F$ are never deleted, and therefore the number of $(\geq 6)$-faces never drops. We conclude that a constant number of iterations may take place before modesty is violated and the algorithm terminates. Each iteration consists of a constant number of runs of the algorithm of Lemma 45, which runs in time $\mathcal{O}(|G|)$. Altogether at most a small constant number of freeness tests occurs, giving an overall time complexity of $\mathcal{O}(|G|)$. ☐

To use our theory of coloring graphs in the cylinder efficiently, we need to find a short separating cycle, if it exists. We again use tiles to find such cycle.

**Lemma 47.** *Let $G$ be a graph embedded in the torus, $v$ a vertex of $G$, $k$ an integer and $H$ a grid of infinite size tiled by $G$. If there exists a path $P$ of length at most $k$ connecting two distinct preimages $v_0, v_1$ of $v$ in $H$, then $G$ contains a non-contractible cycle of length at most $k$.*

*Proof.* We may assume that $P$ is a shortest path connecting $v_0$ and $v_1$. Clearly, any path in $H$ connecting two preimages of $v$ projects into a closed walk in $G$. We will argue that every vertex of $G$ has at most one preimage in $V(P)$, except $v$ which has exactly two preimages in $V(P)$. It then follows that $P$ is a cycle and projects into a cycle $C$ of $G$. Since $P$ is not a cell, by Observation 42, $C$ does not bound a cell and is therefore non-contractible.

Suppose $x_1, x_2 \in V(P)$ are two preimages of $x \in V(G) \setminus \{v\}$, in the order as they appear on $P$. Then we may cut out the segment of $P$ between $x_1$ and $x_2$, and mimic the segment from $x_2$ to $v_1$ to instead go from $x_1$ to some $v_2$ along preimages of the same edges. If $v_2 \neq v_0$, then the obtained path is a shorter path connecting two distinct preimages of $v$, a contradiction. If $v_2 = v_0$, we instead consider the segment $S$ of $P$ connecting $x_1$ and $x_2$. Clearly, $S$ satisfies the assumptions of the Lemma, with even lower $k$, and has less preimages of $x$, we proceed inductively. $\square$

We will need to solve the following problem. Given a planar graph $H$ and a (small) constant $k$, we need to find shortest paths between all pairs of vertices that are at distance at most $k$ in time $\mathcal{O}(\|H\|)$. We use the following result based on an oracle data structure.

**Lemma 48** (Kowalik and Kurowski [61])**.** *For a fixed integer $k$, there exists a datastructure which for a planar graph $G$ takes preprocessing time $\mathcal{O}(|G|)$ and in time $\mathcal{O}(1)$ answers queries whether given two vertices are at distance at most $k$ and if so computes a corresponding shortest path.*

Their structure additionally allows updates upon deleting edges or vertices in time $\mathcal{O}(1)$, and in time $\mathcal{O}(\log^k |G|)$ upon adding an edge. In the construction of coloring algorithm later on, it can therefore be kept updated as the interiors of separating cells are deleted and as 4-faces are collapsed.

**Corollary 49.** *There exists an algorithm such that for any $G$ embedded in the torus decides whether $G$ has a non-contractible cycle of length at most $k$ and if so outputs such cycle. The algorithm runs in time $\mathcal{O}(|G|)$ for fixed $k$.*

*Proof.* Suppose $C$ is a non-contractible cycle of length at most $k$ in $G$. Let $H$ be a grid of size $k$ tiled by $G$. Let $v \in V(C)$, $v_0$ be a preimage of $v$ in the central tile of $H$ and $e_1, e_2$ edges incident with $v$ in $C$. Applying Observation 42 to $C - e_1$ and $C - e_2$ we get that $C$ has a preimage in $H$ that is a path beginning in $v_0$ and ending in some other preimage of $v$. Together with Lemma 47 we get that $G$ contains a non-contractible cycle of length at most $k$ such that $v \in V(C)$ if and only if $H$ contains a path $P$ of length at most $k$ connecting $v_0$ and another preimage of $v$.

We apply Lemma 48 to test each vertex of the central tile of $H$ together with all other preimages of the same vertex of $G$. If we find no shortest path of length at most $k$, then we conclude that $G$ does not contain a non-contractible cycle of length at most $k$. Otherwise let $P$ be the shortest of all found paths, then $G$ contains a non-contractible cycle of length $l \leq k$, and it must project to one of the found paths of length exactly $l$. Therefore the projection of $P$ into $G$ is exactly a shortest non-contractible cycle.

Since $|H| = \mathcal{O}(k^2|G|)$ and $k$ is fixed, we can perform all test in time $\mathcal{O}(|G|)$. $\square$

We are finally ready to put together the test of 3-colorability.

**Theorem 50.** *There exists an algorithm that, given a triangle-free graph $G$ with a 2-cell embedding in the torus, decides whether $G$ is 3-colorable. The algorithm runs in time $\mathcal{O}(|G|)$.*

*Proof.* At any point we may assume that $G$ is of minimum degree 3, connected and not planar, otherwise we can reduce $G$ or decide that $G$ is 3-colorable (by Grötzch's Theorem).

We use Lemma 46 to either conclude that $G$ is 3-colorable or obtain subgraph $H$ of $G$ such that $S(H)$ is modest and $H$ is 3-colorable if and only if $G$ is 3-colorable.

We find the shortest non-contractible cycle $C$ of length at most 5 in $H$, if it exists, using the algorithm from Corollary 49. If the shortest cycle is longer than 5, then by Lemma 40 $H$ is 3-colorable. (Alternatively we may rely on the theoretical result of Lemma 25, giving limit 20)

We use the cycle $C$ to split $H$ into a graph $H'$ embedded in the cylinder. We iterate through all possible precolorings of $C$, inducing a precoloring of the rings of $H'$. For each precoloring we run the algorithm from the Theorem 15 which decides whether the precoloring can be extended. Since $S(H)$ is modest, and $C$ is short, we get that the complexity factors of the algorithm from the Theorem 15 depending on $S(H')$ are limited by a small constants and therefore the algorithm finishes in linear time. $\square$

### 2.4.2 Algorithm for Obtaining a 3-coloring

The task of actually producing a 3-coloring turns out to be substantially more complex than just testing whether a 3-coloring exists. This is partly due to the fact that several of the tools used in the 3-colorability test are non-constructive, in the sense that they do not produce coloring when a graph is decided to be 3-colorable, but rather focus on certifying non-colorability. In the positive cases, the approaches do not provide any obvious ways to break-down the task of actually coloring the graph.

An algorithm for obtaining a 3-coloring of triangle-free embedded graphs is already shown to exist in [31]. The contribution of our approach is that unlike the algorithm in [31], we use a significantly more constructive approach. The complexity of our algorithm has only small multiplicative constants and can be used in practice.

A standard approach is to use the test of 3-colorability as a black-box, and to reduce the graph step by step. For instance, if we can adjust $G$ to force the same color for some pair of vertices, we can then test colorability of the adjusted graph and regardless of the test result, we can narrow our search to colorings that assign the same or distinct colors to the pair. The former can be done for instance by unifying the two vertices, the latter by introducing a new edge connecting them. In our approach we follow a similar idea, using 4-faces to narrow our search by collapsing them. Given the structural limitations on inputs of our testing method however, we also need to deal with several additional special cases where we run into non-constructive dead-ends.

As we reduce the input graph, we not only collapse 4-faces, but also remove interiors of short separating cells. Once the graph becomes trivial and we obtain a 3-coloring, we need to extend the coloring back into the original graph. We therefore need an algorithm which takes a planar graph with a precolored outer face (up to some length) and extends the precoloring into the whole graph (if it exists). Another obvious issue is that we may run out of 4-faces, or be given a graph with no 4-faces to begin with. In this case, we have a graph embedded without contractible ($\leq 4$)-cycles (note that it may still contain non-contractible 4-cycles). We deal with this special case by adapting existing results. The last issue is that in collapse of a 4-face we may produce triangles. This issue is unavoidable, but given our tools and existing results relatively straight-forward to deal with, although a little tedious. Dealing with triangles will take most of the remainder of this section.

For completeness, since a graph embedded in the torus may be planar, we refer to the following result dealing with the planar case.

**Theorem 51** ([62]). *There exists an algorithm running in linear time which outputs a* 3*-coloring for any input triangle-free planar graph.*

### No Contractible Cycles of Length at Most 4

The first ingredient towards a complete coloring algorithm is the following characterization of graphs of high girth.

**Theorem 52** (Thomassen [23]). *Every graph of girth at least* 5 *embedded in the torus is* 3*-colorable.*

Fortunately for us, in order to apply induction, Thomassen actually proves the following stronger claim:

**Lemma 53** (Thomassen [23]). *Every graph embedded in the projective plane or in the torus so that all contractible cycles have length at least* 5 *is* 3*-colorable.*

And to prove this, he first proves the following helpful characterization

**Lemma 54** (Thomassen [23]). *Let $G$ be a plane graph of girth at least* 5*. Then $G$ is* 3*-colorable. Furthermore, if $G$ has an outer face bound by a cycle $C$ of length at most* 9*, then any proper precoloring of $G[V(C)]$ extends to a* 3*-coloring of $G$, unless $C$ has length* 9 *and $G - C$ has a vertex joined to three vertices of $C$, which are precolored by three distinct colors.*

The proofs of both lemmas are structured as follows. Suppose we are given a graph $G$ which is a minimal counterexample. We observe that various structures cannot appear in $G$, as otherwise we can reduce $G$ into a smaller graph $G'$ satisfying the assumptions of the lemma and therefore 3-colorable by assumption of minimality, and the 3-coloring of $G'$ can be extended to a 3-coloring of $G$ (in the proof of Lemma 53 sometimes using the Lemma 54). After collecting several of these forbidden structures, we observe that $G$ not containing any of these cannot exist.

By inspection of the proofs, one can derive an algorithm to actually obtain the 3-coloring as follows. First search for the forbidden structures, as they appear in the proof. If found, follow the construction reducing $G$ into $G'$, recursively color $G'$ and expand the 3-coloring of $G'$ into a 3-coloring of $G$. By the nature of the argument, at least one such structure must appear in $G$, and so $G$ recursively reduces until it is trivial, at which point we color $G$ by brute force.

The forbidden structures used in the arguments are easy to detect. In the case of Lemma 54, it is short generalized chords, single vertices neighboring the outer cycle, and facial walks of lengths at most 6, possibly with prescribed degrees of vertices. In the case of Lemma 53, subgraphs isomorphic to $K_4$ and facial walks of lengths other than 5, with or without certain chords. The presence of all of these structures can be tested in linear time. We therefore claim the following two statements to be true:

**Lemma 55.** *There exists an algorithm which for any plane graph $G$ of girth at least 5 and an outer cycle $C$ of length at most 9 and a precoloring $\psi$ of $G[V(C)]$ such that $G - C$ has no vertex connected to three vertices of $C$ colored by distinct colors by $\psi$ constructs a 3-coloring of $G$ extending $\psi$ in time $\mathcal{O}(|G|^2)$.*

**Lemma 56.** *There exists an algorithm which for any graph $G$ embedded in the torus without contractible cycles of length at most 4 constructs a 3-coloring of $G$ in time $\mathcal{O}(|G|^2)$.*

The argument we give here is only a sketch of derivation of the algorithms and their complexity. Let us call the algorithm corresponding to Lemma 55 as algorithm $A$, and an algorithm corresponding to Lemma 56 as algorithm $B$.

First we construct the algorithm $A$. As mentioned above, finding a forbidden structure can be achieved in linear time. For any structure, we have a prescription how to make a local adjustment of the graph that does not increase the number of vertices and allows us to either recurse on a subgraph, or to split it into multiple subgraphs disjoint up to their boundary and recurse on each. Since the proof is by induction on $|V(G)|$, the graphs we recurse on have strictly less vertices and therefore the depth of recursion is at most $|V(G)|$.

As mentioned, the recursion of the algorithm $A$ may branch, so we need to analyze the recursion more closely. We observe that when the recursion splits the working graph $G$ into graphs $G_1, G_2, \dots$ to branch, it essentially recurses into graphs induced by distinct areas of the embedding that only overlap in their boundaries (bound by cycles). Similarly, if it does not branch, it recurses into a strictly smaller area. Therefore, in each level of recursion, each edge appears in at most two graphs. Since each recursion node processing a graph $G_i$ runs in time $\mathcal{O}(|G|)$, this gives an overall complexity of $\mathcal{O}(|G|)$ for each level of recursion, and $O(|G|^2)$ for the whole algorithm $A$. This analysis of course ignores the local adjustments in each recursion node, as arguing that these do not change this outcome would require going through each possible construction.

The algorithm $B$ has a significantly simpler structure. As before, finding a forbidden structure can be achieved in linear time. For each structure found in $G$, we have a prescription of a local adjustment of $G$ into $G'$ which does not increase the number of vertices and allows us to use recursion. In this case, we always recurse by at most one call of $B$, possibly after deleting interior of some separating cycle. We then extend the obtained coloring into this cycle by a call of the algorithm $A$, or into some small structure adjusted for the recursive call. The argument is based on induction in $|V(G)|$, so each recursive call of $B$ strictly decreases the number of vertices. We conclude that the depth of the recursion is at most $|V(G)|$, and each recursion node of $B$ (excluding the call of $A$) runs in time $\mathcal{O}(|G|)$.

Similarly to the analysis of the algorithm $A$, we observe that each recursive call essentially recurses into a smaller subgraph, and if the recursion node calls both algorithms $A$ and $B$, then the graphs in their inputs are disjoint up to the bounding cycle of the graph presented to $A$. In particular, each edge appears as an interior edge in at most one call of $A$, each call contains at most 9 other edges, and the total number of calls is $\mathcal{O}(|V(G)|)$. We conclude that the overall complexity of all calls of the algorithm $A$ is $O(|G|^2)$, and so is the complexity of the whole algorithm.

### Interiors of Separating Cells

Other ingredients we need are ways to deal with the interiors of separating cells that are inconsequential to colorability of the whole graph, yet still need to be colored in the final coloring.

**Lemma 57** (Dvořák, Kawarabayashi and Thomas [62] algorithm 4.6)**.** *There is an algorithm which for any triangle-free plane graph $G$ with outer cycle $C$ of length at most 5 and a precoloring $\psi$ of $C$ decides in time $\mathcal{O}(|G|)$ whether $\psi$ extends to a 3-coloring of $G$ and outputs such coloring in the affirmative case.*

Note that in the above lemma, a 3-coloring always exists by Lemma 18. We will also need the following stronger versions of a pair of cases of the Lemma 18.

**Lemma 58** ([27])**.** *Let $G$ be a plane triangle-free graph with outer face bounded by an induced cycle $C = c_1, \ldots, c_6$ of length 6. The graph $G$ is $C$-critical and $\psi$ is a 3-coloring of $C$ that does not extend to a 3-coloring of $G$ if and only if $G$ contains no separating cycles of length at most four, every internal face of $G$ is a 4-face and for $i \in \{1, 2, 3\}$, $\psi$ satisfies $\psi(c_i) = \psi(c_{i+3})$.*

**Lemma 59** (Dvořák and Lidický [52])**.** *Let $G$ be a plane triangle-free graph with outer face bounded by a cycle $C = c_1, \ldots, c_7$ of length 7. The graph $G$ is $C$-critical and $\psi$ is a 3-coloring of $C$ that does not extend to a 3-coloring of $G$ if and only if $G$ contains no separating cycles of length at most five and one of the following propositions is satisfied up to relabeling of vertices (see Figure 2.6 for an illustration).*

*(a) The graph $G$ consists of $C$ and the edge $c_1c_5$, and $\psi(c_1) = \psi(c_5)$.*

*(b) The graph $G$ contains a vertex $v$ adjacent to $c_1$ and $c_4$, the cycle $c_1c_2c_3c_4v$ bounds a 5-face and every face drawn inside the 6-cycle $vc_4c_5c_6c_7c_1$ has length four; furthermore, $\psi(c_4) = \psi(c_7)$ and $\psi(c_5) = \psi(c_1)$.*

*(c) The graph $G$ contains a path $c_1uvc_3$ with $u, v \notin V(C)$, the cycle $c_1c_2c_3vu$ bounds a 5-face and every face drawn inside the 8-cycle $uvc_3c_4c_5c_6c_7c_1$ has length four; furthermore, $\psi(c_3) = \psi(c_6), \psi(c_2) = \psi(c_4) = \psi(c_7)$ and $\psi(c_1) = \psi(c_5)$.*

*See Figure 2.6 for illustration.*

### Non-contractible Triangles and Census

In this section we derive an analogue of Theorem 19 allowing us to limit census of critical graphs in the cases where non-contractible triangles are present. The overall idea is to use the triangles to cut the working graph into slices of the torus, each of which is critical in respect to precoloring of its triangular boundaries shared with the neighboring slices. The goal of this section is to show that each slice has a small limited census. Deriving this fact from the base case with no 4-faces is however surprisingly far from straight forward.

We say that a graph $G$ is $\{H_1, H_2\}$*-critical* if $H_1, H_2$ are subgraphs of $G$ and for every proper subgraph $H$ of $G$ (containing $H_1$ and $H_2$) there exists a precoloring

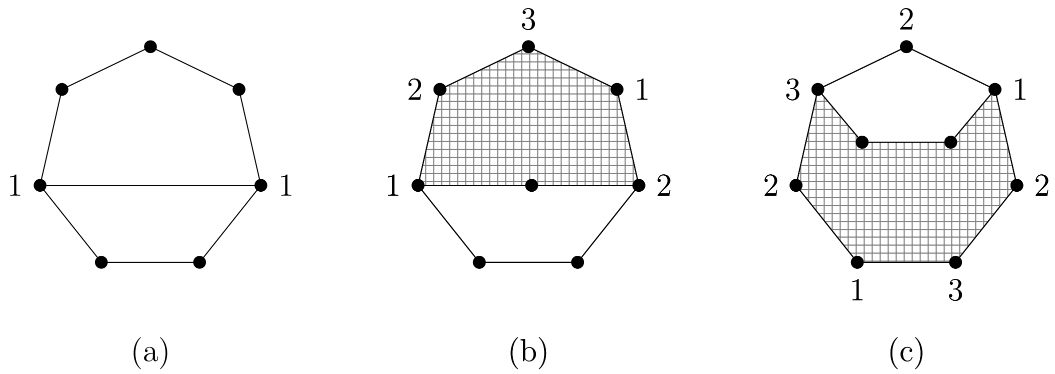



Figure 2.6: Illustrations of the cases of Lemma 59

of $H_1 \cup H_2$ that does not extend into a coloring of the whole graph $G$, but does extend into $H$.

Consider a ring $R$ of $G$ embedded in the cylinder. By a stretch of the definition we allow a pair of neighboring vertices or a single vertex to be a ring, of lengths 2 and 1 respectively. We say that $R$ is a *vertex-like* ring if at most one of its vertices is incident with vertices outside of $R$. Note that if $R$ is vertex-like, then precoloring of the ring is effectively equivalent to a precoloring of its single vertex.

**Lemma 60** ([63])**.** *Let G be an $\{R_1, R_2\}$-critical graph embedded in the cylinder, where $R_1, R_2$ are the rings of G and $|R_1|, |R_2| \leq 3$. If every cycle of length at most 4 in G is non-contractible, then one of the following claims holds:*

- *G consists of $R_1, R_2$ and an edge between them, or*
- *neither $R_1$ nor $R_2$ is vertex-like and G consists of $R_1, R_2$ and two edges between them, or*
- *neither $R_1$ nor $R_2$ is vertex-like and G consists of $R_1, R_2$ and two adjacent vertices of degree three, each having a neighbor in $R_1$ and in $R_2$.*

Observe that by the lemma above, given the lengths of $R_1$ and $R_2$, the census of $G$ is uniquely determined for each case. If $|R_1| = |R_2| = 3$, then the census of $G$ is $\{8\}$, $\{5, 5\}$ and $\{6, 5, 5\}$ for each case respectively. Later on we will need, that in all of these cases, $S(G)$ is an element of $\mathcal{S}_{4,10}$.

First we need a simple characterization showing the reasoning behind refinement operation from Section 2.3. Recall that by *collapse* of a 4-face we mean the operation where two opposite vertice of a 4-face are unified and the resulting two 2-faces suppressed by unifying parallel edges. The two edges formed in this way form a path and we say that the 4-face *collapses into* this path. Note that there are two distinct directions in which a 4-face can be collapsed. Also recall that by *reduction* of a critical graph, or by reduction of its 4-face, we mean a graph obtained by first collapsing a 4-face and then taking a critical subgraph of the result.

Since in this section we work with graphs critical in respect to some precolored vertices, we need to be a bit more careful. For the purposes of the general logic of the constructions in this section, we assume that the precolored vertices of a reduction are inherited from the original graph in the natural way. We use the same notation to refer to the precolored vertices in both graphs, and often refer to a vertex obtained as a unification of two vertices one of which was precolored by the name of the original precolored vertex. We avoid unification of two precolored vertices. While it would be a valid operation under the condition that both input vertices are precolored using the same color, the precolorings we work with are often only implicit.

To make full sense of the following lemma, we consider the relation of $H$ being a subgraph of $G$ as witnessed by an injection of $H$ into $G$.

**Lemma 61.** *Let $G, H$ be plane graphs 4-critical in respect to two precolored rings of length at most 3, and without triangles except for their rings. Suppose $H$ is a reduction of $G$, let $f$ be the collapsed face of $G$, $F$ the path into which $f$ collapses and $x$ the middle vertex of $F$. If $S(G)$ is not a refinement of $S(H)$, then $H$ contains no edges of $F$ and in $H \cup F$ both edges of $F$ are leaves within the same face of $H$. In particular, $x$ is an articulation of $H$ or $x$ is precolored in $H$ and $H$ is a subgraph of $G$, however the injection of $H$ into $G$ does not map $x$ to a precolored vertex of $G$.*

*Proof.* We obtain this result by reproducing the proof of Lemma 27, with weaker assumptions and a slightly different approach. Let $G'$ be the graph obtained from $G$ by collapsing $f$ into the path $F$. Note that $S(G) = S(G')$ as the lengths of all facial walks (except $f$) are preserved.

We argue that $x \in V(H)$. If not, then $H \subset G$. This is not possible if both graphs have the same vertices precolored (in respect to the injection of $H$ into $G$), in particular if no vertices are precolored. Since we assume that $H$ inherits its precolored vertices from $G$, both $G$ and its subgraph $H$ can be critical only if $x$ is precolored in $H$ but the injection of $H$ into $G$ does not map $x$ to a precolored vertex (and therefore the other non-mapped vertex merging into $x$ was precolored), however then necessarily $x \in V(H)$. See Figure 2.7a for illustration.

Consider the graph $H \cup F$. If we uncollapse $F$ back into $f$, we get a graph $H'$ which is a subgraph of $G$. Similarly to comparing $S(G)$ and $S(G')$, we have that $S(H') = S(H \cup F)$ and since $G$ is obtained from $H'$ by filling some of its faces by plane graph critical in respect to their boundary, $S(G)$ is a refinement of $S(H')$.

Let us compare $S(H \cup F)$ and $S(H)$. If $H$ contains both edges of $F$, then clearly $S(H) = S(H \cup F)$. Let $e_1, e_2$ be the two edges of $F$. Let us assume that $e_1 \notin E(H)$ but $e_2 \in E(H)$. Then $H \cup F$ is either obtained from $H$ by adding a chord into one of its faces, or a leaf. In the former case, $S(H \cup F)$ is a refinement of $S(H)$ as cycle with a chord is always critical in respect to its boundary and therefore the refinement operation describes this operation in terms of census. In the latter case, the adjusted face grows in length by 2. Consider uncollapsing $F$ back into $f$, the leaf turns into a vertex of degree 2 incident with two faces, one of which is $f$. Since $G$ is critical and does not contain a vertex of degree 2, we see that during subsequent transition from $H'$ to $G$, the interior of the enlarged face is replaced by a plane graph critical in respect to its boundary. The refinement operation describes these two consecutive steps. We conclude that $S(G)$ is a refinement of $S(H)$, even though the intermediate step $S(H \cup F)$ is not.

Suppose neither $e_1$ nor $e_2$ are edges of $H$. By the previous reasoning, if at least one of them is added as a chord into $H$, or each is added into as a leaf into a different face of $H$, then $S(G)$ is a refinement of $S(H)$. If both are added as a


(a) An example of $G$ and its reduction $H$ such that $H \subset G$ but both are critical in respect to inherited precolored vertices. Precolored vertices are indicated in black, $f$ is the reduced face in $G$. Both graphs are colorable if and only if colors do not repeat on the precolored vertices.


(b) The arrow graph (on left) and the crinkle (on right)

leaf into the same face, it grows in length by 4. Note that if $e_1$ and $e_2$ neighbor around $x$, then uncollapsing $F$ back into $f$ produces $H'$ such that $H \subset H' \subset G$. By the reasoning above, this is not possible unless $x$ is precolored but does not map to a precolored vertex via the injection of $H$ into $G$. If $e_1$ and $e_2$ do not neighbor around $x$, but are in the same face, then $x$ is an articulation. □

Let *arrow graph* be the graph on 4 vertices composed of a triangle and an edge connecting the fourth vertex to the triangle.

Let *crinkle* be the plane graph obtained from a 4-face $f$ by attaching a pair of paths of length 3 connecting its opposite vertices (outside of $f$), necessarily crossing to form a single triangle $T$ sharing an edge with the 4-face. We say that the 4-face $f$ and the triangle $T$ form a crinkle. See Figure 2.7b for reference.

**Lemma 62.** *Let $G$ be a plane graph with some vertices of $G$ precolored, in particular all vertices of triangles are precolored, let $f$ be a 4-face of $G$ and $T$ a triangle sharing a single edge with $f$. Suppose further that all precolored vertices that are not vertices of $T$ are at distance at least 2 from $T$. Then there exists a 4-face $f_0$ sharing edge with $T$ (possibly $f_0 = f$) that can be collapsed without forming a new triangle or unifying any precolored vertices. Furthermore, after the collapse, the distance between $T$ and the closest precolored vertex decreases by at most 1.*

*Proof.* Since $f$ shares an edge with $T$, all of the other vertices of $f$ are at distance at most 1 from $T$, and therefore none of them are precolored. Suppose $f$ itself does not satisfy the requirements of the lemma, then each direction of collapse of $f$ must form a triangle. Consider the two implied paths of length 3 connecting the opposite vertices of $f$. By the definition of a crinkle, $f$ forms a crinkle, necessarily with $T$ as no other triangle can share an edge with $f$ by the assumptions of the lemma.

Let $f_0$ be either of the 4-faces of the crinkle other than $f$. We observe that collapsing $f_0$ by unifying the vertex $x$ shared by $f$ and $T$ with a vertex of the outer face cannot form a triangle. Indeed, any path of length 3 connecting the opposite vertices of $f$ would necessarily use at least one of the edges of $f$ or $T$ to reach the outer face, but then in all cases forms a triangle in the outer face.

Since $f_0$ shares an edge with $T$, no other vertices of $f$ are precolored. We conclude that $f_0$ satisfies the statement. Note that in both cases, namely collapsing $f$ or $f_0$ as indicated, the vertex unified with a vertex of $T$ is at distance 1 from $T$, which implies the final part of the statement. $\square$

**Lemma 63.** *Let $G$ be a plane graph with triangle $T$ bounding a ring, otherwise with no triangles and $\{T, v\}$-critical where $v \in V(G) \setminus V(T)$. If $d(T, v) \leq 1$ or all 4-cycles in $G$ separate $T$ from $v$, then $G$ is the arrow graph.*

*Proof.* Suppose $d(T, v) = 1$ and let $uv$ be the edge connecting $T$ and $v$. We double the edge $uv$ and split the vertex $u$ into two vertices so that the 2-face enclosed by the two parallel copies of $uv$ merges with the ring bound by $T$. We obtain a plane graph with precolored 5-cycle $C$ bounding the new ring, critical in respect to $C$. According to Lemma 18, such graph is equal to the cycle $C$. We conclude that $G$ contains no other elements other than $T$, $v$ and $uv$, and is therefore an arrow graph.

For the second case, we use Lemma 60. Let us consider $T$ and $v$ as two rings of length at most 3, one of which is clearly vertex-like. We conclude that $G$ necessarily follows the first case of Lemma 60, implying it is an arrow graph. $\square$

**Lemma 64.** *Let $G$ be a plane graph with a triangle $T$ and otherwise without triangles. For every precoloring $\psi$ of $T$ and every $v \in V(G) \setminus V(T)$ there exists a color $c$ such that if $\psi'$ is obtained from $\psi$ by additionally setting $\psi'(v) = d$ for an arbitrary color $d \neq c$, then $\psi'$ extends to coloring of $G$.*

*Proof.* We proceed by induction on the number of vertices. For contradiction, let $G$ be the smallest counterexample, in particular $G$ is $\{T, v\}$-critical. Let us first note that by Lemma 18, any precoloring of the triangle $T$ extends to the whole $G$. If there is an edge connecting $v$ to $T$, then the statement holds by Lemma 63, which implies that if $v$ is colored consistently with its only neighbor on $T$, then the coloring always extends. We may therefore assume that $d(T, v) \geq 2$. If every 4-cycle in $G$ separates $T$ from $v$, then by Lemma 63 the same conclusion holds as well.

By the above, we may fix a 4-face $f$ of $G$. If $f$ can be reduced so that the resulting graph $H$ is triangle-free apart from $T$, and $v$ is not unified with a vertex of $T$, then by the induction hypothesis, the claim holds on $H$ for $v$ and some color $c$. Consider the two colorings of $H$ obtained by extending precoloring with colors other than $c$ on $v$. By the natural extension of these two colorings, the claim also holds on $G$. We may therefore assume that no 4-face can be collapsed without forming a triangle or unifying $v$ with $T$.

Each direction of collapse of $f$ must either produce a triangle or merge $v$ with $T$. If both directions produce a triangle, then $f$ must be part of a crinkle in $G$, together with $T$. By Lemma 62, there exists a face $f_0$ sharing an edge with $T$ such that at least in one direction of its collapse does not produce a triangle and maintains that $d(T, v) \geq 1$. We conclude that $f_0$ can be reduced, a contradiction. It follows that for each 4-face $f$, collapse in one direction must unify $v$ with a vertex of $T$ and the collapse in the other direction must produce a triangle.

Let $v, x, a, y$ be the facial walk of $f$, where $a \in V(T)$ and let $P$ be a path of length 3 connecting $x$ to $y$, and let $C$ be the 5-cycle composed of $P$ and $v$. Without loss of generality, $T$ lies in the interior of $C$. Consider the exterior of $C$, since it contains no precolored vertices, it must be critical in respect to $C$ and is therefore just a 5-face, by Lemma 18. Consider splitting the vertex $a$ into two so that the face bound by $T$ and the face $f$ merge into a single 7-face $f'$. Since all precolored vertices now lie on the boundary of $f'$, the adjusted graph $G'$ must be critical in respect to $f'$ and therefore $G$ is critical in respect to the corresponding cell $D$ of length 7. By Lemma 18, the census of the exterior of $D$ is exactly $\{5\}$, and therefore the triangle bound by $T$ and the outer 5-face are the only faces in $G$ of length other than 4. In particular, the interior of $C$ contains a 4-face other than $f$ which is not incident with $v$ and therefore can be reduced, as neither direction of its collapse can unify $v$ with a vertex of $T$, a contradiction. □

**Lemma 65.** *Let $G$ be embedded in the cylinder with rings $T_1, T_2$ of lengths 3 and otherwise without triangles. If $G$ is $\{T_1, T_2\}$-critical, then any articulation in $G$ is a vertex of $T_1$ or $T_2$ and separates $T_1$ from $T_2$.*

*Proof.* We proceed by contradiction. Let $v$ be an articulation in $G$. Suppose $v$ is not separating $T_1$ from $T_2$. By criticality of $G$, each side of the cut can be colored independently. The colors of the part with no precolored vertices can be permuted, obtaining a coloring compatible with the coloring of the other part, and consequently a coloring of $G$, which is a contradiction.

Otherwise suppose $v$ is an articulation separating $T_1$ from $T_2$ but not a vertex of either of the triangles. We use Lemma 64 on both sides of the cut and obtain two colorings for each side with different colors on $v$. Clearly, there is a pair of colorings matching on $v$ combining into a coloring of the whole graph $G$, a contradiction. □

**Lemma 66.** *Let $G$ be a plane graph with a ring $T$ of length 3, otherwise without triangles and containing an articulation. If $G$ is $\{T, v\}$-critical where $v \in V(G) \setminus V(T)$, then $G$ is the arrow graph. In particular $S(G) = \{5\}$ and the distance between $T$ and $v$ is 1.*

*Proof.* Let $w$ be the articulation of $G$. Consider gluing a triangle $T_2$ to $v$ and precoloring it. The resulting graph is $\{T, T_2\}$-critical and therefore by Lemma 65 $w$ is a vertex of $T$. We split $G$ into two subgraphs $G_1$ and $G_2$ along $w$, so that $T$ is contained in $G_1$, $v$ is contained in $G_2$ and $w$ is part of both subgraphs. The subgraph $G_1$ is $T$-critical, and therefore by Lemma 18 $G_1 = T$. Clearly, $G_2$ is $\{w, v\}$-critical, but then by Lemma 63, $G_2$ is exactly the edge connecting $u$ and $v$. We conclude that $G$ is the arrow graph and consequently $S(G) = \{5\}$. □

**Lemma 67.** *Let $G$ be a plane graph with a ring $T$ of length 3 and otherwise without triangles. If $G$ is $\{T, v\}$-critical where $v \in V(G) \setminus V(T)$, then $S(G) \in \mathcal{S}_\Delta = \{\{5\}, \{5, 5, 5\}\}$.*

*Proof.* We proceed by induction on the number of vertices. By Lemma 66 we can assume that $G$ has no articulation and by Lemma 63 $G$ has at least one 4-face.

We repeatedly use the following argument. Suppose $G$ contains a 4-face $f$ such that collapsing it in at least one direction neither produces a new triangle nor lowers the distance between $T$ and $v$ down to 1. Let $H$ be a graph obtained by a reduction in this direction. Then by Lemma 66, $H$ does not have an articulation and by the induction hypothesis, $S(H) \in \mathcal{S}_\Delta$. By Lemma 61 $S(G)$ is a refinement of $S(H)$ unless one of the unified vertices was $v$. If it was not, in particular if $v$ is not incident with $f$ at all, we conclude that therefore $S(H) = S(G)$, a contradiction. We refer to this argument by saying that $f$ *admits amplification*.

We now prove a series of claims.

(1) The distance of $T$ and $v$ is at least 3

By Lemma 63 the distance $d(T, v)$ is at least 2. Suppose it is exactly 2. We proceed similarly to the proof of Lemma 63. We double both edges along a shortest path from $T$ to $v$ and then split the two vertices other than $v$ so that the two 2-faces merge with the ring bound by $T$ into a new face of length 7, which is a new ring. The resulting graph is critical in respect to the 7-cycle bounding the ring, and by Lemma 18, the census of the graph is $\{5\}$. Since this surgery did not change any faces other than the ring, we conclude that $S(G) = \{5\}$.

(2) $T$ does not share an edge with a 4-face

Suppose for contradiction $f$ is a 4-face sharing an edge with $T$. By (1), after collapsing $f$ in either direction $d(T, v) \geq 2$. Unless $f$ admits amplification, both directions of collapse of $f$ must produce a triangle and therefore $T$ and $f$ form a crinkle. By Lemma 62, there is a face $f_0$ sharing edge with $T$ which admits amplification, a contradiction. Hence (2) holds.

We say that a 4-face $f$ is *unsafe* if one pair of its opposite vertices $a, b$ is connected by a path $P$ of length 3 and in the other pair, one of the vertices is $v$ and the other vertex $x$ is part of a separating closed walk $C$ of length at most 5 with $T$ located on the other side of $C$ than $f$, and $C \cap P = \emptyset$. Without loss of generality, we assume $T$ is in the interior of $C$. See Figure 2.8a for illustration.

(3) There exists a safe 4-face

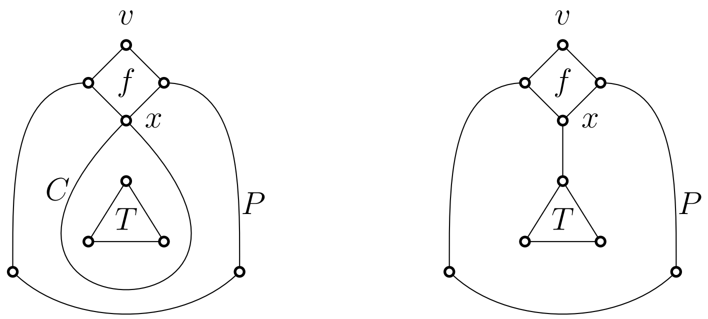
Suppose every 4-face is unsafe and let $f$ be an unsafe 4-face. We use the notation as specified in the definition above. Note that indeed the cycle formed by $P$ and $x$ separates $v$ from $T$ as otherwise it would itself be a separating 5-cycle with all precolored vertices on the same side, which is not possible according to Lemma 18. Similarly, the exterior of the cycle formed by $P$ and $v$ is critical in respect to the 5-cycle and is therefore necessarily empty, in other words an outer 5-face. On the other hand, the interior of the cycle formed by $P$ and $x$ does not contain any 4-face as such 4-face cannot be incident with $v$ and would therefore be safe, contradicting assumption of the subclaim.

The facial walk of $f$ together with $C$ form a cell $W$ of length at most 9, such that the whole exterior of $W$ is critical in respect to $W$. We use Lemma 17 which for setting $k \leq 9$ implies that the exterior of $W$ contains at most one vertex. This contradicts presence of $P$, which has two such vertices.

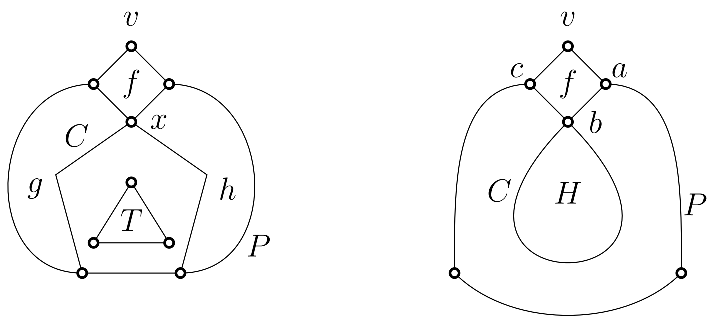
(4) There is no safe 4-face incident with $v$

For contradiction suppose $f$ is a safe 4-face incident with $v$. Let us denote the facial walk of $f$ as $vabc$. If both directions of reduction of $f$ form a triangle, $f$ and $T$ must form a crinkle, which contradicts (1). If one direction of collapse reduces the distance between $T$ and $v$ to 1, then $T$ is connected via an edge to $b$, and the other direction of collapse must form a triangle. However, this configuration cannot happen as then $f$ would be a special case of unsafe 4-face, see Figure 2.8a for illustration.

We can therefore reduce $f$ without forming a triangle and maintaining that




(a) Unsafe 4-faces $f$. General case, as defined in Lemma 67 on the left, and a special simple case on the right.




(b) Construction illustrations. On the left $H'$ in Lemma 67, subclaim (4), case where $C \cap P$ is an edge. On the right, $H_1$ in the final argument of Lemma 68

$d(T, v) \geq 2$. However since we may assume $f$ does not admit amplification by unifying $a$ and $c$, there is a path $P$ of length 3 connecting $a$ and $c$. We reduce $f$ by unifying $b$ and $v$ obtaining a graph $H$ in which the distance between $T$ and $v$ is at least 2, and therefore $H$ is 2-connected by Lemma 66. Let $F$ be the path into which $f$ collapses in $G$ during reduction. Let us return $F$ back into $H$. Since $f$ does not admit amplification, by Lemma 61 it must be the case that both edges of $F$ are missing in $H$ and both are added into the same face of $H$. Note that this face is of length at most 5, by induction hypothesis. Without loss of generality we assume it is the outer face of $H$.

Let $H'$ be obtained from $H$ by uncollapsing the face $f$ from $F$ and adding the path $P$ connecting $a$ and $c$. Note that $H'$ contains $H$ as a subgraph, bound by a cell $C$ of length at most 5. Also note that the cycle formed by $P$ and $v$ bounds an outer 5-face, as otherwise it would be a separating 5-cycle with all precolored vertices on the same side. If $C$ and $P$ are disjoint, we conclude that $f$ is in fact an unsafe 4-face, contradicting the assumption of the claim.

Let $g, h$ be the two faces of $H'$ in the exterior of $C$ other than $f$ and the outer 5-face. We bound their lengths by counting the number of edges of $P, C$ and $f$ that may bound them. If $P \cap C$ contains an edge, then $|g| + |h| \leq 8$ and therefore both are 4-faces, see Figure 2.8b for illustration. In particular, equality must hold and consequently $C$ has length 5, implying the outer face of $H$ is a 5-face. We can see that $S(G) = S(H)$ as $G$ is obtained from $H$ by filling its outer 5-face with three 4-faces and a 5-face. If $P \cap C$ is a vertex, then $|g| + |h| \leq 10$. Without loss of generality let $h$ be incident with the middle vertex $w$ of $P$ that is not a vertex of $C$. The degree of $w$ must be at least 3 by criticality and therefore $h$ is not a face of $G$, but rather a separating cell. By Lemma 18, $|h| \geq 6$, we conclude that $|h| = 6$, $|f| = 4$, and the interior of $h$ is quadrangulated in $G$. As before, the length of $C$ must be exactly 5 and $S(G) = S(H)$. We have a contradiction with $G$ being an counterexample and therefore (4) must hold.

We may therefore assume that (4) holds on $G$ as otherwise the claim of the lemma holds for $G$.

(5) There is no safe 4-face incident with $T$

For contradiction suppose $f$ is a safe 4-face incident with $T$. By (2) the incidence is a single vertex $x$. Let us denote the facial walk of $f$ as $xabc$. We may assume that $f$ does not admit amplification by unifying $a$ and $c$, and therefore there is a path $P$ of length 3 connecting $a$ and $c$. Note that $T$ and $v$ must be on the opposite sides of the 5-cycle formed by $P$ and $b$.

Suppose further that $b$ is connected to $v$ via an edge. Consider the graph $H$ consisting of $T$, $f$, $P$ and the edge $bv$. The face incident with $v$ is of length 7 and is not a face of $G$ since $b$ is not an articulation. Since it is also critical in respect to its boundary, the census of the interior of the face is exactly $\{5\}$ by Lemma 18. The cell incident with $T$ is of length 8 and is similarly not a face of $G$ since $x$ is not an articulation, and therefore its census is either $\{5, 5\}$ or $\{6\}$ or $\emptyset$. By (2) the census cannot be $\emptyset$ and if the census was $\{6\}$, then by (2) the triangle $T$ neighbors only with the 6-face implying $x$ is an articulation, which is not possible. We conclude that $S(G) = \{5, 5, 5\}$, a contradiction.

In the remaining case, $b$ is not connected to $v$ via an edge. Since $f$ and $T$ do not form a crinkle, we can reduce $f$ by unifying $x$ and $b$ without any triangle forming, and additionally the distance between $T$ and $v$ remains at least 2. This shows that $f$ does admit amplification, which is a contradiction and therefore (5) holds.

Finally, we formally draw the final contradiction to prove the lemma. Together the subclaims (3), (4) and (5) imply that there exists a face $f$ in $G$ not incident with any precolored vertices. In particular, reducing such a face cannot decrease the distance of $T$ and $v$ down to 1. Also, at most one direction of collapse of $f$ may produce a triangle, as otherwise $f$ forms a crinkle, necessarily touching $T$. We conclude that $f$ admits amplification, and therefore $S(G) \in \mathcal{S}_\Delta$, a contradiction. □

Note that there is only one case of the proof where the census of $G$ is $\{5,5,5\}$. We believe that this case can be further analyzed to show that in fact the census is always $\{5\}$. However for our purposes it suffices that the largest element of the census is 5 and that the number of elements is small. Therefore we do not craft further expansion of the already tedious argument.

We say that a multiset of integers is *quasimodest* if it can be obtained as a union of three parts, a subset of $\{5,5,5\}$, a subset of an element of $\mathcal{S}_{4,8}$ and a subset of an element of $\mathcal{S}_{4,9} \cup \mathcal{S}_{4,10}$. In particular, if $X$ is a quasimodest multiset, then its maximum element is at most 8, and its size is at most 12.

**Lemma 68.** *Let $G$ be embedded in the cylinder with rings $T_1, T_2$ of lengths 3 and otherwise without triangles. If $G$ is $\{T_1, T_2\}$-critical, then $S(G)$ is quasimodest.*

*Proof.* Let us proceed by induction on the number of vertices. Let $G$ be a minimal counterexample. If $G$ has no 4-face, then according to Lemma 60 $S(G) \in \mathcal{S}_{4,10}$ and is quasimodest by definition. Therefore, we may assume that $G$ has at least one 4-face.

We now prove a series of claims.

(1) If $G$ is as in the statement of the lemma and contains an articulation $x$, then $S(G) \in \{\{8\},\{8,5,5\},\{7,5\},\{7,5,5,5\}\}$.

According to Lemma 65 and without loss of generality $x$ is a vertex of $T_1$. Let us split $G$ using the cut vertex $x$. The side containing $T_1$ contains no precolored or cut vertices other than $T_1$, therefore it is critical in respect to $T_1$, and by Lemma 18 equal to $T_1$, implying that $T_1$ is a vertex-like ring in $G$. Consider deleting the two vertices of $T_1$ other than $x$ from $G$. We obtain a graph which is $\{T_2, x\}$-critical and by Lemma 67 has census $\{5\}$ or $\{5,5,5\}$. We obtain $G$ by gluing a triangular ring back to $x$, increasing length of one face by 3 (possibly of a 4-face which is not represented by an element of the census). We conclude that $S(H) \in \{\{8\},\{8,5,5\},\{7,5\},\{7,5,5,5\}\}$, all of which are quasimodest and therefore (1) holds.

Similarly to Lemma 67, we say that a 4-face $f$ *admits amplification* if in at least one direction of reduction no new triangles or new articulations are formed. Suppose $H$ is a graph obtained by a reduction in this direction. By the induction hypothesis, $S(H)$ is quasimodest and by Lemma 61 $S(G)$ is a refinement of $S(H)$. Since quasimodesty is closed under refinement, we conclude that $S(G)$ is also quasimodest, a contradiction. Note that to avoid articulations, it suffices if none of the two unified vertices is a vertex of $T_1$ or $T_2$, by Lemma 65, as applied to $H$.

(2) $d(T_1, T_2) \geq 3$

Suppose for contradiction it is at most 2. We double all edges along the shortest path and split all of its vertices so that the 2-faces and both rings merge into a single face of length at most 10, which becomes the new ring. The rest of the graph is critical in respect to this ring and therefore, depending on its length, by Lemma 18 we have that $S(G)$ is an element of $\mathcal{S}_{4,10}, \mathcal{S}_{4,8}$ or $\mathcal{S}_{4,6}$, the latter two of which are subsets of $\mathcal{S}_{4,10}$ and therefore $S(G)$ is necessarily quasimodest, a contradiction.

(3) There is no 4-face sharing edge with $T_1$ or $T_2$ in $G$.

Suppose $f$ shares an edge with $T_1$, without loss of generality. We may assume $f$ does not admit amplification. If collapsing $f$ in each direction produces a triangle, then $f$ and $T_1$ form a crinkle and by Lemma 62 there exists a face $f_0$ sharing edge with $T_1$ and $f_0$ can be collapsed without forming a triangle; we take $f_0$ instead of $f$. Again, we may assume that $f$ does not allow amplification and therefore reducing $f$ in at least one direction must produce $H$ with an articulation $x$. Since $G$ is minimal counterexample, $S(H)$ is quasimodest. Since quasimodesty is closed under refinement, if $S(G)$ is a refinement of $S(H)$, we have that $G$ is not a counterexample and we reach a contradiction. Let $F$ be the path into which $f$ collapses in $G$. By Lemma 61 it must hold that neither edge of $F$ is an edge of $H$. However one edge of $F$ is an edge of $T_1$, a contradiction. We conclude that (3) holds.

Let $f$ be a 4-face, by (3) it has at most one precolored vertex, implying $f$ does not form a crinkle and therefore collapsing it forms a triangle in at most one direction of collapse. On the other hand, collapsing $f$ may produce an articulation in at most one direction as that can only happen when a vertex of $T_1 \cup T_2$ is unified with another, by Lemma 65. Let the facial walk of $f$ be $xabc$. We may assume that $f$ does not allow amplification. By the previous, we may assume $x \in V(T_1)$ without loss of generality and that there is a path $P$ of length 3 connecting $a$ and $c$.

Let us consider reducing $f$ by unifying $x$ and $b$ and denote the resulting graph $H$. Since we assume $f$ does not admit amplification, the resulting unified vertex $x$ must be an articulation in $H$. By (1), $S(H) \in \{\{8\}, \{8,5,5\}, \{7,5\}, \{7,5,5,5\}\}$.

Let $F$ be the path into which $f$ collapses. We may assume that $S(G)$ is not a refinement of $S(H)$, as otherwise the claim holds. By Lemma 61, we have that $H$ contains no elements of $F$ other than $x$. Let $H_0$ be the graph obtained from $H$ by returning $F$ into $H$, and adding back the path $P$ connecting $a$ and $c$. We further have that all of the new elements are contained within one 7- or 8-face of $H$, splitting it into two parts which we denote $A$ and $B$ where $A$ is the face incident with $T_1$. Let us determine the sizes of $A$ and $B$. The addition of $F \cup P$ adds 5 edges, and each edge contributes one unit of length to each of the faces $A$ and $B$. The face $A$ is bound by $T_1 \cup F \cup P$ and is therefore of length 8, while $B$ is of length either 9 or 10 depending on the length of the initial face being 7 or 8 respectively.

Let $H_1$ be a graph obtained by uncollapsing $F$ back into $f$ in $H_0$, see Figure 2.8b for illustration. Clearly $S(H_1) = S(H_0)$, while $H_1$ is a subgraph of $G$ and therefore $S(G)$ is a refinement of $S(H_1)$. Since $G$ has no articulation, neither face $A$ nor $B$ can be a face of $G$, as then either $x$ or $b$ would be an articulation in $G$. By filling both faces $A$ and $B$ in $H_1$ appropriately, we obtain $H_2$ which is a subgraph of $G$. The census $S(H_2)$ is union of three parts, first part is a subset of $\{5,5,5\}$ depending on the census of $H$, second part is an element of $\mathcal{S}_{4,8}$ representing the filling of $A$, and the last part is an element of either $\mathcal{S}_{4,9}$ or $\mathcal{S}_{4,10}$

representing the filling of $B$. In particular, $S(H_2)$ is quasimodest by definition. Since each part is on its own closed under refinement, and $S(G)$ is obtained as a refinement of $S(H_2)$, we conclude that $S(G)$ is also quasimodest.

$\square$

### Non-contractible Triangles, Algorithms

To process graphs critical in respect to two precolored triangles, we need analogue of the algorithm from Lemma 45. We say that a graph $G$ is a *segment* if $G$ is a graph 2-cell embedded in the cylinder with both rings of length 3 and otherwise triangle-free.

**Lemma 69.** *There exists an algorithm which for any segment $G$ with disjoint rings, $F$ a set of faces of $G$ (other than rings), $k$ an integer such that $k \leq 9$ satisfying one of the following conditions:*

- $|F| = \{f\}$, $k \leq |f| + 1$
- $|F| \leq 3$, *every $f \in F$ is 4-free*

*decides whether $F$ is $k$-free. If $F$ is not $k$-free, the algorithm returns a certifying separating cell $C$ such that $|C| \leq k$ and $C$ itself is $|C|$-free. The algorithm runs in time $\mathcal{O}_k(|G|)$.*

*Proof.* The theoretical part of the construction is identical to that of Lemma 45, although technical details differ.

Similarly to the construction in Lemma 45, we want to unravel the cylindrical graph into a repeating structure. This time, we need to repeat in only one axis, as cylinder is cyclic in only one direction. Let $P$ be a (shortest) path connecting the rings $T_1$ and $T_2$ of $G$. We cut $G$ along $P$, obtaining a planar graph embedded into a rectangle bound by two copies of $P$ and walks of $T_1$ and $T_2$, a *semi-tile* $T$. Analogously to the definition of a tiled grid of size $k$, we define a *tiled strip* of size $k$ as follows. We take $2k + 1$ copies of $T$ arranged into a strip so that the edges of the embedding rectangles coincide and unify the overlapping embeddings of the copies of $P$. The strip has a natural projection back into $G$. Analogously to Observation 43, we observe that a closed walk of length $k$ beginning in the central tile is contained within tiles at distance at most $k/2$.

Let $H$ be a tiled strip of size $k/2$ and $c$ its central tile. We build the network from $H^\star$ analogously to the construction in Lemma 45, with the source being the outer face $g$ of $H$, all capacities 1, and the sink connected via an edge of limited capacity to various substructures.

As in the Lemma 45, we branch into several networks in the case when $F = \{f\}$ and $k \geq |f|$. We iterate over at most $k + 1$ neighbors of $f$, however exclude $g$ in the case when $f$ is incident with a ring in $G$.

Similarly, in the case when $|F| \geq 2$, we iterate over various projections of elements of $F$ other than the first one into $H$. Let us note that in the case where $|F| = 3$, we may restrict iteration over all projections where the two faces projected with the largest distance are mutually within a span of the potential separating closed walk implied by the analogue of the Observation 43.

Once we run the max-flow algorithm, we either obtain a flow of size $k + 1$ or we extract the minimum cut $S$ of size at most $k$ closest to $g^\star$. Unlike in the

Lemma 45, $S$ may consist of edges incident with $g^\star$ originating from the rings of $G$. This does not constitute a problem as these edges correspond to a valid border of $G$. We obtain a closed walk in $H$ corresponding to $S$ and project it back into $G$. By the same argument as in Lemma 45, we argue that the obtained structure is a separating cell. □

**Lemma 70.** *There exist an algorithm which for any given segment $G$ with disjoint rings $T_1, T_2$, terminates in time $\mathcal{O}(|G|)$ and either*

- *Produces a quasimodest subgraph $H$ such that $T_1, T_2 \subset H$ and every precoloring $\psi$ of $T_1 \cup T_2$ extends to a 3-coloring of $H$ if and only if it extends to $G$, or*
- *determines that every precoloring $\psi$ of $T_1 \cup T_2$ extends to a 3-coloring of $G$.*

*Proof.* The algorithmic approach is practically identical to that of Lemma 46. We build a set $F$ of faces with high freeness, so that if the set exceeds certain limit, we may certify that $G$ does not contain a $\{T_1, T_2\}$-critical subgraph and therefore all precolorings extend. The main difference is that we are now guided by Lemma 68 and quasimodesty. The key observation bring that if $G$ contains a critical subgraph $H$, with all faces of length at most 8, then any 8-free ($\geq 5$)-face of $G$ contributes to a census of $H$.

Lemma 18, describes the only cases when a ($\geq$ 5)-face may be contained in a separating cell of length at most 8. In particular, the faces in $F$ should maintain the following properties:

- Every 5-face is 6-free
- Every pair of 5-faces is 7-free
- Every triple of 5-faces is 8-free
- Every 6-face is 7-free
- Every pair consisting of a 6-face and a ($\geq 5$)-face is 8-free
- Every ($\geq 7$)-face is 8-free

Before running the procedure, we remove interiors of separating 4-cycles, if any exist, using the algorithm from Lemma 44. We start with the initial $G$ and an empty set $F$ and iteratively use Lemma 69 to test unprocessed ($\geq 5$)-faces $f$ of $G$ for $\min\{|f| + 1, 8\}$-freeness. If a separating cell is found, we delete its interior and add the new resulting face into $F$, otherwise we add $f$ into $F$. If the face being added is a ($\geq 9$)-face, the quasimodesty is violated and we terminate instead. Note that the added face is always at least free.

After adding the new face into $F$, we need to maintain the above listed properties of single faces, pairs and triples. We always test the conditions on single faces first, only moving to pairs once all individual conditions holds, and then to triplets once all conditions on pairs hold. When a new face is obtained from a separating cell, it is only free, and so we test whether its freeness is high enough, possibly replacing it again with a longer face and repeating until the test is successful. If the new face is a 5-face or a 6-face, we test it in combination with the faces already present in $F$. As before, if a short separating cell is found, we delete its interior and continue with the new face instead. Note that whenever the tests of pairs or triples detect a short separating cell, it is always longer than each of the faces from $F$ contained in its interior, at least by 2 due to their individual freeness.

If the set $F$ satisfies all of the properties above, and it is not quasimodest, we terminate and conclude that $G$ admits all precolorings to extend. Otherwise we continue processing faces of the remainder of the graph until all ($\geq 5$)-faces are in $F$ and we return the remainder of the graph.

We show that $F$ grows lexicographically in every iteration. Notice that the first step of each iteration either adds a new element into $F$ or terminates the algorithm. Consider the multiset of lengths of faces in $F$, sort them from highest to lowest value and compare resulting lists lexicographically. Clearly, whenever the set is extended by a new element, it also grows in the lexicographical sense. It may happen that some elements of $F$ are removed when the algorithm deletes the interior of a separating cell. Since all faces in $F$ are free, the separating cell must be of length strictly higher than each of the removed elements, implying that $F$ grows in the lexicographical sense.

At the beginning of each iteration, $|F| \leq 12$, as otherwise the quasimodesty is violated and we terminate. Each iteration consists of several freeness tests, where each test runs in time $\mathcal{O}(|G|)$ (for faces of limited size) and the number of tests depends on the current size of $F$. Therefore, each iteration runs in linear time. As observed, the set $F$ grows lexicographically in each iteration, and therefore the number of iterations is limited by a constant. We conclude that the algorithm terminates in time $\mathcal{O}(|G|)$. $\square$

Let $G$ be a graph with a 2-cell embedding in the torus such that $G$ is 3-colorable and all triangles in $G$ are non-contractible, and let $T$ be an arbitrary triangle. Let $\psi$ be a partial coloring of $G$ such that all (non-contractible) triangles in $G$ that are homotopically equivalent to $T$ are colored by $\psi$ and $\psi$ extends to a 3-coloring of $G$. We say that the tuple $(G, \psi)_T$ is a *segmented toroidal graph.* We may take the set of all non-contractible triangles of $G$ homotopically equivalent to $T$ and cut $G$ along each. We obtain a set of cyclically ordered segments.

The triangle $T$ is only a technical way to ensure that the definition is consistent with its use, that is, we intend to use previously developed tools to work with individual segments and precolorings of their rings and only their rings. A triangle homotopically non-equivalent to $T$ might contain a vertex in the interior of a segment, causing problems if precolored by $\psi$. Also, cutting along such triangle would split $G$ into pieces that are not cylindrical.

Let $T_1, T_2$ be two homotopically equivalent triangles. We say that $T_1$ and $T_2$ are *matching* if the order of their vertices wraps around the surface (torus or cylinder) in the same order. More formally, if represented as oriented closed curves, these can be brought together so that their orientations match. The triangles are *mismatching* otherwise. We say that triangles have *matching* 3-colorings if the order of colors on the triangles matches, and mismatching 3-colorings if the order of colors does not match. In particular, if $T_1 \cap T_2 \neq \emptyset$, then $T_1 \cup T_2$ has at most two types of coloring, up to permutation of colors, one matching and possibly one mismatching.

**Observation 71.** *Let $G$ be a segment with rings $T_1, T_2$ such that $T_1 \cap T_2 \neq \emptyset$. Then a precoloring of $T_1 \cup T_2$ does not extend into a 3-coloring if $G$ if and only if $T_1$ and $T_2$ intersect in a single vertex, the colorings of $T_1$ and $T_2$ are mismatching, and $G$ is a quadrangulation (except its rings).*

*Proof.* First suppose that the rings $T_1$ and $T_2$ intersect in a single vertex $x$. The triangles together form a boundary $C = T_1 \cup T_2$ of a cell of length 6. Assuming that a precoloring of $C$ does not extend into $G$, Lemma 58 confirms the conclusion unless $C$ has a chord improperly colored by the precoloring. We observe that all vertices of $C$ are at distance at most 2 from each other and so any such chord would form a triangle distinct from $T_1$ and $T_2$, which contradicts $G$ being a single segment.

Let us now assume that $T_1$ and $T_2$ intersect in an edge. By the same approach, the rest of the graph is then a equivalent to a triangle-free planar graph critical in respect to its boundary of length 4, however no such graph exists according to Lemma 18. Therefore $G = T_1 \cup T_2$ and the given precoloring colors the whole $G$, a contradiction. □

**Lemma 72.** *There exists an algorithm such that given a segment $S$ and a precoloring $\psi$ of the rings of $S$ decides whether $\psi$ extends into a 3-coloring of $S$ in time $\mathcal{O}(|S|)$.*

*Proof.* First we use the algorithm from Lemma 46 on $S$ and if we obtain a quasi-modest reduced version of the segment $S'$, we proceed by algorithm from Theorem 15 applied to $S'$, otherwise $\psi$ is guaranteed to extend. Both algorithms run in time $\mathcal{O}(|S|)$. □

**Lemma 73.** *There exists an algorithm such that given a segmented toroidal graph $(G, \psi)_T$ produces a 3-coloring of $G$ in time $\mathcal{O}(|G|^2)$.*

Note that the produced 3-coloring does not necessarily extend $\psi$.

*Proof.* We proceed via recursion. If $\psi$ colors all vertices of $G$, we terminate. First we use the algorithm from Lemma 44 to delete interiors of all separating 4-cycles. If $G$ has no contractible 4-cycle $F$, we use the algorithm from Lemma 56 to obtain a 3-coloring. Note that in this case the obtained coloring does not extend $\psi$. Otherwise, we have a contractible 4-cycle $F$, which bounds a 4-face $f$. Note that $f$ is contained within a single segment $S$ of $G$.

By collapsing $f$ in (at least) one of the two possible directions we obtain a graph $G'$ such that $\psi$ extends to a 3-coloring of $G'$. Let $S$ be the segment containing $f$, $T_1, T_2$ the rings of $S$, and $S'$ the cylindrical subgraph of $G'$ corresponding to $S$. To decide whether $\psi$ extends into $G'$, we only need to test whether $\psi$ extends into $S'$.

If $S'$ contains a contractible triangle, we observe that $G$ contains a separating 5-cycle $C$ containing $f$ in its interior. According to Lemma 18 we may delete the interior of $C$ in $G$ instead of collapsing $f$. We recurse on the obtained graph instead and once its 3-coloring is obtained, we use the algorithm from Lemma 55 to extend it into the interior of $C$. We may therefore assume from now on that no contractible triangles appear in $S'$.

First suppose that there is no new non-contractible triangle homotopically equivalent to $T$ in $G'$, therefore $S'$ is a segment. We test whether $\psi$ extends to $S'$

using the algorithm from Lemma 72. If the precoloring does extend, we recurse on $(G', \psi)$. If the precoloring does not extend, we collapse $f$ in the opposite direction. We are of course guaranteed that the precoloring extends in this other direction, although we might end up in a different case of this algorithm.

Otherwise, let $x$ be the new vertex resulting from the unification of opposite vertices of $f$ and $\mathcal{T}$ be the set of all new non-contractible triangles homotopically equivalent to $T$. Note that each triangle in $\mathcal{T}$ contains $x$ and the triangles can be ordered by distance from $T_1$. We get a sequence of segments ordered from $T_1$ to $T_2$, separated by triangles in $\mathcal{T}$. We proceed in a manner of dynamic programming. In principle, we may test each segment to obtain the precolorings of its rings which extend into the segment, and combine these in order from $T_1$ to $T_2$ to obtain a consistent coloring of $\mathcal{T}$ that together with $\psi$ extends to $S'$.

First we deal with the (at most two) segments such that at least one of their rings does not intersect $x$, we say that these segments are *non-trivial*. To test which precolorings extend, for each we iterate over all precolorings of the ring that is not precolored by $\psi$ and test extendability using the algorithm from Lemma 72.

If $|\mathcal{T}| \geq 2$, it remains to deal with the trivial segments bound by two rings intersecting in $x$. We observe that for the purposes of combining with the non-trivial segments, all colorings of $\mathcal{T}$ are of only two distinct types, those that color the first and the last triangle in the matching way, and those that do not. It suffices to decide for each type whether such coloring exists. We use Observation 71. If all triangles in $\mathcal{T}$ are precolored in a matching way, the precoloring always extends into all intermediate segments. A coloring such that the first and the last triangle in $\mathcal{T}$ are colored in a mismatching way exists if and only if at least one trivial segment extends a mismatching precoloring of its rings; if so, we construct a precoloring extending into all segments where all up to one pair of consecutive triangles are precolored in a matching way. According to Observation 71 this fails if and only if all of the trivial segments are quadrangulated and each consecutive pair of triangles in $\mathcal{T}$ intersects in exactly $x$.

We combine all types of precolorings extending into the non-trivial segments consistent with $\psi$ together with the (at most) two options of precoloring of $\mathcal{T}$. If we get a consistent combination, we extend $\psi$ into $\psi'$ accordingly and recurse on $(G', \psi')$. Otherwise we conclude that $\psi$ does not extend to $G'$ and we collapse $f$ in the other direction.

Once we obtain the 3-coloring from the recursive call, we adjust it to $G$ by coloring vertices of $f$, and we possibly use the algorithm from Lemma 55 to extend the coloring into the deleted interiors of separating 4-cycles.

Each recursion except the last one runs in time $\mathcal{O}(|G|)$, where the leading contributor to the complexity is the testing of extendability. Note that when multiple segments are tested, testing on each segment is performed for a constant number of precolorings, and each test runs in time linear in respect to the size of the segment. The sizes of the segments sum up to at most $2\|G\|$. Each time, the size of the graph is reduced by at least one vertex, implying that the depth of the recursion is at most $\mathcal{O}(|G|)$. In the last recursion node, either the graph was reduced to a trivial size, or we call the algorithm from Lemma 56, which runs in time $\mathcal{O}(|G|^2)$. We conclude that the overall time complexity is $\mathcal{O}(|G|^2)$. $\square$

### Coloring in Torus

**Theorem 74.** *There exists an algorithm that given a triangle-free graph G with a 2-cell embedding in the torus, decides whether G is 3-colorable and produces a 3-coloring in time $\mathcal{O}(|G|^2)$.*

*Proof.* First we may test whether $G$ is 3-colorable, using the algorithm from Theorem 50, and reject $G$ if it is not colorable. If $G$ is planar, we run the algorithm from Theorem 51.

As an overarching strategy, we iteratively reduce $G$ by collapsing its 4-faces, until we are able to obtain a coloring of the reduced graph. Then we unfold the reduced graph back to $G$, extending the 3-coloring. Along the way, we remove vertices of degree 2 and interiors of separating contractible ($\leq 5$)-cycles, which can then be colored during the unfolding.

If $G$ has no contractible 4-cycle $F$, we use the algorithm from Lemma 56 to obtain a 3-coloring. Otherwise, we may assume that $F$ bounds a 4-face $f$, by Lemma 18.

Collapsing $f$ in at least one direction must produce a 3-colorable graph $G'$. First assume that $G'$ contains no triangles. We use the algorithm from Theorem 50 to determine whether $G'$ is 3-colorable and if it is, we recurse on $G'$, otherwise we reduce $f$ in the other direction.

If $G'$ contains a contractible triangle, then we observe that $G$ contains a separating 5-cycle $C$ containing $f$ in its interior. We remove the interior of $C$ and recurse on the remaining graph instead. Once the recursive call produces a 3-coloring, we extend it into the interior of $C$ using the algorithm from Lemma 57.

Finally, suppose $G'$ contains a non-contractible triangle $T$. Note that if $G'$ is 3-colorable, then any proper 3-coloring of $T$ extends to $G'$. We fix any such coloring $\psi$. If $T$ is the only triangle in $G'$ homotopically equivalent to $T$, $(G', \psi)_T$ is a segmented toroidal graph, and we use the algorithm from Lemma 73 to finish the 3-coloring. If there are multiple triangles in $G'$ homotopically equivalent to $T$, we extend $\psi$ into precoloring of all of the triangles analogously to the approach used in Lemma 73, except that in this case we have only one non-trivial segment. Note that we may also get triangles that are not homotopically equivalent to $T$, however these triangles constitute no issue for our approach to coloring if segmented toroidal graphs, as no such triangle can exist as a whole within a single segment.

Each recursion node except the last runs in time $\mathcal{O}(|G|)$, where the leading contributors to the complexity are the testing of 3-colorability (or extendability of segments in the case of multiple homotopically equivalent triangles) and possibly extension of a 3-coloring into a separating cell of length at most 5. In each recursive call, the size of the graph is reduced by at least one vertex, implying that the depth of the recursion is at most $\mathcal{O}(|G|)$. In the last recursion node, we call one of the other algorithms, each of which runs in time $\mathcal{O}(|G|^2)$. We conclude that the overall time complexity is $\mathcal{O}(|G|^2)$. □

### 2.4.3 Algorithmic Remarks

There are essentially only a few types of procedures to our construction of algorithms. We group them into four types and discuss each in turn.

The core component is the testing algorithm from Theorem 15 testing 3-colorability, which is however only efficient if the input is ”4-critical-like”, that is, having a limited census and shortest non-contractible cycle. Naturally, the specific limits we put on the census are not essential for the working of the algorithm, however the complexity scales steeply when the limitations are loosened. In a typical application, in particular in the decision algorithm from Theorem 50, we iterate over all precolorings of the rings, where the number of configurations scales exponentially with the length of the rings. If we use analysis analogous to that from Section 2.2 to narrow down the prospective precolorings, however, the complexity of the test itself scales only linearly with the lengths of the rings.

We could in principle of course eliminate many more inputs in the Theorem 50 based on other properties observed in Sections 2.2 and 2.3 to be necessary for a graph to be non-3-colorable. For example, we may eliminate inputs where the ($\geq 5$)-faces are not close together. This may however require more nuanced processing tools, as a 4-critical graph may contain only two faces of lengths 5 and 7 that do indeed touch, but its modest super graph (obtained by Lemma 46) we work with may have only two 5-faces, one of which is deep inside the separating 7-cycle, not touching the other 5-face.

The second kind of procedures are those reducing the problem down to the ”4-critical-like” graphs, dealing with separating cells and separating cycles. In all constructions using the tiling of a grid, the sizes of the grids used are worst-case scenarios based on reasoning along the lines of Observation 42. In the argument we consider a walk across the grid where each edge of the walk is contained in some tile at distance $d$ from the central tile, and connects a vertex incident with a tile at distance $d-1$ to some tile at distance $d+1$. If no such shortcuts using either one or some small constant number of edges to cross the tile exist, we can greatly reduce the grid sizes. The presence of such shortcuts in a fixed tile can of course be tested in linear time (in respect to the size of the tile), and the grid designed dynamically, with different number of tile repetitions in each direction. We of course put no effort into constructing tiles in any special way, there is therefore freedom in designing the tile so that such shortcuts are avoided unless the input graph is somewhat degenerated.

As mentioned just after Lemma 48, the oracle construction used to detect short non-contractible cycles is even more powerful than we need to achieve the correct asymptotic complexity. If we are only interested in testing of 3-colorability, it is enough if we can either find a short non-contractible cycle of length at most $k_1$ or certify no non-contractible cycle of length at most $k_2$ exists. Note that while we set $k_1 = k_2 = 5$, this is not necessary. We only need $5 \leq k_1$ to safely eliminate only inputs that are 3-colorable by Lemma 40, while $k_2$ can be any constant, although as discussed above, the complexity of 3-colorability testing scales steeply with $k_2$ without additional improvements.

If we are interested in obtaining a 3-coloring, the full power of the oracle is useful in the sense that whenever we reduce the working graph by deleting the interior of a separating cell, we can update the oracle in constant time (per element), and when collapsing a 4-face, we can update it in time $\mathcal{O}(\log^k |G|)$.

Note that this requires us to maintain the tiled grid graph rather than the graph embedded in the torus.

On the other hand, when building a 3-coloring, we do not need the algorithm from Lemma 44 to detect and delete interiors of separating 4-cycles in linear time and we may use a simpler method instead. It suffices to iterate through all 4-faces in quadratic time, by checking each pair of vertices whether they have more than one common neighbor. We then find interior (if it exists) and delete it analogously to Lemma 44. It is then straight-forward to maintain that the working graph has no separating 4-cycles by checking only new 4-cycles containing the new vertex after each collapse, avoiding repeated runs of the procedure.

Third set of procedures are those used to reduce inputs to smaller subproblems in order to find a 3-coloring. When it comes to collapsing 4-faces, we choose arbitrarily the next 4-face to be processed. Given a large input graph without short non-contractible cycles, it is in principle possible to systematically collapse 4-faces so that no short non-contractible cycles appear and therefore we avoid the necessity to repeatedly test 3-colorability; it is guaranteed by non-existence of short non-contractible cycle. For example, once we construct a tile, we can perform any collapse such that the vertices being unified are at distance at least 3 from the borders of the tile. Since a non-contractible cycle passing through the new vertex must have preimage in the grid connecting two preimages of the new vertex to boundaries of distinct tiles, it cannot then be of length less than 6.

With more sophisticated approach in the spirit of templates from Section 2.3, we can even remove whole quadrangulated cells at once, replacing them with quadrangulations of at most quadratic size in respect to the length of the cell boundary. While this does not affect 3-colorability, the quadrangulation replacement must be chosen carefully so that any obtained coloring then extends into the interior of the original cell. The specific problems are described by Lemma 30.

When we work with the segmented toroidal graphs, it is in principle possible to use not only non-contractible triangles to separate the individual segments, but also longer cycles. Since we can reduce the graph easily until it has at least one non-contractible cycle of length at most 5, this might simplify the architecture of the algorithm, however the price would be a much higher limit on census, and consequently exponentially higher running time of the core testing algorithm in the worst case. Similar approach was employed by Dvořák and Lidický in [64], where the the graphs in consideration are segmented by ($\leq$ 4)-cycles. Their analysis of graphs critical in respect to two rings of length at most 4 suggests that the census analysis for segmentation by 5-cycles would be significantly more difficult.

The last type of procedures are the procedures producing 3-colorings in special cases. In particular, we use algorithms for 3-coloring planar triangle-free graphs and graphs of girth at least 5 or with no contractible ($\leq$ 4)-cycles. These algorithmic counterparts of classical results by Grötzsch are not described in this work in depth, however various constructions and improvements exist across literature. The efficiency and applicability of these procedures may differ heavily depending on the chosen variation.

## Final Remarks

Our approach does not seem to allow a particularly straight-forward generalization into higher surfaces. The most obvious struggle with a general surface $\Gamma$ is to determine its equivalent of modest census values, and more crucially methods to reduce general graphs down to the case with limited census. Recall that the general bound on census given by Dvořák, Král' and Thomas in [28] only implies upper-bound on census complexity for 4-critical graphs without non-contractible 4-cycles. While this is not an issue in the torus, 4-critical graphs in higher surfaces can contain unlimited number of (generalized) segments with non-empty census separated by non-contractible 4-cycles. A description of such graphs, called Thomas–Walls graphs, and their analysis can be found for instance in [29].

While these graphs can be analyzed using the algorithm from Theorem 15, even if the goal is to only test 3-colorability, our approach leads towards splitting the graphs into (generalized) segments, with potentially much richer structure of interaction in between segment boundaries.

Our ambition was to generalize our approach to decide 3-colorability in the Klein bottle as well, since graphs embedded in the Klein bottle after cutting along a single non-contractible cycle simplify into graphs embedded in the cylinder, similarly to the torus case. However, as can be seen from the effort needed to deal with segments separated by non-contractible triangles, dealing with Thomas-Walls graphs is a lot of additional effort, in our opinion not worth persuing without a more general result in mind or at least a particular interest in the Klein bottle 3-coloring problem.

In a recent, not-yet published result, Dvořák, Bang, Heath and Lidický came up with generalization of the methods behind Theorem 15 to allow testing 3-colorability of near-quadrangulations on general orientable surfaces. Recall, that in our setting, we describe vorticity as integer values, and aim to find a big-enough set of non-contractible cycles to guarantee that a nowhere-zero flow can be adjusted to be consistent with a given precoloring (prescribing a specific vorticity value). A direct generalization of such approach would represent vorticity as elements of a homology group and the necessary structure of cycles to guarantee sufficient adjustments would be rather complex, as contribution of flipping flow along each would represent an element in the homology group. It therefore came as a surprise to us, that a rather elegant solution exists through a change of perspective. This is a great step towards fast 3-coloring algorithms on general surfaces.

# 3. Induced Odd Cycle Packing Number

## 3.1 Induced Odd Cycle Packing Number and Geometric Representations

In this section we will explore the connection between the induced odd cycle packing number, a relatively new graph parameter, and geometric representability of a graph. The *induced odd cycle packing number* of a graph $G$, denoted $\mathrm{iocp}(G)$ is the maximum integer $k$ such that $G$ contains an induced subgraph consisting of $k$ pairwise vertex-disjoint odd cycles. Note that whenever two cycles are connected by an edge, their vertices do not induce such a subgraph.

To simplify the exposition, we will often require the iocp to be bounded in the complement of a graph. A *complement induced odd cycle packing number* of a graph $G$, denoted as $\mathrm{ciocp}(G)$ is defined as $\mathrm{iocp}(\bar{G})$ where $\bar{G}$ is the complement of $G$.

Bonnet et al. [65] proved that the intersection graphs of disks in the have their ciocp at most 1. They exploit this insight to derive several algorithmic applications to efficiently solve problems that are hard in general. Later, Bonamy et al. [66] proved that the intersection graphs of unit balls in 3-dimensional space also satisfy ciocp at most 1 with similar algorithmic consequences. Additionally, it fairly easy to observe the following:

**Observation 75.** *If $G$ is planar, then $\mathrm{ciocp}(G) \leq 1$.*

*Proof.* Suppose for contradiction that $\bar{G}$ has an induced subgraph $\bar{H}$ consisting of at least two odd cycles $C_1, C_2$. For $i \in \{1, 2\}$, let $S_i \subseteq V(C_i)$ such that $|S_i| = 3$. Clearly, $S_1 \cup S_2$ induces a supergraph of $K_{3,3}$ in $G$ implying it is not planar. □

We show that this phenomenon is more common. In particular we show broad classes of shapes, and sufficient geometric properties, that induce graphs of limited ciocp and are therefore algorithmically approachable via methods we later present in Section 3.2 as well as motivate further development of similar methods. We conjecture that the property of limited ciocp is in fact much more common than we can currently show.

### 3.1.1 Geometrically Represented Graph Classes

Let $\mathcal{X}$ be a finite set of geometric shapes in the plane. We say that $\mathcal{X}$ is a *geometric representation* of a graph $G$ if $V(G) = \mathcal{X}$ and for all pairs of $X, Y \in V(G)$, $XY \in E(G)$ if and only if $X \cap Y \neq \emptyset$. Conversely we say that $G$ is the *intersection graph* of $\mathcal{X}$.

For any *class of shapes* $\mathcal{C}$ we may define its *intersection class* of graphs, as the class of all graphs obtained as intersection graphs of arrangements of shapes from $\mathcal{C}$. We only consider classes $\mathcal{C}$ of reasonably behaved shapes, that is, composed of connected closed subsets of the plane, obtainable by deformation of closed full discs, or lines and simple (non-self-intersecting) continuous curves, possibly closed curves. We understand an *arrangement* of shapes from $\mathcal{C}$ to be obtained by arbitrary translations of arbitrary shapes from $\mathcal{C}$. We do not necessarily allow rotations or scaling. If we do allow rotations or scaling, we express this using $\mathcal{C}$. In such a case we say that $\mathcal{C}$ is closed under various transformation.

We say that a shape $S$ has *crossection* 0 if there exists a finite set of points $P$ such that $S \setminus P$ is disconnected. We say that $S$ has a *positive crossection* otherwise.

We want to explore properties of touching shapes in sufficient generality, for this reason we need to define touch of shapes in a rather technical manner. For basic geometric shapes of positive crossection, such as discs and squares, it is sufficient to say that a pair of shapes is touching if it intersects, but the intersection is contained in the boundaries of the shapes or equivalently is of volume 0. Such definition is insufficient for shapes of crossection 0, such as lines, line segments or circles.

We say that two shapes $A, B$ *touch* if they intersect, their intersection is of volume 0, and there exist two shapes $S_A, S_B$ topologically equivalent to full open discs such that $A$ and $B$ are contained in the closures of $S_A$ and $S_B$ respectively, and either $S_A \cap S_B = \emptyset$ or $S_A \subset S_B$ or $S_B \subset S_A$. Note that the formulation of relation between $S_A$ and $S_B$ cannot be narrowed to $S_A \cap S_B = \emptyset$ to accommodate shapes with holes.

If each pair of intersecting shapes in $\mathcal{X}$ only touches, we say that the represented graphs are *touching graphs* and we define the *touch classes* analogously to the intersection classes. Note that under our definition, every touch representation is an intersection representation. Often times in literature, an intersection representation is assumed to contain no pairs that only touch. Depending on $\mathcal{C}$, this can usually be enforced by scaling up the shapes by a small factor. Most shape classes we assume allow such scaling and avoid this nuanced distinction.

It is fairly easy to see that the touch classes almost always form subclasses of planar graphs.

**Observation 76.** *Let $G$ be a touching graph of connected closed shapes of positive crossection such that no three shapes touch in a single point. Then $G$ is planar.*

*Proof.* Suppose $G$ is touch-represented by connected closed shapes of positive crossection. We may construct a planar embedding of $G$, or rather of the incidence graph of vertices and edges of $G$. To represent edges, choose any point in the intersection of the relevant shapes, and represent each vertex as any point of the associated shape. It is now possible to connect each point representing a vertex to all point representing incident edges (which now lie on the boundary of the shape) by curves so that no two curves cross. Clearly, the same representation is also a planar drawing of $G$. □

Similar conclusion can be reached for touching graphs of lines, line segments and closed curves, by first continuously deforming the representation to ensure that all shapes have positive diameter. Note that since we do not restrict the drawing, it is not an issue that the shapes get distorted in the process.

It is natural to ask, whether the converse is true, that is, whether all planar graphs can be represented as touch graphs of some specific class of shapes.

Perhaps surprisingly, there are many classes of shapes that indeed allow representation of all planar graphs. Among the most prominent are discs and triangles. As shown by the Schramm's Monster theorem [67], a much more general settings of shapes, than the ones we choose to work with here, are sufficient. This demonstrates a very tight two-directional connection between planarity and touch-representability.

### 3.1.2 Intersection Classes

The intersection classes are substantially more complex. Clearly we may expect a typical intersection class to also contain all planar graphs (at least under the definition we use), which are in a sense already complex. Additionally, large cliques and even chordal graphs are typically representable subclasses. In the case of touch representations, is it easy to see that if we do not require the shapes to be connected, non-planar graphs may become representable. Similarly, we may ask interesting questions about representability via intersections, such as what natural conditions on the class of shapes, or their arrangements, impose an exploitable structure on the represented graphs, and how to quantify complexity of this structure. Conversely, what natural conditions still allow a wide-enough classes of graphs to be represented? Put together, a very interesting prospect is to represent usefully general graph classes, especially classes structured by some practical conditions, and exploit their structure or representation algorithmically. Naturally, these often exhibit as various generalizations of planar classes, and generalizations of algorithms for planar graphs.

We will now define a few interesting properties and later show their connection to the ciocp parameter. To avoid technical difficulties in this section, we define the types of shapes we work with in a more combinatorial way. We say that a shape is *simple*, if the shape is formed by the (closed) interior of a simple closed curve (by *simple* we mean that the curve does not self-intersect or touch itself). We call this curve a *boundary curve* of the shape. We say that a collection of shapes is *simple* if all shapes in the collection are simple and the curves never touch, that is, each time two boundary curves intersect, they also cross.

A representation is *non-pass*, if for every pair of shapes $A, B \in \mathcal{X}$ the part of $A$ outside $B$ (formally $A - (A \cap B)$) is connected. Intuitively, the non-pass property says that no shape can have part of it pass through another shape and emerge on the other side.

A representation is *strongly non-pass*, if for every shape $A$ and union of any number of shapes $\mathcal{B} = \cup_i B_i$, we analogously have that $A \backslash \mathcal{B}$ is connected.

If for two shapes $A, B$ we have $A \subset B$, we say that $A$ is *nested* in $B$. If any shape is nested, we say that the representation has *nested shapes*. We say that a representaion has *no nested (pairwise) intersections* if for every triplet of distinct shapes $A, B, C$, if $A$ and $B$ intersect, then $(A \cap B) - C$ is non-empty. We usually drop the 'pairwise' specification, as we always limit only the pairwise intersection in this sense.

Let $\mathcal{X}$ be a set of shapes in the plane. A point $p$ in the plane is of *thickness* $t$ if $p$ is contained in exactly $t$ shapes.

A representation has *no hidden intersections* if for every pair of intersecting shapes $A, B$, there exists a point in $A \cap B$ of thickness 2. In other words, the intersection is not nested in the union of all of the other shapes.

**Observation 77.** *A simple strongly non-pass representation with no hidden intersections and no nested shapes is equivalent to a touch representation (using more general shapes) and consequently the represented graph $G$ is planar.*

*Proof.* Let $A, B$ be two distinct shapes and $\mathcal{U}$ the union of all shapes other than $A, B$. Let $\mathcal{X}$ be the set of all points of thickness at least 3. For every shape $S$ let $S'$ be the restriction of $S$ to $S \setminus \mathcal{X}$. By the strong non-pass property and the fact that no shapes are nested, $A \setminus \mathcal{U}$ and $B \setminus \mathcal{U}$ are both connected, and so are $A'$ and $B'$.

Suppose $A \cap B \neq \emptyset$. Since no intersections are hidden, both restrictions $A'$ and $B'$ also contain a point from $A \cap B$. We conclude that $A'$ and $B'$ intersect. Since $A, B$ were chosen arbitrarily, the representation consisting of all of the restricted shapes $S'$ represents the same graph.

Let us now consider further restriction of shapes by deleting all points of thickness 2, restricting every shape $S'$ further to $S''$. The shape $A''$ remains connected and if $A$ intersects $B$, $A''$ touches $A' \cap B'$. We may therefore extend each such pair of shapes arbitrarily into $A' \cap B'$ so that they touch. We obtain a touch representation of connected shapes representing the same graph.

To obtain the embedding, for every shape $S$ we choose an interior point $c_S$ of thickness 1 representing its vertex. For each edge $e$ we choose a point $p_e$ where the corresponding shapes touch. Clearly, within each shape $S$ we can draw curves representing half-edges connecting $c_S$ and all of the points $p_e$ which lie on the boundary of the shape. All used points are point of thickness at most 2 and at least 1 in the original representation. $\square$

**Observation 78.** *Let $\mathcal{G}$ be a hereditary graph class. For any $G \in \mathcal{G}$ there exists $G' \in \mathcal{G}$ such that $ciocp(G) = ciocp(G')$ and no shapes in $G'$ are nested.*

*Proof.* Let $G$ be the represented graph and $H$ its induced subgraph equal to the complement of disjoint union of odd cycles. If $u, v$ are vertices of $H$, then each has exactly two non-neighbors in $H$ and this pair of non-neighbors is unique. If the shape representing $u$ is contained in the shape representing $v$, then $N[u] \subseteq N[v]$, in particular every non-neighbor of $v$ is a non-neighbor of $u$, which is not possible. $\square$

In particular, the observation implies that when upper-bounding $\mathrm{ciocp}(\mathcal{G})$ by considering an arbitrarily chosen graph $G \in \mathcal{G}$, we may assume $G$ has no nested shapes without loss of generality.

**Lemma 79.** *If a simple representation (of a finite graph) satisfies (weak) non-pass property, has no nested intersections and each intersection is connected, then it also satisfies the strong non-pass property.*

*Proof.* For contradiction, let $A$ together with some collection $\mathcal{S}$ of shapes violate the strong non-pass property, that is, $A \setminus (\cup \mathcal{S})$, has (at least) two connected components $A_1, A_2, \ldots$. For the purposes of the proof we assume that the pair $(A, \mathcal{S})$ is minimal, that is, no $(B, \mathcal{S}')$ violates the strong non-pass property, where $|\mathcal{S}'| < |\mathcal{S}|$. Our goal is to derive that $|\mathcal{S}| = 1$, showing that the weak non-pass property is violated.

For the rest of the proof, by *shapes* (with no further specification), we mean shapes from $\{A\} \cup \mathcal{S}$. By the minimality, each shape in $\mathcal{S}$ intersects $A$. Furthermore we show that no shape $S$ is contained in $A$. Consider otherwise. If $S$ intersects another shape $S_0 \in \mathcal{S}$, then $S \cap S_0$ is nested in $A$, violating the assumptions. We conclude that $S$ does not intersect any shape other than $A$. Either $S$ itself cuts $A$ into at least two components, contradicting the choice of $\mathcal{S}$, or $S \cap A$ is contained within one component of $A \setminus (\cup(\mathcal{S} \setminus S))$, contradicting the minimality of $\mathcal{S}$. In here we use the fact that $A$ has no holes, implying that $A \setminus (S_1 \cup S_2)$ cannot be disconnected if $A \cap (S_1 \cup S_2)$ is disconnected.

We define an auxiliary (multi-)graph $H$ as follows. Vertices of $H$ correspond exactly to points of pairwise intersections of the boundaries of shapes. The edges of $H$ represent segments of the boundaries in between the intersection points. The geometric representation of shapes (more specifically their boundaries) is a planar drawing of $H$. Note that every shape $S$ naturally corresponds to a cycle in $H$ (possibly of length 2), we say that shape $S$ *induces* this cycle. We say that a shape $S$ *holds* an element $x$ (vertex, edge or face) of $H$ if $x$ represents part of plane contained in the interior of $S$ or on its boundary.

For technical convenience we limit ourselves to a subgraph $G$ of $H$ containing only vertices and edges held by $A$. Let $\bar{A}$ denote the outer face of $G$. The boundary of $\bar{A}$ is the cycle induced by $A$, we call this cycle the *horizon*. Note that every shape $S$ other than $A$ induces a path in $G$ with both ends on the horizon. Let $a_1, a_2$ denote the two faces of $G$ corresponding to $A_1$ and $A_2$.

In the geometric representation, we say that a curve $c$ is *cutting*, if it is contained in $A \cap (\cup \mathcal{S})$, ends and begins on the boundary of $A$ and $A \backslash c$ splits into (at least) two components such that distinct components contain $A_1$ and $A_2$ respectively. An existence of cutting curve is equivalent with the assumption that $A \backslash \mathcal{S}$ is disconnected. We say that a walk $W$ in $G$ is cutting, if its drawing is a cutting curve. In other words, $W$ separates $a_1$ from $a_2$ within $A$.

For the purposes of this proof, we understand walks as defined by a list of edges, with vertices naturally implied. Let $W_a$ and $W_b$ be walks such that the last vertex of $W_a$ is the first vertex of $W_b$ and $W_c$ is their concatenation. We express relations of these walks as $W_C = W_A + W_B$ (concatenation) and $W_B = W_C - W_A$ (prefix removal). Note that for the lengths (number of edges) analogous relations hold, $|W_C| = |W_A + W_B| = |W_A| + |W_B|$.

We define a potential $\Phi(W, S)$ of a cutting walk $W$ and a shape $S$ as the length of walk $W - W_S$ where $W_S$ is maximal initial segment of $W$ induced by $S$. The potential $\Phi(W)$ of the walk $W$ is then defined as the minimal $\Phi(W, S)$ over all choices of $S$ from $\mathcal{S}$. We say that $S$ *witnesses* $\Phi(W)$ via the (initial) segment $W_S$ if $\Phi(W, S) = \Phi(W) = |W_S|$. Intuitively, $|W - W_S|$ quantifies by how much $S$ fails to induce $W$ all the way to the other side of the horizon. We want to show that actually there exists $S$ such that it carries $W$ all the way ($\Phi(W, S) = 0$).

Since the representation of shapes satisfies the weak non-pass property, for every pair of shapes there are at most two vertices of $G$ representing the points where their respective boundaries intersect. By the same reasoning, each shape induces a limited amount of edges of $G$ (at most $2|\mathcal{S}|$). Clearly it is possible to construct a cutting walk $W$ with a finite potential by considering a facial walk of $a_1$ and connecting its ends to the horizon if the facial walk does not intersect the horizon.

Let $W$ be a walk with minimum potential. Assume $\Phi(W) > 0$. First we observe that the first edge of $W$ is not induced by $A$, as otherwise $\Phi(W) = |W|$ (as no shape induced the first edge of $W$) and by removing the first edge we obtain a shorter cutting walk, clearly with lower potential. Let $S_1$ be witnessing $\Phi(W)$ via the initial segment $W_1$ of $W$ (necessarily $|W_1| > 0$).

We observe that if the maximal initial segment $X$ of $W$ held by $S_1$ is longer than $W_1$ (the walk $W$ continues inside $S_1$ after leaving its the boundary), then $\Phi(W)$ is not minimal. Let $q$ be the last vertex of $X$. Consider $W'$ obtained as $X' + (W - X)$ where $X'$ is chosen as either of the two paths induced by $S_1$ connecting $q$ to the horizon. We observe that $W'$ is cutting, since it can be obtained from $W$ by a continuous deformation within $S_1$. Clearly, $\Phi(W', S_1) < \Phi(W, S_1)$, as $W - X = W' - X'$ and $X'$ is induced by $S_1$ while $X$ is not.

Let $S_2$ be the shape inducing the edge $e$ immediately following $W_1$ on $W$ and let $W_2 = W - W_1$ be the rest of $W$. Consider one of the two paths induced by $S_1 \cap S_2$ connecting $q$ to the horizon and denote it as $W_3$. We observe that $W_3+W_2$ is cutting, by the same reasoning as before since $W_3$ is held by $S_1$. Clearly, $\Phi(W', S_2) < \Phi(W, S_1) = |W_2|$. We get a contradiction with the assumption that $\Phi(W) > 0$.

It remains to note that if $\Phi(W) = 0$, then the boundary of some $S$ induces a cutting walk, implying that $|\mathcal{S}| = 1$. We conclude that $A$ and $S$ violate the weak non-pass property. □

A collection of shapes is a collection of *pseudocircles* if each shape is simple, and for each pair of shapes either their boundaries intersect in exactly two points and their intersection is connected, or their boundaries do not intersect at all.

**Observation 80.** *The following holds:*

- *If $G$ is represented via pseudocircles with no hidden intersections, then ciocp$(G) \leq 1$*
- *If $G$ is represented via pseudocircles, then $\bar{G}$ does not have an induced subgraph consisting of two disjoint triangles.*

*Proof.* Note that a representation via pseudocircles has weak non-pass property and connected intersections by definition.

If we also assume that given representation has no hidden intersections, and as a consequence no nested intersections, then by Lemma 79 the strong non-pass property holds, and by Observation 77, $G$ must be planar, implying $\text{ciocp}(G) \leq 1$ by Observation 75.

For the second point, let us assume that $\bar{G}$ contains an induced pair of disjoint triangles. The same vertices induce $K_{3,3}$ in $G$. Since $K_{3,3}$ is triangle-free, its induced representation has no points of thickness at least 3 and therefore no nested intersections. By Observation 77, the represented graph is planar, which is a contradiction. □

### 3.1.3 Representations Using Geometric Shapes

In this section we present several examples of geometrically represented classes. Apart from being interesting in isolation, we use these examples to support our conjecture connecting ciocp and properties of geometric representations.

For the following lemma, we define *circle* as the set of boundary points of a disk. In other words, for a given center point $c$ and radius $r$, the set of points at distance exactly $r$ from $c$.

We say that a family of shapes is *rotable*, if it is closed under rotation. Similarly, a family of shapes is *stretchable* if it is closed under arbitrary rescaling along arbitrary (non-fixed) direction. For example, a stretchable family containing all disks must also contain all ellipses.

**Lemma 81.** *The following intersection representation classes have unlimited ciocp*

- *representation via lines or line segments*
- *representation via circles*
- *representation via arbitrarily stretchable and rotable shapes*

*Proof.* For the representation via lines, note that two distinct infinite lines intersect if and only if their slopes differ. For arbitrary $k$ we construct a representation with ciocp $= k$ as follows; fix $k$ arbitrary slopes and construct 3 lines for each slope. In the complement, each triplet of lines induces an independent triangle. For line segments, we construct effectively the same representation. Depending on the choice of the slopes and the minimal length of a segment, there exists an $\epsilon$ such that if the centers of all line segments are at distance at most $\epsilon$ from a fixed point, then they intersect if and only if their slopes are different.

For the case of scallable rings, we use a slight variation of the same construction. For arbitrary $k$, consider $k$ unit rings, all intersecting in one point $x$. For some $\epsilon$, and every unit ring with center $c$, add two more rings with the center at $c$ and radii $1 + \epsilon$ and $1 - \epsilon$. If $\epsilon$ is sufficiently small, two rings intersect if and only if they are defined by distinct centers.

In the last case, we may stretch any initial shape until its width is negligible in respect to its length. We may then use the same construction as for the line segments. $\square$

**Lemma 82.** *The intersection representations via axis aligned rectangles have ciocp at most* 2. *Furthermore, axis aligned rectangles can represent a graph of ciocp at least* 2.

*Proof.* Let $G$ be a graph represented by axis aligned rectangles, and for contradiction assume that $C_1, C_2, C_3$ are complements of cycles, disjoint subgraphs of $G$ showing that *ciocp*$(G) \geq 3$. Let $x_1, x_2$ be two consecutive vertices on $C_1$ and $w_1, w_2$ two consecutive vertices on $C_2$. Note that these four vertices induce a $K_{2,2}$ in $G$. Let $A_i$ be the rectangle representing $v_i$ and $B_j$ the rectangle representing $w_j$. By convexity, we have two lines $p_1$ separating $A_1$ from $A_2$ and $p_2$ separating $B_1$ from $B_2$. By the assumption on axis alignment, we may choose each of $p_1, p_2$ to be either horizontal or vertical and without loss of generality let $p_1$ be vertical. If $p_2$ is also vertical, without loss of generality let $p_2$ be to the right of $p_1$. Then one of $A_1, A_2$ is to the left of $p_1$ and one of $B_1, B_2$ is on the right of $p_2$. This is a contradiction, as each $A_i$ intersects each $B_j$. We conclude that $p_2$ is horizontal and intersects $p_1$ in some point $p$. Let $z$ be a vertex of $C_3$. As it is complete to all $v_1, v_2, w_1, w_2$, it must intersect both $p_1$ and $p_2$. By the alignment assumption, the point $p$ must belong to the rectangle representing $z$. As $z$ was chosen arbitrarily from $C_3$, all rectangles representing vertices from $C_3$ contain $p$ and therefore pairwise intersect. This is a contradiction with the choice of $C_3$ as a complement of a cycle.

To observe that ciocp 2 is obtainable for axis aligned rectangles, we use a construction adapted from Lemma 81. Consider three horizontal and three vertical line segments arranged so that they represent a $K_{3,3}$ (every non-parallel pair of line segments intersecting). Clearly the same structure may be represented by rectangles. $\square$

**Lemma 83.** *The following intersection representations have ciocp at most* 1.

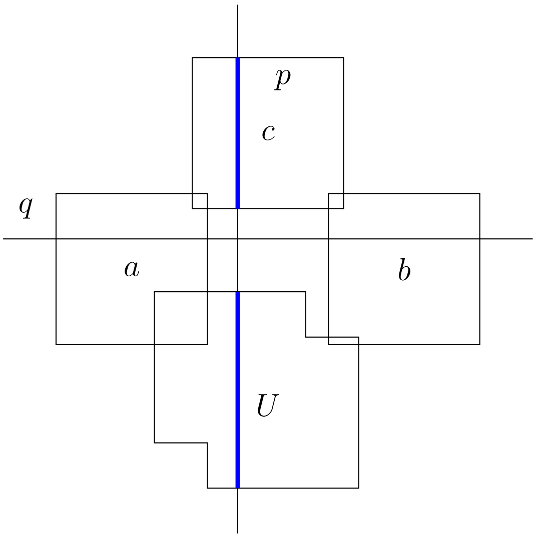
- *disks*
- *unit spheres (in 3D-space)*
- *unit circles*
- *axis aligned unit squares*

*Proof.* The first two cases were shown by Bonamy et al. [66]. For the rest of the proof, let us use the following notation. Let $v$ be a vertex of the represented graph, then $s(v)$ is the shape representing $v$.

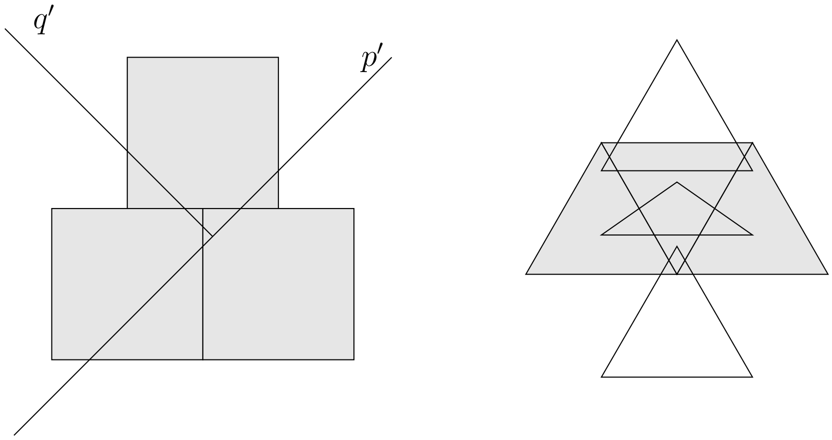
To show the second case, observe that by completing every ring into a full circle (of the same radius) we obtain a representation by circles such that two rings in the original representation intersect if and only if their respective circles intersect.

For the case of unit squares, let us first assume that the complement of $G$ contains a packing of size 2 with one triangle, represented by three non-intersecting unit squares. Using convexity, consider three lines, each separating distinct pair of squares, by alignment, all lines can be chosen as either horizontal or vertical. Similarly to the aligned rectangle representation, if we can choose the lines so that at least one is horizontal and at least one is vertical, then their intersection must belong to every square representing the second odd cycle, which is a contradiction. Without loss of generality, let all three lines be forced to be vertical by the arrangement. Fix any pair of squares and note that if the third square was entirely below or above all y-coordinates shared by the pair of squares, one of the lines could be chosen as horizontal. We conclude that there exists a horizontal line intersecting all three squares, and therefore the horizontal distance between two of the squares is more than 1. This shows that no unit square can intersect all three, contradicting the existence of second odd cycle.

Let us consider the case of unit squares and a packing with a $C$ cycle of length at least 7. Let $a, b, x, y$ be four vertices of $C$ such that $a, b$ and $x, y$ are pairs of consecutive vertices and together $a, b, x, y$ induce a $C_4$ (that is, there is no edge of $C$ between these pairs). Consider a line $p$ separating $s(a)$ and $s(b)$, without loss of generality $p$ is vertical. Since $\{x, y\}$ is complete to $\{a, b\}$, both $s(x), s(y)$ intersect $p$, but not each other. We conclude that any axis-aligned line $q$ separating $s(x)$ and $s(y)$ must be horizontal. Consequently, square representing any vertex complete to $C$ must contain $p \cap q$, which contradicts existence of the second cycle in the packing.




(a) Example of unit-square argument in Lemma 83




(b) Examples of intersection-represented $\overline{K_{3,3}}$ in Lemma 84

Figure 3.1: Examples of constructions

Finally, if the packing contains a 5-cycle $C$, we adjust the argument slightly. See Figure 3.1a for illustration. Let $abcde$ be the vertices of $C$ in order as they appear on $C$. Without loss of generality, $s(a)$ and $s(b)$ are separated by a vertical line $p$. Observe that $s(d)$ intersects $s(a), s(b)$ and therefore also intersects $p$. Consider now together the union $U = s(c) \cup s(e)$. For the same reason as $s(d)$, $U$ intersects $p$. This implies that either $s(c)$ or $s(e)$ intersects $p$. As both are also disjoint from $s(d)$, we have two disjoint squares intersecting $p$ and therefore separable by a horizontal line. As before, this contradicts existence of any other cycle in the packing.

□

**Lemma 84.** *The following intersection representations allow ciocp at least* 2.

- *unit squares*
- *isosceles triangles with one side horizontal, and side length in the range* $[1-\epsilon, 1]$ *for any* $\epsilon$.

*Proof.* See figure 3.1b for illustration of the constructions.

In the first case, consider stacking three squares in a shape of symmetric pyramid, negligibly close to touching. We refer to the squares as the tip square and the two base squares according to their relative positions. Let $p$ be the line separating the tip square from the base squares, let $q$ the half-line perpendicular to $p$ separating the two base squares and let $x$ be their intersection. Note that the positions of $p, q$ fully describe the representation via pyramid. Consider a line $p'$ with slight slope (rising towards the right side), intersecting $p$ slightly to the right of $x$ so that it intersects all three squares. Define $q'$ as perpendicular to $p'$, passing slightly to the left of $x$, thus pointing upward and intersecting all three squares ($q'$ intersects a corner of left base square above $p'$). We observe that all squares of the pyramid defined by $p', q'$ pairwise intersect with all squares of the original pyramid. Clearly, if the squares of the new pyramid were made large enough, both new base squares intersect every square intersected by $q'$ and the new tip square intersects every square intersected by $p'$. Since $p' \cap q'$ can be constructed arbitrarily close to $x$, all of the pairwise intersections are witnessed by points arbitrarily close to $x$ and thus unit squares are sufficient.

For the second construction, let $h$ be the height of a unit-side triangle. For convenience we take $\epsilon' = \epsilon/5$. We describe a touching representation and argue it can be turned into an intersection representation. We define each triangle by its horizontal base and the direction its tip points to. For the first triples, we choose two triangles $T_1, T_3$ with bases $[(-1, 0), (0, 0)]$ and $[(0, 0), (1, 0)]$, pointing upwards, and a triangle $T_2$ with base $[(-1/2, h), (1/2, h)]$ pointing downwards. For the second triplet, let us have triangles with $T_A, T_B, T_C$ with bases $[(-1/2, h), (1/2, h)]$;$[(-1/2, 0), (1/2, 0)]$ and $[(-h, -1/2), (-h, 1/2)]$, and all three triangles pointing upwards. By shifting $T_1, T_3$ to the left and right respectively by a small shift $\epsilon'$, the triangles $T_1, T_2, T_3$ no longer touch. By shifting the triangle $T_A$ down by $2\epsilon'$ and $T_C$ up by $2\epsilon'$ they now intersects all three of $T_1, T_2, T_3$. The triangle $T_B$ already intersects $T_1, T_2, T_3$, so it remains to shrink it in order to avoid intersections with $T_A$ and $T_C$. We shrink $T_B$ so that its height is less than $h - 5\epsilon'$ (e.g. set scale down by a factor of $1 - 5\epsilon'$) and shift it so its base is at height of $5/2\epsilon'$. $\square$

### 3.1.4 IOCP Representation Conjecture

**Conjecture 85.** *There exists a constant c such that if G has simple representation with weak non-pass property (and possibly all intersections are assumed to be connected), then ciocp$(G) \leq 2$, in particular, if G is represented by pseudocircles, then ciocp$(G) \leq 2$.*

We further conjecture, that the bound in both cases is in fact 1. From our investigation we believe that a constant upper-bound exists, and though upper bound 1 seems as a very strong claim, we have reasons to believe that an upper bound of 2 holds. Some of the evidence boils down to a sharp cut-off when relaxing conditions in the geometric classes, where we do not know of any class that would allow ciocp at least 3 but not arbitrary ciocp, unless the class achieves this somewhat artificially (for example lines with 3 different slopes).

An example of rotable and stretchable rectangles, according to Lemma 81, shows that violation of weak non-pass property (even with convexity, connected intersections, and no hidden intersections) admits arbitrary ciocp. Similarly, every example of representation with ciocp $\geq 2$ given in Section 3.1.3 violates the non-pass property and we do not know of any such example that would satisfy the non-pass property, even artificial or exploiting some wild class of shapes. Conversely, the examples from Lemma 84 are such where the chosen shapes are able to violate non-pass property while otherwise very constrained geometrically; still a representation admitting ciocp $\geq 2$ is easily obtained. It would therefore seem that very low ciocp (in particular ciocp $\leq 1$) is closely tight to the non-pass property, and not directly dependent on any other considered properties.

We now show that a rather straight-forward argument can be used to show that if in the conjecture above we assume that one of the odd cycles in the packing is a triangle, then the conjecture holds.

**Observation 86.** *Let $\{A, B, C\}$ and $\{X, Y\}$ be two sets of simple shapes such that two shapes are non-intersecting if and only if they are from the same set. Furthermore, let the shapes satisfy the non-pass property. Then the boundaries of $X$ and $Y$ intersect shapes $A, B, C$ in opposite (cyclic) permutations.*

*Proof.* For contradiction and without loss of generality, let both $X, Y$ follow permutation of intersection $ABC$. We proceed similarly to the construction of drawing in Observation 77. We obtain an planar drawing of the intersection graph $G$ of $A, B, C, X, Y$ with vertices $a, b, c, x, y$ such that the cyclic clockwise permutation of neighbors around both $x$ and $y$ is $abc$. Consider clockwise walk around a face of $G$, starting on the left side of edge going from $a$ to $x$, by the permutation of neighbors the walk visits vertices in the (cyclic) order $axbycxaybxcy$. Clearly, this face is the only face of $G$. However $G$ has 5 vertices and 6 edges and therefore by Euler's formula must have exactly 3 faces. This contradicts $X$ and $Y$ intersecting $A, B, C$ in the same order. Since only two cyclic permutations are possible on three elements, the permutations of intersections must be opposite. $\square$

**Lemma 87.** *If $G$ is representable with a non-pass property, then its complement allows no induced odd cycle packing of size at least two where one of the cycles is a triangle.*

*Proof.* Let $a, b, c$ and $x_1, x_2, \ldots, x_k$ represent two induced independent cycles in the complement of $G$, cycle $C$ of length 3 and $X$ of length $k$. Each (cyclically) consecutive pair $x_i, x_{i+1}$ of vertices of $X$ together with $a, b, c$ form a configuration satisfying the assumption of Observation 86, implying that shapes representing $x_i$ and $x_{i+1}$ can be described by opposite permutations of intersection with shapes representing $a, b, c$. We conclude that $k$ is necessarily even. $\square$

There are various ways similar arguments can be used to achieve the same result. However none was successfully extended to the general case. The main difficulty here seem to be hidden intersections. In the case of one triangle $T$ present in the packing, $T$ itself cannot exhibit hidden intersections. It is possible to extend the proof to arbitrary length of $T$ when assuming that no pairs of shapes from $T$ have hidden intersections. However, this assumption is rather unreasonable when the cycles in question get long ($\geq 7$).

## 3.2 Induced Odd Cycle Packing Number, Independent Sets and Chromatic Number

The graph classes defined by forbidden cycles or induced cycles of certain lengths figure prominently in the structural graph theory, motivated in particular by the Strong Perfect Graph Theorem [38] which shows that perfect graphs are characterized by forbidden odd holes and their complements. The most interesting graph parameters in the context of these graph classes are the chromatic number, the independence number, and the clique number: while they are NP-hard to approximate within any fixed precision [41] in general graphs, using semidefinite programming they can be determined in polynomial time for perfect graphs [40].

Perfect graphs also motivate the concept of $\chi$-boundedness. A class of graphs is *$\chi$-bounded* if the chromatic number of the graphs from the class can be bounded by a function of the clique number (of course, there is no such function in general, due to numerous known constructions of triangle-free graphs of arbitrarily large chromatic number). The notion of $\chi$-boundedness was introduced by Gyarfás [68], who also proposed a number of influential questions on this topic. As an example, he conjectured that graphs without odd holes are $\chi$-bounded; this conjecture was only recently confirmed by Scott and Seymour [69]. In a similar vein, Bonamy, Charbit, and Thomassé [70] showed that graph classes that forbid induced cycles of length that is a multiple of 3 have bounded chromatic number. Unlike the study of perfect graphs, we work with all odd cycles, not just with odd holes (i.e., iocp takes into account also triangles).

The main motivation for our work comes from recent algorithmic results exploiting low ciocp appearing in the context of geometric graph classes. Specifically, Bonnet et al. [65] proved that the intersection graphs of disks in the plane have the complement induced odd cycle packing number at most 1 and gave a QPTAS and an exact subexponential-time algorithm for the independence number of graphs with iocp $\leq 1$. In combination, this gives a QPTAS and an exact subexponential-time algorithm for the clique number of intersection graphs of disks. Bonnet et al. [65] only explicitly give the algorithms for intersection graphs of disks, but the inspection of their algorithms shows that they only use the above mentioned property, and no other properties specific to intersection graphs of disks.

Building upon these results, Bonamy et al. [66] proved that the intersection graphs of unit balls in 3-dimensional space also satisfy ciocp $\leq 1$. Moreover, they gave a randomized algorithm to approximate the clique number arbitrarily well in polynomial time (a randomized efficient polynomial-time approximation scheme) for intersection graphs of disks and of unit balls; however, in addition to the limited ciocp, they use further properties derived from the geometry of the problems, namely bounded VC dimension and a linear lower bound on the independence number.

**Theorem 88** (Bonamy et al. [66])**.** *There exists a randomized algorithm taking as an input integers $k, b, c, a \geq 1$ and graph $G$ of VC dimension at most $c$ such that iocp$(G) \leq k$ and $\alpha(G) \geq |V(G)|/b$, and in time $O_{k,b,c,a}(|V(G)|^2)$ returns an independent set of $G$ of size at least $(1 - 1/a)\alpha(G)$.*

In unpublished results, they further proved that one can remove the assumption that the VC dimension is bounded, at the expense of making the exponent in the time complexity depend on the desired precision, i.e., obtaining a PTAS with time complexity $O(|V(G)|^{f(k,b,a)})$ for some function $f$ rather than an EPTAS. In this section we show that it is not necessary to make this sacrifice, obtaining an EPTAS without the assumption of bounded VC dimension.

Then we focus our attention on coloring. We show that graph classes with bounded iocp are $\chi$-bounded by a function polynomial in the clique number (with the degree of the polynomial depending linearly on the iocp parameter). Furthermore, our proof of this fact can be turned into a coloring algorithm running in polynomial time for fixed maximum clique size. We complement this result partially by a lower bound on the $\chi$-bounding function for the case when $k = 1$.

Finally, we apply the $\chi$-boundedness to show how these results can be combined to obtain a QPTAS for the maximum independent set in graphs with bounded iocp and no other assumptions. This generalizes a result of Bonnet et al. [65], who gave such a QPTAS for graphs with induced odd cycle packing number at most one. It needs to be noted that their approach easily generalizes to the more general case of bounded induced odd cycle packing number; indeed, the joint journal version [71] of the conference papers [66] and [65], which was published while these results were in the review process, states this generalization explicitly. Still, we believe the alternate approach we provide could be of interest; let us note that the time complexity of our algorithm is $O(\exp(\log^2 n))$, while the algorithm of [71] has time complexity $O(\exp(\log^6 n))$.

### 3.2.1 EPTAS for High Odd-Girth

It is a well known result that a largest independent set in a bipartite graph can be found in polynomial time (by reduction to a maximum-flow problem). The natural approach to solving the problem when odd cycles are present is to remove the odd cycles without distorting the optimal solution too much. When the iocp parameter is fixed the structure of odd cycles allows a simple approach; whenever an odd cycle together with its neighborhood is removed from a graph, the iocp parameter decreases, eventually reaching the bipartite case.

Our EPTAS is structured as follows. First, we deal with the case of high odd-girth as a base case. Then we investigate graphs that cannot be reduced to this base case without distorting the solution too much due to many disjoint short odd cycles. We show that in such case, it is possible to either reduce iocp or (with high probability) the number of vertices of the input graph without distorting the solution too much, producing a recursive algorithm.

The *odd girth* of a graph $G$ is the length of shortest odd cycle appearing in $G$. Let us start with a mild variation on a crucial part of the argument of Bonamy et al. [66]; we describe it here in detail, as they do not state the result separately, only as a part of a longer argument. We also give the result in larger generality than we actually need, in a weighted setting, as this is quite natural (unfortunately, the rest of our argument only works in unweighted setting). Essentially, we show that it is possible to remove a small "wedge" from the close neighborhood of the cycle (destroying the odd cycle in the process) so that the remainder of the neighborhood is bipartite.

**Lemma 89.** *Let $g$ be an odd integer, let $G$ be a graph of odd girth at least $g$, and let $C$ be a shortest odd cycle in $G$. Let $t \leq (g-1)/2$ be a non-negative integer, and suppose that every vertex of $G$ is at distance at most $t$ from $C$. Let $z$ be a vertex of $C$, let $A = N_t[z] \cap C$, and let $R = N_t[A]$. Then $G - R$ is bipartite.*

*Proof.* Note that $C$ is geodesic in $G$, i.e., the distance between any two vertices of $C$ is the same in $C$ as in $G$ (otherwise, $G$ would contain a path $Q$ between two vertices $x$ and $y$ of $C$ shorter than the distance between $x$ and $y$ in $C$, and $C \cup Q$ would contain an odd cycle shorter than $C$), and in particular $C$ is an induced cycle.

Let $F$ be a forest of shortest paths from vertices in $G$ to $V(C)$, and for each $v \in V(G)$, let $f(v)$ denote the vertex in which the component of $F$ containing $v$ intersects $C$. Note that for each $v \in V(G)$, the distance in $F$ from $v$ to $f(v)$ is at most $t$. Observe that $(F \cup C) - z$ is also a forest, and thus it has a proper 2-coloring $\psi$.

We claim that the restriction of $\psi$ is a proper 2-coloring of $G - R$. Suppose for a contradiction there exists an edge $uv \in E(G - R)$ with $\psi(u) = \psi(v)$. Since $u, v \notin R$, we have $f(u) \neq z \neq f(v)$, and thus there exists a unique path $P$ between $u$ and $v$ in $(F \cup C) - z$. Since $\psi(u) = \psi(v)$, the cycle $P + uv$ has odd length, and since $C$ is a shortest odd cycle in $G$, we have $|E(P)| + 1 \geq |C| \geq g \geq 2t + 1$. In particular, $P$ contains both $f(u)$ and $f(v)$. Let $P'$ be the subpath of $P$ between $f(u)$ and $f(v)$; we have $|E(P')| \geq |E(P)| - 2t \geq |C| - 2t - 1$. Since $P'$ is a subpath of the path $C - z$ and $|E(C - z)| = |C| - 2$, we can by symmetry assume that the distance between $f(u)$ and $z$ is at most $t$, and thus $f(u) \in A$. But then $u \in R$, which is a contradiction. $\square$

Note that if $|C| \gg t$, then the above lemma can be used to obtain many disjoint sets whose removal makes the graph bipartite, and thus one of them must contain only a small fraction of vertices, or, in a weighted setting, contain only a small fraction of the total weight. We now use this observation to decrease the induced odd cycle packing number without decreasing the weight of the heaviest independent set too much.

Given an assignment $w : V(G) \to \mathbb{Z}^+$ of weights to vertices of $G$, let for each set $X \subseteq V(G)$ define $w(X) = \sum_{v \in X} w(v)$ and let $\alpha_w(G)$ be the maximum of $w(X)$ over all independent sets $X$ in $G$.

**Lemma 90.** *There exists an algorithm that, given an integer $b \geq 1$, an $n$-vertex non-bipartite graph $G$ of odd girth at least $2b(8b-3)$, and an assignment $w : V(G) \to \mathbb{Z}^+$ of weights to vertices, returns in time $O(bn^2 + n^3)$ induced subgraphs $G_1, \ldots, G_{2b}$ of $G$ such that iocp$(G_i) \leq$ iocp$(G) - 1$ for $i \in \{1, \ldots, 2b\}$, and*

$$\max\{\alpha_w(G_i) : i \in \{1, \ldots, 2b\}\} \geq (1 - 1/b)\alpha_w(G), \text{ and}$$
$$\max\{w(V(G_i)) : i \in \{1, \ldots, 2b\}\} \geq (1 - 1/b)w(V(G)).$$

*Proof.* We find (in time $O(n^3)$ by BFS from each vertex) a shortest odd cycle $C$ in $G$, necessarily of length at least $2b(8b-3)$. The argument from the proof of Lemma 89 shows that $C$ is geodesic in $G$. Let $Z = \{z_1, \ldots, z_{2b}\}$ be a set of vertices of $C$ at distance at least $8b-3$ from one another and for $i \in \{1, \ldots, 2b\}$, let $R_i = N_{4b-2}[z_i]$ denote the set of vertices of $G$ at distance at most $4b-2$ from $z_i$

(the choice of $z_1, \ldots, z_{2b}$ implies these sets are pairwise disjoint). Let $L_i$ denote the set of vertices of $G$ at distance exactly $i$ from $C$. Let $G_i = G - L_i - R_i$.

Consider any $i \in \{1, \ldots, 2b\}$. We claim that $\mathrm{iocp}(G_i) \le \mathrm{iocp}(G) - 1$. Indeed, let $G_1$ be the subgraph of $G$ induced by vertices at distance less than $i$ from $C$ and $G_2$ the subgraph induced by vertices at distance greater than $i$ from $C$, so that $G - L_i$ is the disjoint union of $G_1$ and $G_2$. Since $G_1$ contains the odd cycle $C$ whose vertices have no neighbors in $G_2$, we have $\mathrm{iocp}(G_2) \le \mathrm{iocp}(G - L_i) - 1 \le \mathrm{iocp}(G) - 1$. Furthermore, by Lemma 89 (for $t = 4b - 2$), the graph $G_1 - R_i$ is bipartite, and thus $\mathrm{iocp}(G_i) = \mathrm{iocp}(G_1 - R_i) + \mathrm{iocp}(G_2 - R_i) \le \mathrm{iocp}(G) - 1$.

Consider a heaviest independent set $I$ in $G$. Since the sets $L_1, \ldots, L_{2b}$ are pairwise disjoint, and the sets $R_1, \ldots, R_{2b}$ are pairwise disjoint as well, we have

$$\sum_{i=1}^{2b} w(I \cap (L_i \cup R_i)) \le 2w(I),$$

and thus there exists $i \in \{1, \ldots, 2b\}$ such that $w(I \cap (L_i \cup R_i)) \le w(I)/b$. Hence, $\alpha_w(G_i) \ge (1 - 1/b)\alpha_w(G)$. Similarly,

$$\sum_{j=1}^{2b} w(L_j \cup R_j) \le 2w(V(G)),$$

and thus there exists $j \in \{1, \ldots, 2b\}$ such that $w(L_j \cup R_j) \le w(V(G))/b$ and $w(V(G_j)) \ge (1 - 1/b)w(V(G))$. $\square$

Let us remark on a lower bound on $\alpha_w(G)$ in graphs satisfying the assumptions of Lemma 90.

**Corollary 91.** *Let $k \ge 0$ and $b \ge 1$ be integers and let $G$ be a graph of induced odd cycle packing number at most $k$ and odd girth at least $2b(8b - 3)$. For any assignment $w : V(G) \to \mathbb{Z}^+$ of weights to vertices, we have*

$$\alpha_w(G) \ge \frac{(1 - 1/b)^k w(V(G))}{2} \ge \frac{(1 - k/b)w(V(G))}{2}.$$

*Proof.* We prove the claim by induction on $k$. If $G$ is bipartite, then one of the parts of bipartition of $G$ has weight at least $w(V(G))/2$. Hence, we can assume that $G$ is not bipartite, and thus $k \ge 1$. By Lemma 90, there exists an induced subgraph $G'$ of $G$ of induced odd cycle packing number at most $k - 1$ such that $w(V(G')) \ge (1 - 1/b)w(V(G))$, and by the induction hypothesis,

$$\alpha_w(G) \ge \alpha_w(G') \ge \frac{(1 - 1/b)^{k-1} w(V(G'))}{2} \ge \frac{(1 - 1/b)^k w(V(G))}{2}.$$

$\square$

Similarly, iterating Lemma 90, we obtain an approximation for the maximum-weight independent set in graphs of bounded induced odd cycle packing number.

**Lemma 92.** *There exists an algorithm that, given integers $k \ge 0$ and $b \ge 1$, an $n$-vertex graph $G$ of induced odd cycle packing number at most $k$ and odd girth at least $2b(8b - 3)$, and an assignment $w : V(G) \to \mathbb{Z}^+$ of weights to vertices, returns in time $O((2b)^k n^3)$ an independent set $X \subseteq V(G)$ such that*

$$w(X) \ge (1 - 1/b)^k \alpha_w(G) \ge (1 - k/b)\alpha_w(G).$$

*Proof.* We prove the claim by induction on $k$. If $G$ is bipartite, then we can find an independent set in $G$ of largest weight via a maximum flow algorithm in time $O(n^3)$; and considering the heavier of the two color classes of $G$, we have $\alpha_w(G) \geq w(V(G))/2$. Hence, we can assume that $G$ is not bipartite, and in particular $k \geq 1$.

Let $G_1, \ldots, G_{2b}$ be the induced subgraphs of $G$ of induced odd cycle packing number at most $k-1$ obtained using the algorithm from Lemma 90. We now recurse on $G_1, \ldots, G_{2b}$ (with $k$ replaced by $k-1$) obtaining by the induction hypothesis in time $2b \cdot O((2b)^{k-1}n^3) = O((2b)^k n^3)$ independent sets $X_1, \ldots, X_{2b}$ such that $w(X_i) \geq (1-1/b)^{k-1}\alpha_w(G_i)$ for each $i \in \{1, \ldots, 2b\}$. We return the heaviest of these independent sets; we have

$$\begin{aligned}\max\{w(X_i) : i \in \{1, \ldots, 2b\}\} \\ &\geq (1-1/b)^{k-1} \max\{\alpha_w(G_i) : i \in \{1, \ldots, 2b\}\} \\ &\geq (1-1/b)^{k-1} \cdot (1-1/b)\alpha_w(G) = (1-1/b)^k \alpha_w(G)\end{aligned}$$

□

### 3.2.2 EPTAS assuming linear independence number

Let us now move on to the general case of graphs with bounded induced odd cycle packing number. From now on, we work in unweighted setting. In order to make use of Lemma 92, it is necessary to destroy all short odd cycles in the input graph graph. We show that unless simply deleting a maximum packing of such cycles erases only a small portion of the graph (and hence yields a useful approximation), there exist other algorithmically useful structures. Either there exists a cycle which can be used to decrease iocp without decreasing the size of the largest independent set too much or there are many high-degree vertices which can be used to guess part of the solution while significantly reducing the remainder of the graph.

For a set $\mathcal{S}$ of graphs, an *$\mathcal{S}$-packing* in a graph $G$ is a set $\mathcal{X}$ of pairwise vertex-disjoint induced subgraphs of $G$, each isomorphic to a graph belonging to $\mathcal{S}$ (note that we allow edges between members of $\mathcal{X}$, unlike in the definition of iocp). Let $V(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} V(X)$. For an integer $g \geq 3$, let $\mathcal{S}_g$ denote the set of all odd cycles of length less than $g$. A maximal $\mathcal{S}_g$-packing $\mathcal{X}$ in an $n$-vertex graph $G$ can be found in time $O(n^4)$ by repeatedly finding a shortest induced odd cycle and deleting it from $G$; observe that $G - V(\mathcal{X})$ has odd girth at least $g$. Lemma 92 is used to deal with graphs without odd cycles of length less than $g$; so, it remains to handle the graphs containing an $\mathcal{S}_g$-packing covering a large fraction of the vertices.

**Lemma 93.** *There exists an algorithm that, for input integers $k, p \geq 1$ (with $p \geq k$) and an $n$-vertex graph $G$ of induced odd cycle packing number at most $k$, returns in time $O(n^4 + (4p)^k n^3)$ an independent set $I$ and induced odd cycles $C_1$, ..., $C_m$ (for some $m \leq n$) in $G$, such that at least one of the following claims holds:*

*(a) $|I| \geq (1-k/p)\alpha(G)$, or*

*(b) there exists $i \in \{1, \ldots, m\}$ such that $\alpha(G - N(C_i)) \geq (1-1/p)\alpha(G)$, or*

(c) *there are at least* $\frac{k}{81920p^6}\alpha(G)$ *vertices* $v \in V(G)$ *of degree at least* $\frac{k}{81920p^6}n$ *such that* $\alpha(G - N(v)) = \alpha(G)$.

*Proof.* Let $g = 4p(16p-3)$. Let $\mathcal{X}$ be a maximal $\mathcal{S}_g$-packing in $G$. Let $I$ be an independent set in $G - V(\mathcal{X})$ found using the algorithm from Lemma 92 with $b = 2p$. Let $\mathcal{X} = \{C_1, \ldots, C_m\}$ and for $i \in \{1, \ldots, m\}$, let $H_i = G - N(C_i)$.

Suppose first that $|V(\mathcal{X})| \le \frac{k}{10p}n$. Since $\frac{k}{10p} \le 0.1$, Corollary 91 implies

$$\alpha(G) \ge \alpha(G - V(\mathcal{X})) \ge (1 - k/b)|V(G - V(\mathcal{X}))|/2 \ge 0.45(1-k/b)n \ge 0.225n,$$

and thus $|V(\mathcal{X})| \le \frac{k}{10p}n \le \frac{k}{2p}\alpha(G)$. Consequently, $\alpha(G - V(\mathcal{X})) \ge \alpha(G) - |V(\mathcal{X})| \ge \left(1 - \frac{k}{2p}\right)\alpha(G)$, and thus the set $I$ returned by the algorithm from Lemma 92 satisfies

$$|I| \ge \left(1 - \tfrac{k}{2p}\right)\alpha(G - V(\mathcal{X})) \ge (1 - k/p)\alpha(G),$$

implying that (a) holds.

Hence, we can assume $|V(\mathcal{X})| > \frac{k}{10p}n$. Let $J$ be a largest independent set in $G$. If there exists $i \in \{1, \ldots, m\}$ such that $|N(C_i) \cap J| \le |J|/p$, then (b) holds. Hence, assume that $|N(C_i) \cap J| > |J|/p$ for every $i \in \{1, \ldots, m\}$; and consequently, for each such $i$, there exists $b_i \in V(C_i)$ such that $|N(b_i) \cap J| > \frac{|J|}{pg}$. Let $B = \{b_1, \ldots, b_m\}$ and note that $m \ge |V(\mathcal{X})|/g > \frac{k}{10pg}n$. By double-counting the number of edges of $G$ between $B$ and $J$, we have

$$\begin{aligned}\sum_{v \in J} \deg v \ge \sum_{v \in J} |N(v) \cap B| &= \sum_{b \in B} |N(b) \cap J| \\ &> \frac{m|J|}{pg} > \frac{k}{10p^2g^2}|J|n.\end{aligned}$$

Let $J'$ consist of vertices of $J$ of degree greater than $\frac{k}{20p^2g^2}n \ge \frac{k}{81920p^6}n$. Since $J$ is a largest independent set in $G$, for every $v \in J$ we have $\alpha(G - N(v)) = \alpha(G)$. Note that

$$\begin{aligned}\sum_{v \in J'} \deg v &= \sum_{v \in J} \deg v - \sum_{v \in J \setminus J'} \deg v \\ &> \frac{k}{10p^2g^2}|J|n - \tfrac{k}{20p^2g^2}n \cdot |J \setminus J'| \ge \frac{k}{20p^2g^2}|J|n,\end{aligned}$$

and since each vertex has degree less than $n$, we have

$$|J'| > \tfrac{k}{20p^2g^2}|J| = \tfrac{k}{20p^2g^2}\alpha(G) \ge \tfrac{k}{81920p^6}\alpha(G)$$

□

The possible outcomes of Lemma 93 offer a natural recursive approximation algorithm. Supposing a suitable setting of the parameters, if the case (a) holds, then the algorithm provides a good independent set. If (b) holds, then we may restrict the problem to an induced subgraph of lower iocp while preserving the solution well enough. If (c) holds, then we are guaranteed many vertices of large degree such that deleting their neighbors does not affect the solution. A random sampling of vertices with high degree provides a good probability of hitting one of these vertices. We show that this approach yields a randomized EPTAS under assumption that $\alpha(G) = \Omega(|V(G)|)$.

**Lemma 94.** *There exists a randomized algorithm that, for input integers $k \geq 0$ and $t \geq 1$ and an $n$-vertex graph $G$ of induced odd cycle packing number at most $k$, returns in time $O_{k,t}(n^4)$ an independent set of $G$ whose size is at least $\alpha(G) - n/t$ with probability at least $\Omega_{k,t}(n^{-k})$.*

*Proof.* Let $p = kt$, and when $k \neq 0$, let $q = \frac{1}{81920p^6}$ and $d \geq 1$ be the smallest integer such that $(1-q)^d < 1/p$. Let us now describe a recursive procedure that, applied to an induced subgraph $G'$ of $G$ and a non-negative integer $k' \leq k$ such that $\mathrm{iocp}(G') \leq k'$, returns an independent set of $G'$; as we will show later, this set has size at least $\alpha(G') - nk'/p$ with probability $\Omega_{k,t}(n^{-k'})$.

If $|V(G')| \leq nk'/p$, then we return an empty set (or any other independent set in $G'$). If $k' = 0$, then $G'$ is bipartite and we can find a largest independent set in $G'$ via a maximum flow algorithm in time $O(n^3)$. Hence, suppose that $k' \geq 1$. Apply the algorithm from Lemma 93 to $G'$ (using $k'$ as $k$) to obtain an independent set $I$ and cycles $C_1, \ldots, C_m$. Then perform one of the following actions, at random, each with probability $1/3$:

(a) Return the set $I$.

(b) Choose $i \in \{1, \ldots, m\}$ uniformly at random. Note that the induced subgraph $H_i = G' - N[C_i]$ satisfies $\mathrm{iocp}(H_i) \leq k' - 1$. Let $I'$ be an independent set in $H_i$ obtained by a recursive call for $H_i$ with $k'$ replaced by $k' - 1$. Return the union of $I'$ with an independent set of $C_i$ of size $\alpha(C_i) = \lfloor |C_i|/2 \rfloor$.

(c) Choose a vertex $u \in V(G')$ of degree at least $k'q|V(G')|$ uniformly at random (if no such vertex exists, return an empty set, instead). Return the independent set obtained by the recursive call for $G' - N(u)$, with the same $k'$.

Let us analyze the running time of this procedure when applied to $G$ with $k' = k$. Note that at each level of the recursion, we only perform one recursive call, and either $k'$ decreases by one, or the number of vertices decreases by the factor smaller or equal to $1 - \frac{1}{81920p^6}$. Moreover, the recursion stops when the number of vertices is at most $n/p$ (or earlier). Hence, the total depth of the recursion is at most $k + d$, and since $d$ only depends on $k$ and $t$, the running time of the algorithm is $O_{k,t}(n^4)$.

Let $d(G')$ be the smallest non-negative integer such that $(1-q)^{d(G')}|V(G')| < n/p$; in particular, $d(G) = d$. Let $a(G', k') = (3n)^{-k'}\left(\frac{q}{3p}\right)^{d(G')}$. Let us now show that the set returned by the algorithm for $G'$ and $k'$ has size at least $\alpha(G') - nk'/p$ with probability at least $a(G', k')$. Note that when applied to $G' = G$ and $k' = k$, this implies the algorithm returns an independent set of $G$ of size at least $\alpha(G) - n/t$ with probability $\Omega_{k,t}(n^{-k})$, as required.

We prove the claim by induction on $k' + |V(G')|$. If $k' = 0$, then we return an optimal independent set, and if $\alpha(G') \leq nk'/p$, then the claim is trivial. Hence, we can assume that $k' > 0$ and $\alpha(G') > nk'/p$. Let us now distinguish which outcome of Lemma 93 holds.

- If (a) holds, then with probability $1/3 \geq a(G', 1) \geq a(G', k')$, we return the independent set $I$, which has size at least $(1 - k'/p)\alpha(G') \geq \alpha(G') - nk'/p$.

- If (b) holds, then with probability $\frac{1}{3m} \geq \frac{1}{3n}$, the algorithm takes the branch (b) and chooses $i \in \{1, \dots, m\}$ such that $\alpha(G' - N(C_i)) \geq (1 - 1/p)\alpha(G')$. By the induction hypothesis, with probability $a(H_i, k' - 1) \geq a(G', k' - 1)$, the independent set $I'$ returned by the recursive call in $H_i$ has size at least $\alpha(H_i) - n(k' - 1)/p$. Hence, with probability at least $\frac{1}{3n} \cdot a(G', k' - 1) = a(G', k')$, we return an independent set of size at least $\alpha(C_i) + \alpha(H_i) - n(k' - 1)/p = \alpha(G' - N(C_i)) - n(k' - 1)/p \geq \alpha(G') - n/p - n(k' - 1)/p = \alpha(G') - nk'/p$.

- If (c) holds, then with probability at least $\frac{k'q\alpha(G')}{3n} \geq \frac{(k')^2 q}{3p} \geq \frac{q}{3p}$, the algorithm takes the branch (c) and chooses $u \in V(G')$ such that $\alpha(G' - N(u)) = \alpha(G')$. We return the independent set obtained by the recursive call on $G' - N(u)$, which has size at least $(1 - k'/p)\alpha(G' - N(u)) = (1 - k'/p)\alpha(G')$ with probability at least $a(G' - N(u), k')$ by the induction hypothesis. Note that $d(G' - N(u)) \leq d(G') - 1$, and thus the probability we return an independent set of size at least $(1 - k'/p)\alpha(G')$ is at least $\left(\frac{q}{3p}\right) \cdot a(G' - N(u), k') \geq a(G', k')$.

□

We now improve the probability by iteration.

**Theorem 95.** *There exists a randomized algorithm that, for input integers $k \geq 0$ and $t \geq 1$ and an $n$-vertex graph $G$ of induced odd cycle packing number at most $k$, returns in time $O_{k,t}(n^{k+4})$ an independent set of $G$ whose size is at least $\alpha(G) - n/t$ with probability at least $1 - 1/e$.*

*Proof.* Let $q = \Omega_{k,t}(n^{-k})$ be the lower bound on the probability that the algorithm from Lemma 94 succeeds. Run the algorithm $\lceil 1/q \rceil$ times (with independent random choices), and return the largest of the obtained independent sets. The probability that none of them has size at least $\alpha(G) - n/t$ is at most $(1 - q)^{1/q} \leq e^{-1}$, and thus the algorithm succeeds with probability at least $1 - 1/e$. □

Of course, we can further iterate this algorithm $a$ times, reducing the probability of a result worse than $\alpha(G) - n/t$ to at most $e^{-a}$.

### 3.2.3 $\chi$-boundedness

In this section we show that the classes of graphs with bounded induced odd packing number are $\chi$-bounded, that is, their chromatic number is bounded by a function of their maximum clique size. Let us start with the triangle-free case, where we need to show an absolute bound on the chromatic number.

**Lemma 96.** *Every triangle-free graph $G$ satisfies $\chi(G) \leq 2 + 5iocp(G)$. Furthermore, if $G$ has odd girth at least 7, then $\chi(G) \leq 2 + 4iocp(G)$, and if $G$ has girth at least 7, then $\chi(G) \leq 3 + iocp(G)$.*

*Proof.* We prove the claim by induction on the induced odd cycle packing number. If $\mathrm{iocp}(G) = 0$, then $G$ is bipartite and $\chi(G) \leq 2$, hence suppose that $\mathrm{iocp}(G) > 0$ and the claim holds for all graphs with smaller induced odd cycle packing number. Let $C$ be a shortest odd cycle in $G$, which is necessarily induced. Since

$\text{iocp}(G - N[V(C)]) \leq \text{iocp}(G) - 1$, we can color $G - N[V(C)]$ by the induction hypothesis, and it suffices to show how to color $G[N[V(C)]]$ using at most 5 additional colors, respectively 4 or 1 additional color in the special cases.

Let $A = \{z_1, z_2, z_3\}$ be a set consisting of three consecutive vertices of $C$ and let $R = N[A]$. By Lemma 89, $G[N[V(C)] \setminus R]$ is bipartite. Since $G$ is triangle-free, the neighborhood of any vertex is an independent set, and $N(\{z_1, z_3\})$ is disjoint from $N(z_2)$. If $G$ has odd girth at least 7, then $N(\{z_1, z_3\})$ is an independent set as well; hence, we can use two new colors to color $G[N[V(C)] \setminus R]$, one color for $N(z_2)$ (which includes $z_1$ and $z_3$) and one color for $N(\{z_1, z_3\})$ (which includes $z_2$), using four extra colors in total. Otherwise, we use two new colors to color $G[N[V(C)] \setminus R]$, one color for each $N(z_2)$,$N(z_1)$, and $N(z_3) \setminus N(z_1)$, using five extra colors in total.

In the case that $G$ has girth at least 7, we claim that $N(V(C))$ is an independent set. Indeed, suppose for a contradiction vertices $w, z \in N(C)$ are adjacent, and let $w'$ and $z'$ be neighbors of $w$ and $z$ in $C$, respectively. Since $G$ has girth at least 7, the distance between $w'$ and $z'$ in $C$ is greater than three. However, then $C + w'wzz'$ contains an odd cycle shorter than $C$, which is a contradiction. Hence, we can use three of the colors used on $G - N[V(C)]$ to color $C$ and one extra color for $N(V(C))$. Note that in the base case where $G - N[V(C)]$ is bipartite, we need one more color so that three colors are available to be reused on $N[V(C)]$. $\square$

For the general case with triangles, let us first define the bounding function $f$. Let $f(0, \omega) = 2$ for every positive integer $\omega$. For $k \geq 1$, let us inductively define $f(k, \omega) = \omega + (2 + 5k)\binom{\omega}{2} + f(k-1, \omega)\binom{\omega}{3}$.

**Theorem 97.** *Every graph $G$ satisfies $\chi(G) \leq f(iocp(G), \omega(G))$.*

*Proof.* We prove the claim by induction on the induced odd cycle packing number. If $\text{iocp}(G) = 0$, then $G$ is bipartite and $\chi(G) \leq 2$, hence suppose that $\text{iocp}(G) > 0$ and the claim holds for all graphs with smaller induced odd cycle packing number. Let $K$ be a largest clique in $G$, and for each $v \in V(G) \setminus K$, let $A(v)$ denote the set of vertices of $K$ to which $v$ is not adjacent; the maximality of $K$ implies $A(v) \neq \emptyset$. Let $A'(v)$ be an arbitrary subset of $A(v)$ of size $\min(3, |A(v)|)$. For a set $X \subseteq K$ with $|X| \in \{1, 2, 3\}$, let $B(X) = \{v \in V(G) \setminus K : A'(v) = X\}$. If $|X| = 1$, then the maximality of $K$ implies $B(X)$ is an independent set; for each 1-element set $X$, we use one color for all vertices of $X \cup B(X)$. If $|X| = 2$, then the maximality of $K$ implies $G[B(X)]$ is triangle-free, and by Lemma 96, we can use $2 + 5\text{iocp}(G)$ colors to color $G[B(X)]$. Finally, if $|X| = 3$, then $\text{iocp}(G[B(X)]) \leq \text{iocp}(G[B(X)]) - 1$, since all vertices in $B(X)$ are non-adjacent to the triangle induced by $X$; hence, we can use $f(\text{iocp}(G) - 1, \omega(G))$ colors to color $G[B(X)]$ by induction. Summing the numbers of colors over all choices of $X$, we conclude that at most $f(\text{iocp}(G), \omega(G))$ colors are used to color $G$. $\square$

Let us remark that we can obtain the coloring as in Theorem 97 in polynomial time: instead of choosing $K$ as a largest clique, the inspection of the proof shows that it suffices to choose one which cannot be enlarged by adding at most three and removing at most two vertices, and such a clique can be found in polynomial time, e.g. by a straightforward greedy approach.

### 3.2.4 Lower-bound for $\chi$-bounding Function

We showed that every graph $G$ has chromatic number at most $f(\mathrm{iocp}(G), \omega(G))$, where $f$ is of order roughly $\omega(G)^{3\mathrm{iocp}(G)}$. We do not know whether this upper bound is tight. We show that there exist graphs with induced odd cycle packing number one whose chromatic number is almost quadratic in $\omega(G)$ via a probabilistic construction.

In order to carry out the probabilistic calculations, we will use the following bounds.

**Lemma 98** (See [72], Chernoff Bound)**.** *Suppose $X$ is a sum of independent $\{0, 1\}$-variables. For any $\delta > 0$,*

$$\mathit{Prob}\left[X \le (1-\delta)\,\mathsf{E}[X]\right] \le e^{-\frac{\delta^2\,\mathsf{E}[X]}{2}}.$$

**Lemma 99** (See [72], Talagrand's Inequality II)**.** *Let $X$ be a non-negative random variable, not identically 0, which is determined by $n$ independent trials $T_1$, …, $T_n$, and satisfying the following for some integers $c, r > 0$:*

- *Changing the outcome of any one trial can change $X$ by at most $c$.*
- *For any non-negative integer $d$, if $X \ge d$, then there is a set of at most $rd$ trials whose outcomes certify that $X \ge d$.*

*Then for every non-negative $t \le \mathsf{E}[X]$,*

$$\mathit{Prob}\left[|X - \mathsf{E}[X]| > t + 60c\sqrt{r\,\mathsf{E}[X]}\right] \le 4e^{-\frac{t^2}{8c^2 r\,\mathsf{E}[X]}}.$$

We are now ready to give the construction, which is a variation on a standard construction of triangle-free graphs with no large independent set.

**Theorem 100.** *There exists a family of graphs with induced odd cycle packing number at most one and with arbitrarily large clique number such that every graph $H$ in this family satisfies $\chi(H) = \Omega\left(\frac{\omega^2(H)}{\log^2 \omega(H)}\right)$.*

*Proof.* Let $G$ be a random $G(n, p)$ graph for $p = 1/k$, where $k$ is a sufficiently large even integer and $n = k^2/2$.

Suppose $A_1, A_2 \subset V(G)$ are disjoint and have size three, and $G$ contains all nine edges with one end in $A_1$ and the other end in $A_2$; in this case, we say that the set of these nine edges *forms a* $K_{3,3}$. We say a set of edges of $G$ is *bad* if it can be partitioned into subsets of size 9, each of which forms a $K_{3,3}$. Finally we say a set of edges is *subbad* if it is a subset of a bad set.

We construct a graph $G_0$ by deleting a maximal bad set $B$ from $G$. By the maximality of $B$, no set of edges of $G_0$ forms a $K_{3,3}$. Let $H$ be the complement of $G_0$. Consider any disjoint (odd) cycles in $H$. If there were no edge between these cycles, then any pair of triples of vertices taken one from each cycle would induce a complement of a supergraph of $K_{3,3}$ in $G_0$, which is a contradiction. Consequently $\mathrm{iocp}(H) \le 1$. Furthermore, an analogous argument shows that $\alpha(H) \le 5$, and thus $\chi(H) \ge n/5 = \Omega(k^2)$.

Therefore, it suffices to argue that $\omega(H) = \alpha(G_0) = O(k \log k)$ with non-zero probability. Let us consider a set $S \subseteq V(G)$ of size at least $k$ and define the following random variables.

$$X_1 = |E(G[S])|$$
$$X_2 = \max\Big\{|Z| : Z \subseteq E(G[S]),\ Z \text{ is subbad in } G\Big\}$$

The probability that $S$ is an independent set in $G_0$ is at most $\text{Prob}[X_1 \leq X_2]$. Indeed, if $S$ is independent, then $E(G[S]) \subseteq B$ is subbad, and thus $X_1 = X_2$. Let $s = \binom{|S|}{2} = \Omega(k^2)$, so that $\mathsf{E}[X_1] = s/k$. Using the Chernoff bound (Lemma 98), we obtain

$$\text{Prob}\left[X_1 \leq \frac{s}{2k}\right] = \text{Prob}[X_1 \leq \mathsf{E}[X_1]/2] \leq e^{-\frac{\mathsf{E}[X_1]}{8}} = e^{-\frac{s}{8k}}.$$

Let $Z_2 = \{e \in E(G[S]) : \text{some set containing } e \text{ forms a } K_{3,3} \text{ in } G\}$. Clearly, if $Z \subseteq E(G[S])$ is subbad, then $Z \subseteq Z_2$, and thus $X_2 \leq |Z_2|$. For distinct vertices $x, y \in V(G)$, the probability that $xy$ is an edge and some set containing $xy$ forms a $K_{3,3}$ in $G$ is at most $\frac{n^4}{k^9} = \frac{1}{16k}$, and thus $\mathsf{E}[X_2] \leq \mathsf{E}[|Z_2|] \leq \frac{s}{16k}$. Let $\delta = \frac{s}{16k} - \mathsf{E}[X_2] \geq 0$ and let $X_2' = X_2 + \delta$, so that $\mathsf{E}[X_2'] = \frac{s}{16k}$. Note that flipping the existence of a single edge in $G$ changes $X_2'$ by at most 9. Furthermore, when $X_2' \geq d$, there exist at most $9d$ edges in $G$ whose presence certifies this is the case. Hence, we can apply the Talagrand's inequality (Lemma 99) with $c = r = 9$ for $t = \mathsf{E}[X_2'] = \frac{s}{16k}$. Note that $60c\sqrt{r\,\mathsf{E}[X_2']} \leq \mathsf{E}[X_2']$ for $k$ large enough, since $s = \Omega(k^2)$. Hence, we have

$$\text{Prob}\left[X_2' > \frac{3s}{16k}\right] = \text{Prob}[X_2' > 3\,\mathsf{E}[X_2']] \leq 4e^{-\frac{\mathsf{E}[X_2']}{5832}} = 4e^{-\frac{s}{93312k}}.$$

It follows that

$$\begin{aligned}\text{Prob}[X_1 \leq X_2] \leq \text{Prob}[X_1 \leq X_2'] &\leq \text{Prob}\left[X_1 \leq \frac{s}{2k}\right] + \text{Prob}\left[X_2' > \frac{3s}{16k}\right] \\ &\leq e^{-\frac{s}{8k}} + 4e^{-\frac{s}{93312k}} < e^{-\frac{|S|^2}{200000k}}\end{aligned}$$

for $k$ large enough.

Therefore, for any set $S \subseteq V(G_0)$ of size $q \geq k$, the probability that $S$ is an independent set in $G_0$ is at most $e^{-\frac{q^2}{200000k}}$. Hence, the probability that $G_0$ contains an independent set of size $q$ is less than

$$\begin{aligned}\binom{n}{q} e^{-\frac{q^2}{200000k}} &\leq \left(\frac{ne}{q}\right)^q e^{-\frac{q^2}{200000k}} \\ &\leq (ke)^q e^{-\frac{q^2}{200000k}} = \exp\Big(q \log(ke) - \frac{q^2}{200000k}\Big),\end{aligned}$$

that is, smaller than 1 when $q \geq 200000k \log(ke)$. □

As noted in the proof, the probabilistic construction used is unnecessarily restrictive, excluding all disjoint cycles regardless of parity. Similarly, all cycles or paths of length $\geq 8$ are excluded. From the point of view of vertex 6-tuples, all of these structures exhibit very similar properties. It would seem that achieving a distinction between these patterns and the ones that are necessary to avoid requires more global conditions and thus a much more refined approach.

### 3.2.5 QPTAS assuming only bounded iocp

The fact that triangle-free graphs with bounded iocp have bounded chromatic number has the following easy consequence.

**Lemma 101.** *For all integers $p \geq 1$, every graph $G$ with $n$ vertices satisfies at least one of the following conditions:*

- *$G$ has an independent set of size at least $\frac{n}{4+10iocp(G)}$, or*
- *every maximal packing of triangles in $G$ contains a triangle $T$ such that $\alpha(G - N(T)) \geq (1 - 1/p)\alpha(G)$, or*
- *$G$ contains a vertex $v$ of degree at least $\frac{n}{18p}$ such that $\alpha(G - N(v)) = \alpha(G)$.*

*Proof.* Let $\mathcal{X} = \{T_1, \ldots, T_m\}$ be a maximal packing of triangles in $G$. The graph $G - V(\mathcal{X})$ is triangle-free, and by Lemma 96, $\chi(G - V(\mathcal{X})) \leq 2 + 5\mathrm{iocp}(G)$. Consequently, $\alpha(G) \geq \alpha(G - V(\mathcal{X})) \geq \frac{n-3m}{2+5\mathrm{iocp}(G)}$. Suppose that $G$ does not have any independent set of size at least $\frac{n}{4+10\mathrm{iocp}(G)}$, and thus $m \geq n/6$. Let $J$ be a largest independent set in $G$. If $|N(T_i) \cap J| \leq |J|/p$ for some $i \in \{1, \ldots, m\}$, then the second outcome of the lemma holds.

Otherwise, for each $i \in \{1, \ldots, m\}$, there exists $v_i \in V(T_i)$ satisfying that $|N(v_i) \cap J| > \frac{|J|}{3p}$. Consequently, there exists a vertex $v \in J$ such that $|N(v) \cap \{v_1, \ldots, v_m\}| \geq \frac{m}{3p}$, and thus $\deg v \geq \frac{m}{3p} \geq \frac{n}{18p}$. Since $v \in J$, we have $\alpha(G - N(v)) = \alpha(G)$. $\square$

Combining this lemma with Theorem 95, we obtain a QPTAS for the maximum independent set in graphs of bounded induced odd cycle packing number.

**Theorem 102.** *There exists a randomized algorithm that, for input integers $k \geq 0$ and $p \geq 1$ and an $n$-vertex graph $G$ of induced odd cycle packing number at most $k$, returns in time $n^{O(k+p\log n)}$ an independent set of $G$ whose size is at least $(1 - k/p)\alpha(G)$ with probability at least $1/2$.*

*Proof.* If $k = 0$ (so $G$ is bipartite), we return the largest maximum independent set obtained via a maximum flow algorithm. Otherwise, we find a maximal packing of triangles $\mathcal{X}$ in $G$ greedily, and return the largest of the independent sets obtained by

(a) running the algorithm from Theorem 95 $n$ times with $t = (4 + 10k)p$,

(b) for each $T \in \mathcal{X}$, running the algorithm recursively for $G - N[T]$ with $k$ replaced by $k-1$ and adding a vertex of $T$ to the returned independent set, and

(c) for each $v \in V(G)$ of degree at least $\frac{n}{18p}$, running the algorithm recursively for $G - N(v)$.

Each recursive call either decreases $k$ or decreases the number of vertices by a factor of at most $\left(1 - \frac{1}{18p}\right)$, implying the total number of the calls of the procedure is at most $n^{O(k+p\log n)}$. Iterating the algorithm from Theorem 95 $n$ times for an induced subgraph $G'$ of $G$ ensures we fail to find an independent set of size at least $\alpha(G') - n/t$ with probability at most $2^{-n}$. Hence, with probability at least $1 - n^{O(k+p\log n)}2^{-n} > 1/2$ (for $n$ large enough—for small $n$, we can just find the largest independent set by brute force), we can assume that throughout the run of the algorithm, in part (a) for an induced subgraph $G'$ of $G$, at least one of the returned independent sets has size at least $\alpha(G') - n/t$.

If $\alpha(G) \geq \frac{n}{4+10k}$, then in (a) we return an independent set of size at least $\alpha(G) - n/t \geq \alpha(G) - (4+10k)\alpha(G)/t = (1-1/p)\alpha(G)$. If $G$ contains a vertex $v$ of degree at least $\frac{n}{18p}$ such that $\alpha(G - N(v)) = \alpha(G)$, then in (c) the corresponding recursive call gives an independent set of size at least $(1 - k/p)\alpha(G - N(v)) = (1 - k/p)\alpha(G)$.

If neither of these conditions holds, Lemma 101 implies there exists a triangle $T \in \mathcal{X}$ such that $\alpha(G - N(T)) \geq (1-1/p)\alpha(G)$. The recursive call in (b) returns an independent set $I$ of $G - N[T]$ of size at least $(1 - (k-1)/p)\alpha(G - N[T])$, and since $\alpha(G - N(T)) = \alpha(G - N[T]) + 1$, the addition of a vertex of $T$ turns $I$ into an independent set of size at least $(1 - (k-1)/p)(\alpha(G - N(T)) - 1) + 1 \geq (1 - (k-1)/p)\alpha(G - N(T)) \geq (1 - (k-1)/p)(1 - 1/p)\alpha(G) \geq (1 - k/p)\alpha(G)$. Hence, the algorithm is correct. □

Considering Theorem 102, it is of course natural to ask whether the maximum independent set problem admits a EPTAS on graphs with bounded induced odd cycle packing number without any other assumptions.

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