Please send your solutions by April 10 2021. You may send a partial solution. It is ok to work on a homework in a group, but everybody should contribute and everybody needs to write their solution themselves.

[**Assignment 1.1**] **6 points** In a pack of cards there are n red and n black ones. The pack is shuffled randomly. You are playing the following game at any point you may:

- Take the top most card (if any).
- Leave the game.

Your objective is to get as many points as possible.

- You start with 0 points.
- You get a point for each black card drawn and lose a point for each red card drawn.

Your task:

1. What is the optimal expected win when $n = 1, 2, 3, 4$ (when trying to maximize the expected win and knowing n)?
2. Describe an efficient algorithm that given n outputs the optimal expected win (i.e., generalize the previous). (No need to submit code, pseudocode is enough.)

[**Assignment 1.2**] **6 points** Find eigenvalues and eigenvectors of the complete undirected bipartite graph $K_{n,m}$ (where $V(K_{n,m}) = A \cup B$ where $A \cap B = \emptyset$ and $|A| = n, |B| = m$ and $E(K_{n,m}) = A \times B$).

[**Assignment 1.3**] **6 points** A modification of the envelopes game from tutorial 1, problem 2. There are k coins in one of those and ℓ coins in the other ($k, \ell \in \mathbb{N}$ but you do not know k, ℓ in advance). You may open an envelope and (based on what you see) decide to take this one or the other (without looking into both).

Let us formalize a strategy by a function f: when you open an envelope with m coins inside then you keep this one with probability $f(m)$ and chose the other with probability $1 - f(m)$.

1. You know that both k, ℓ are chosen independently at random from geometric distribution with $p = 0.1$ (that is $\Pr[k = n] = (1 - p)^{n-1} p$ for each $n \in \mathbb{N}$, the same for ℓ).
   (a) Describe your strategy.
   (b) Determine your expected win.
2. Show that for any strategy f there is a strategy g and a distribution D of coin amounts where the expected win of g is higher than the expected win of f when the coin amounts are drawn from the distribution D.

[**Assignment 1.4**] **6 points** In this problem we deal with strongly connected oriented graphs with arbitrary in and out-degrees where for each vertex the outgoing edges have the same probability.

1. Describe a graph family with as large hitting time (max between two vertices) as possible. Both exact computation and asymptotic estimate are ok. (No need to argue optimality, just try your best.)
2. Show an upper bound on hitting time (it can be very large, but should be finite).