1. For the dot product $\langle f|g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ show that functions $3x^2-1$ a $5x^3-3x$ are orthogonal.

2. Let A be a symmetric real matrix such that $u^T A u > 0$ for each non-zero vector $u \in \mathbb{R}^n$ we call such matrices positive definite. Let us define a dot product as $\langle u|v\rangle = u^T A v$. Show that this is indeed a dot product if and only if A is positive definite.

   Given a dot product $\langle u|v\rangle$ as a black-box find a way how to find the corresponding positive definite matrix A which defines it.

   Show that a sum of two positive definite matrices is a positive definite matrix. Show that a positive multiple of a positive definite matrix is a positive definite matrix.

3. Show that vectors $v \in \mathbb{R}^3$ satisfying $\langle (1,0,-3)^T|v\rangle = 0$ form a vector space (subspace of $\mathbb{R}^3$). In other words $\left\{\vec{v} \in \mathbb{R}^3 | \langle (1,0,-3)^T|v\rangle = 0\right\}$ form a subspace of $\mathbb{R}^3$.

   Show that vectors in $\mathbb{R}^3$ satisfying $\langle (1,0,-3)^T|v\rangle = 2$ are an affine space.

4. Let $(2,5)^T, (3,1)^T$ be two vectors in the real plane $\mathbb{R}^2$. What multiple of the first vector should we subtract from the second one so that the result is perpendicular to the first vector. What multiple of the second vector should we subtract from the first one so that the result is perpendicular to the second vector.

5. Do Gram-Schmidt on the rows of the following matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 & 2 \\ 4 & 3 & 2 & 4 \\ 6 & -5 & 3 & 6 \\ -4 & 2 & 4 & 2 \end{pmatrix}$$

6. Show that a norm defined by a dot product ($\|v\| = \sqrt{\langle v|v\rangle}$) satisfies the Parallelogram Law $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

   Can the norm $\|x\|_1 = \sum |x_i|$ or the norm $\|x\|_\infty = \max |x_i|$ be given by a dot product?

Bonus Show that there are no four points in the real plane $\mathbb{R}^2$ such that the distance between each two of those is an odd number (these distances may or may not be the same).