1. Which pairs of following vectors are perpendicular with respect to the standard scalar product? $(1, 2, 3)$, $(5, 2, -3)$, and $(-2, -1, -4)$

   Which properties of the relation of perpendicularity hold: reflexivita, symmetry, tranzitivity?

2. Do Gram-Schmidt on the rows of the following matrix: $\begin{pmatrix} 0 & 3 & 4 & 0 \\ 0 & 0 & 5 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}$

3. How far is the point $(1, 2, 0, 1)^T$ from the plane spanned by vectors $(1, 1, 0, 0)^T, (2, -1, 0, 0)^T$?

4. Using projection find the best solution of the following system of equations: $Ax = b$ where $A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 2 & -4 & -1 \\ 1 & -2 & 2 \end{pmatrix}$, $b = (10, 5, 13, 9)^T$

   Notice that the columns of A are perpendicular. How bad is your solution (i.e. compute $b - Ax$)?

   The least squares method is often used when the errors are small – but it is hard to compute with such systems with pen and paper. Is the solution the same as the solution of the system $A^T Ax = A^T b$?

5. Using Gram-Schmidt find an orthonormal basis of the row-space of the following matrix and expand it to an orthonormal basis of $\mathbb{R}^4$. $\begin{pmatrix} 2 & 4 & 2 & 1 \\ -1 & -2 & -2 & -1 \\ 1 & 2 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

6. Show that a norm defined by a dot product ($\|v\| = \sqrt{\langle v|v\rangle}$) satisfies the Parallelogram Law $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

   Can the norm $\|x\|_1 = \sum |x_i|$ or the norm $\|x\|_\infty = \max |x_i|$ be given by a dot product?

7. Show that columns of Hadamard matrices $H_m \in \mathbb{R}^{2^m \times 2^m}$ defined as

$$H_0 = \begin{pmatrix} 1 \end{pmatrix},$$

$$H_m = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{pmatrix},$$

   are orthonormal.

Bonus Controlling matrix multiplication: someone is selling you a program that can multiply two matrices fast. Can you control that it returns correct results? It is enough to check if $Cx = A(Bx)$ where C is the output of the program and x is uniformly random $\{0, 1\}$ vector of the right length. Show that if $C \neq AB$ then with big probability $Cx = ABx$ does not hold.