Fine-Graind Lower Bounds for Dynamic Graph Problems

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DIMACS Tutorial on Fine-Grained Complexity, July 17, 2024

- An area with lots of FG lower bounds
- More tricks with SETH/OV
- "Finding the right conjecture is key"

Dynamic graph algorithms

Given initial graph G, can preprocess it.
Edge updates: insert(u,v), delete(u,v)

Queries: (depend on the problem)
How many SCCs are there? Can u reach v? …

Want to minimize the preprocessing, *update* and *query* times.

$\tilde{O}(m)$ $\tilde{O}(1)$ $\tilde{O}(1)$

- Worst case time
- Amortized time
- Total time (over all updates)

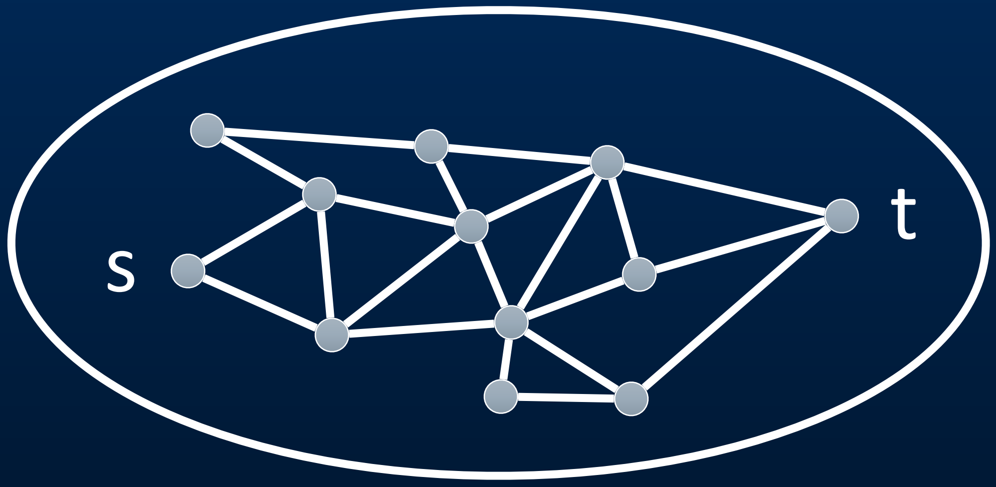
Dynamic Problems

Dynamic (undirected) Connectivity

Input: an undirected graph G

Updates: Add or remove edges.

Query: Are s and t connected?



Trivial algorithm: O(m) updates.

[Henzinger-King '95, Thorup'01]: O(log m (log log m)^3) amortized time per update.

[Pătraşcu - Demaine STOC'05]:
$\Omega(\log m)$ Cell-probe lower bound.

Great!

Dynamic Problems

Dynamic (directed) Reachability

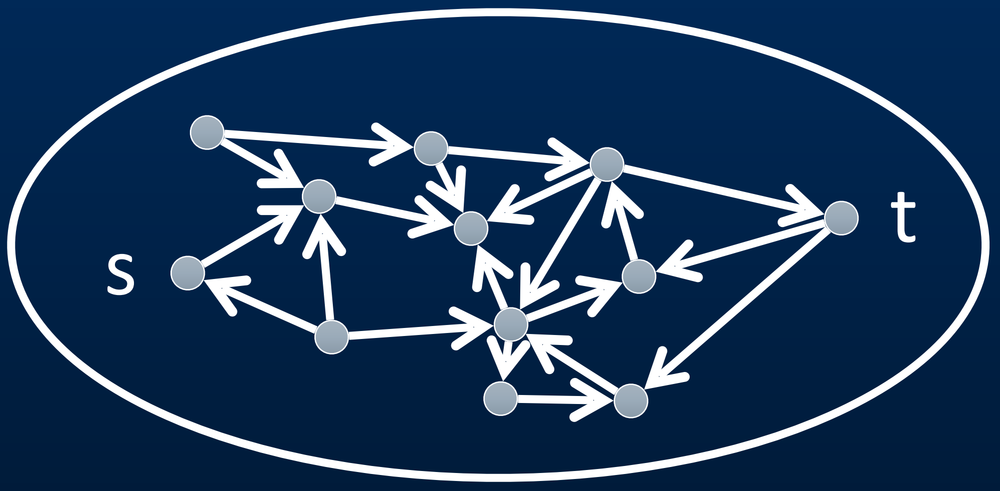
Input: A directed graph G.

Updates: Add or remove edges.

Query:

s,t-Reach: Is there a path from s to t?

#SSR: How many nodes can s reach?



Trivial algorithm: O(m) time updates

Using fast matrix multiplication
[Sankowski FOCS'04] $O(n^{1.57})$

Not great.

Best cell probe lower bound still $\Omega(\log m)$

Many Examples

Connectivity

Minimum Spanning Tree (MST)

Maximal Matching

Reachability

Strongly Connected Components (SCC)

s,t-shortest-path

(Bipartite) Maximum Matching

s,t-Max Flow

Diameter

Conditional Lower Bounds?

Connectivity

Minimum Spanning Tree (MST)

Maximal Matching

Reachability

Strongly Connected Components (SCC)

s,t-shortest-path

(Bipartite) Maximum Matching

s,t-Max Flow

Diameter (static to dynamic self-reduction)

Conditional Lower Bounds?

[Pătraşcu STOC'10]: Polynomial Lower Bounds under the 3-SUM Conjecture.

3SUM ⟶ Triangle Listing ⟶ … ⟶ Dynamic Problems

$\Omega(n^{1/8})$ lower bounds

[A. - Vassilevska Williams FOCS'14]: "Finding the right conjecture is the key…"

Tight lower bounds under SETH/APSP/more.

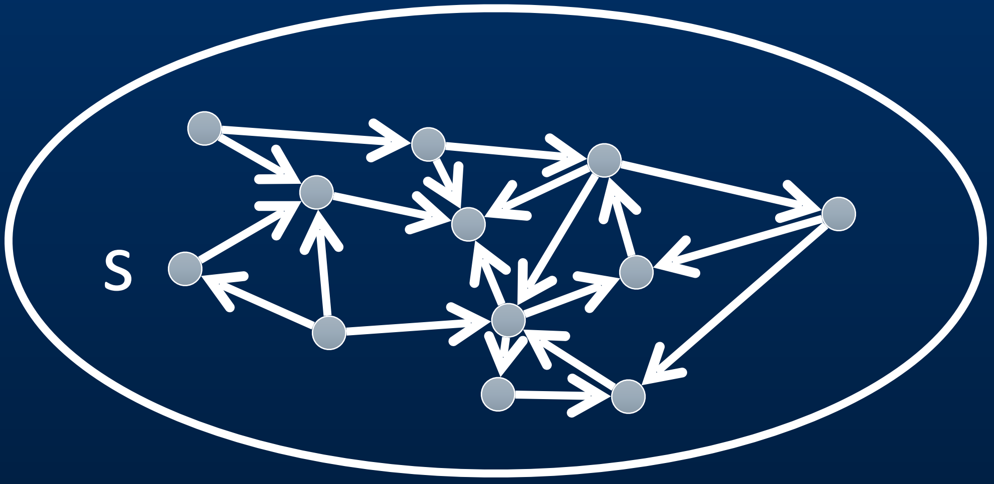
Single Source Reachability

Input: A directed graph G.

Updates: Add or remove edges.

Query:

#SSR: How many nodes can s reach?



Trivial algorithm: O(m) updates.

Theorem:

If **dynamic #SSR** can be solved with $O(m^{0.99})$ update and query times, then **OV** can be solved in $\tilde{O}(n^{1.99})$ time (and SETH is false).

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Proof outline:

Orthogonal Vectors

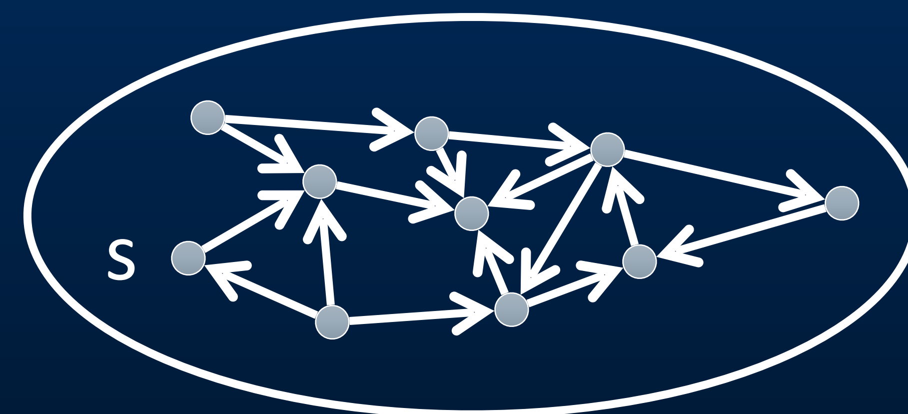
(0,0,1,…,1)	(0,0,1,…,1)
(0,1,1,…,1)	(1,0,1,…,1)
…	…
(1,0,1,…,0)	(1,1,1,…,1)

(1,0,1,…,0)

(0,0,1,…,1)

Given two lists of n vectors in $\{0,1\}^d$ is there an orthogonal pair?

dynamic #SSR



#SSR asks how many nodes can s reach?

Graph G on ***m=O(nd)*** *nodes and edges*, **O(nd)** updates and queries

OVP in ***~O(n^1.9)*** time

(refutes SETH)

O(nd) updates/queries in ***~O(n^1.9)*** time

Amortized ***O(m^0.9)*** update/query time

d=polylog(n), m=~O(n)

Orthogonal Vectors

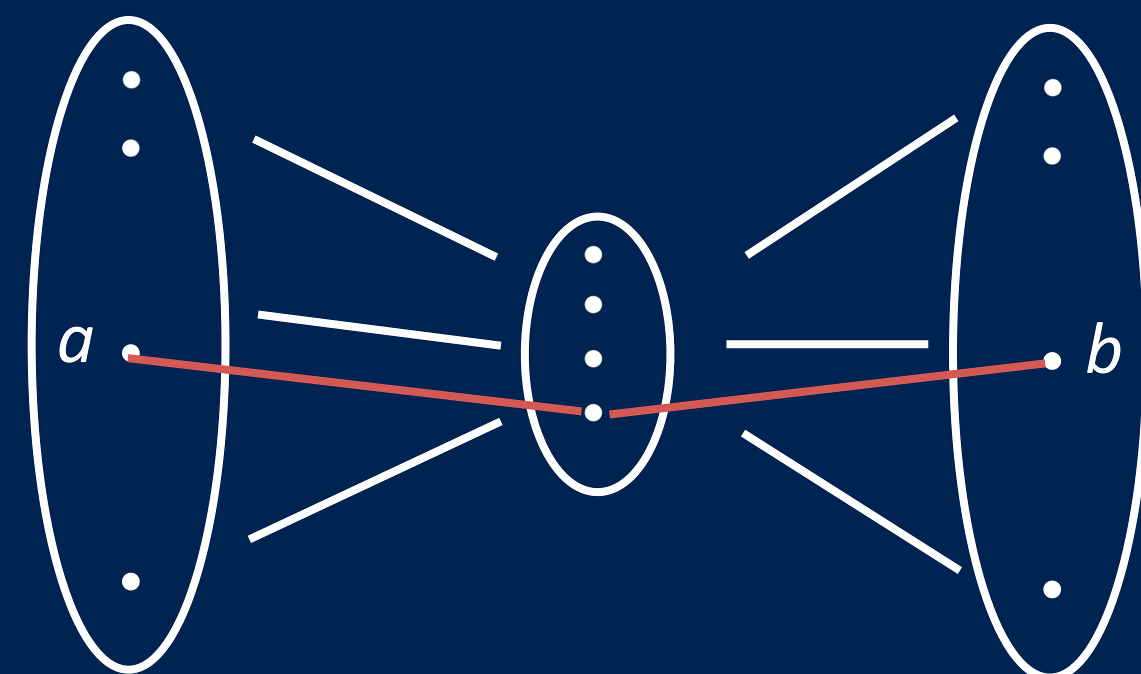
(0,0,1,…,1)
(0,1,1,…,1)
…
(1,0,1,…,0)

(0,0,1,…,1)
(1,0,1,…,1)
…
(1,1,1,…,1)

(1,0,1,…,0)
(0,0,1,…,1)

Given two lists of n vectors in $\{0,1\}^d$ is there an orthogonal pair?

Graph OV



d(a,b) = 2 if not orth.
d(a,b)>2 if orth.

Theorem: If **dynamic #SSR** can be solved with $O(m^{0.99})$ update and query times, then **OV** can be solved in $\tilde{O}(n^{1.99})$ time (and SETH is false).

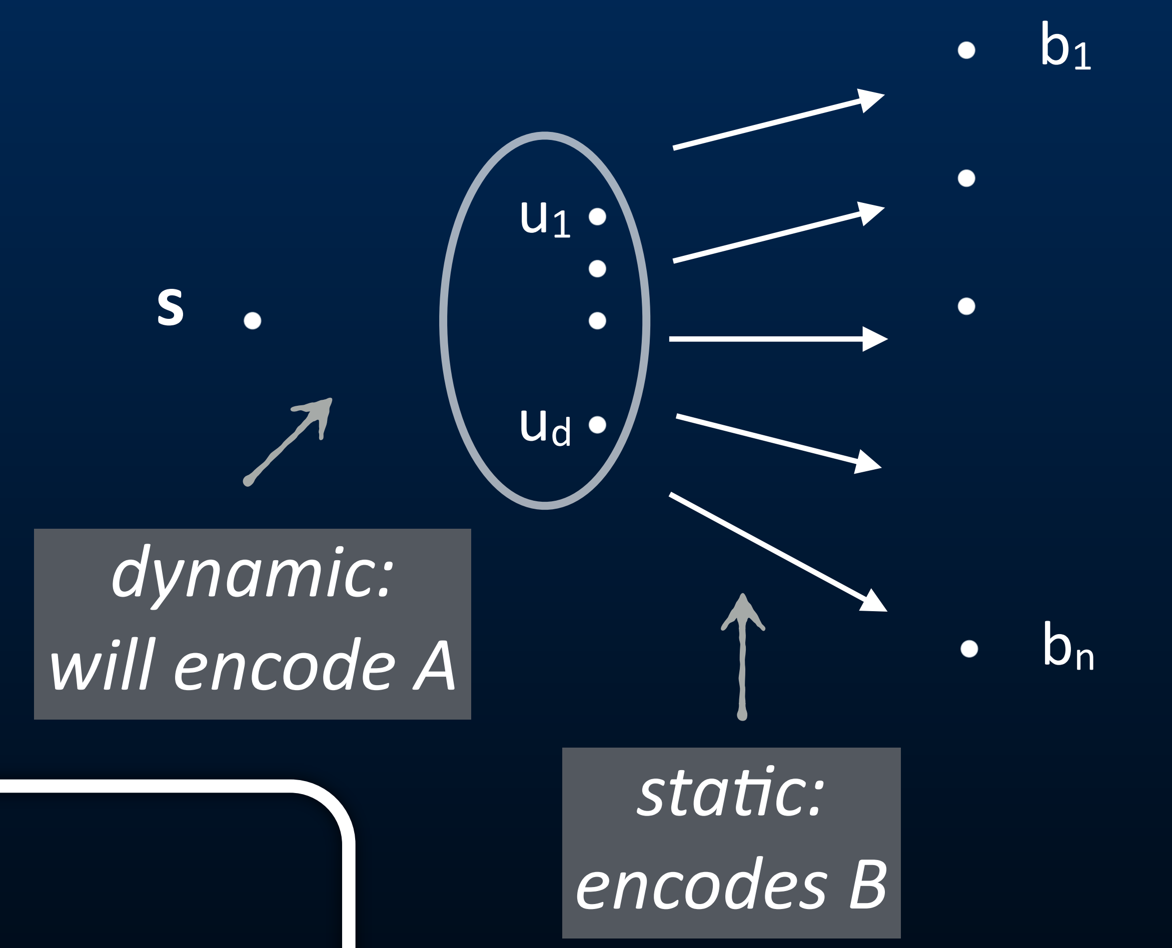
Proof:

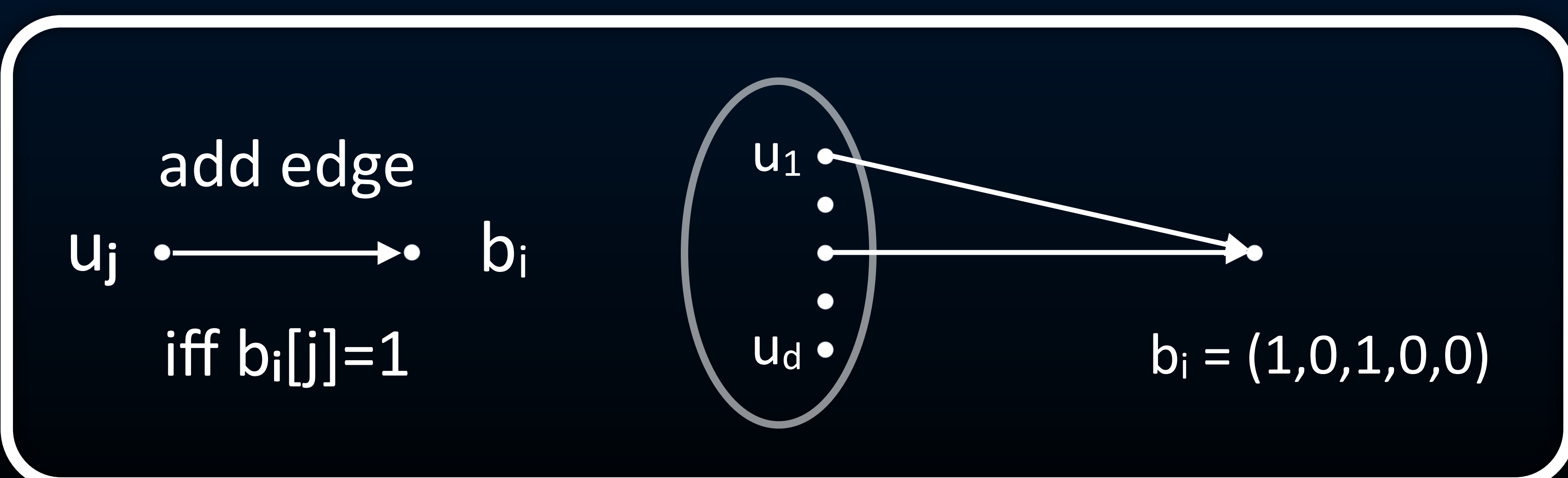
Orthogonal Vectors → **dynamic #SSR**

a_1 (0,0,1,…,1)
(0,1,1,…,1)
…
a_n (1,0,1,…,0)

b_1 (0,0,1,…,1)
(1,0,1,…,1)
…
b_n (1,1,1,…,1)

(1,0,1,…,0)
(0,0,1,…,1)





Theorem: If **dynamic #SSR** can be solved with $O(m^{0.99})$ update and query times, then **OV** can be solved in $\tilde{O}(n^{1.99})$ time (and SETH is false).

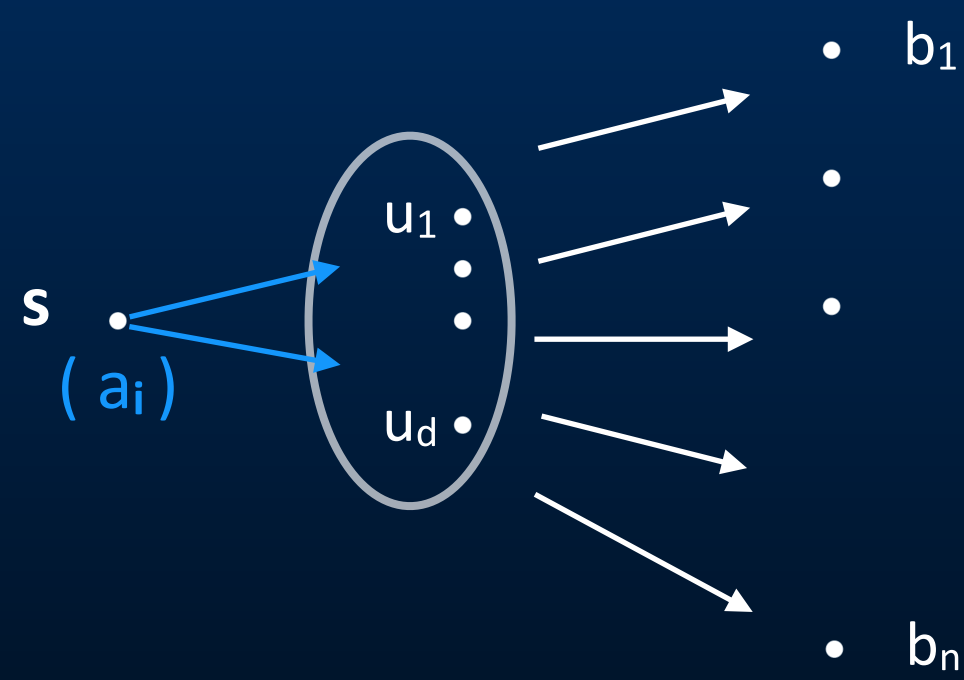
Proof:

Orthogonal Vectors → **dynamic #SSR**

	A	B	
a_1	(0,0,1,…,1)	(0,0,1,…,1)	b_1
	(0,1,1,…,1)	(1,0,1,…,1)	
	…	…	
a_n	(1,0,1,…,0)	(1,1,1,…,1)	b_n

(1,0,1,…,0)

(0,0,1,…,1)



add edge $u_j \to b_i$ iff $b_i[j]=1$

For each a_i:

1. add edges $s \to u_j$ iff $a_i[j]=1$
2. ask #SSR(s)

Theorem: If **dynamic #SSR** can be solved with $O(m^{0.99})$ update and query times, then **OV** can be solved in $\tilde{O}(n^{1.99})$ time (and SETH is false).

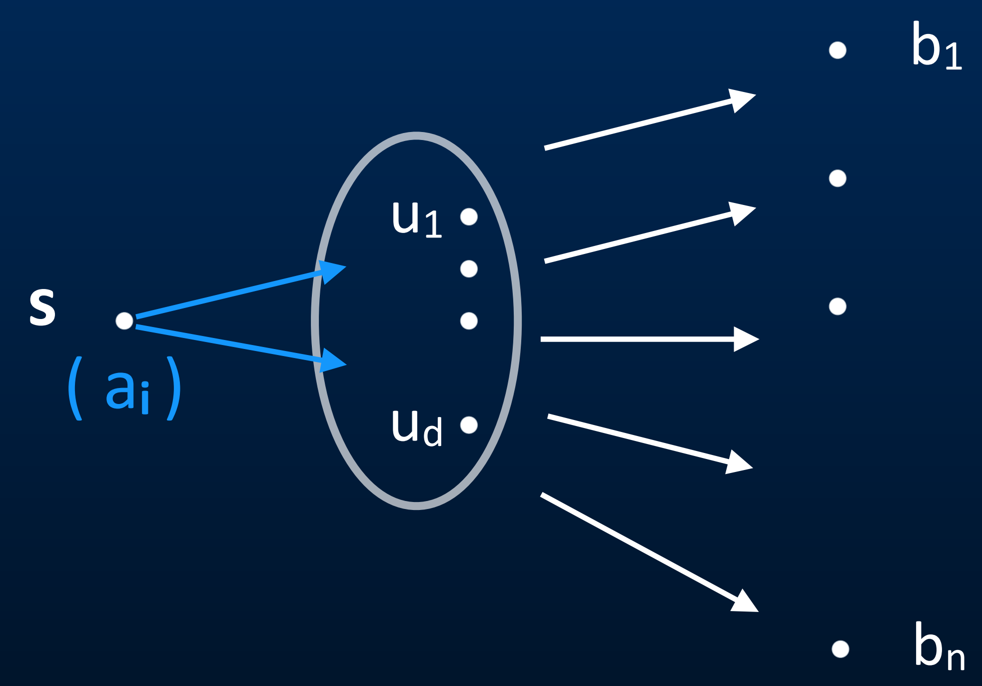
Proof:

Orthogonal Vectors → **dynamic #SSR**

a_1	(0,0,1,…,1)	(0,0,1,…,1)	b_1
→	(0,1,1,…,1)	(1,0,1,…,1)	
	…	…	
a_n	(1,0,1,…,0)	(1,1,1,…,1)	b_n

(1,0,1,…,0)

(0,0,1,…,1)



add edge

u_j → b_i

iff $b_i[j]=1$

Observation:

s cannot reach ***b*** iff a_i and ***b*** are orthogonal.

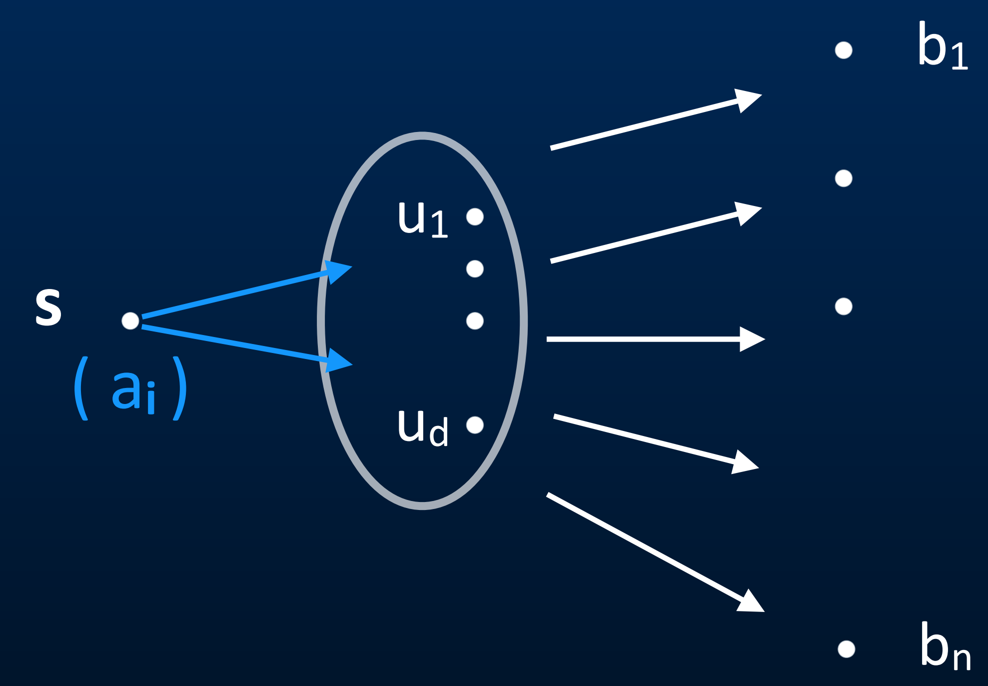
Theorem: If **dynamic #SSR** can be solved with $O(m^{0.99})$ update and query times, then **OV** can be solved in $\tilde{O}(n^{1.99})$ time (and SETH is false).

Proof: **Orthogonal Vectors** → **dynamic #SSR**

a_1	(0,0,1,…,1)	(0,0,1,…,1)	b_1
	(0,1,1,…,1)	(1,0,1,…,1)	
	…	…	
a_n	(1,0,1,…,0)	(1,1,1,…,1)	b_n

(1,0,1,…,0)

(0,0,1,…,1)



add edge

u_j → b_i

iff $b_i[j]=1$

For each a_i:

1. add edges s → u_j iff $a_i[j]=1$
2. ask #SSR(s),

   if $< n + (1s \text{ in } a_i)$, output "yes".
3. remove edges and move on to next a_i

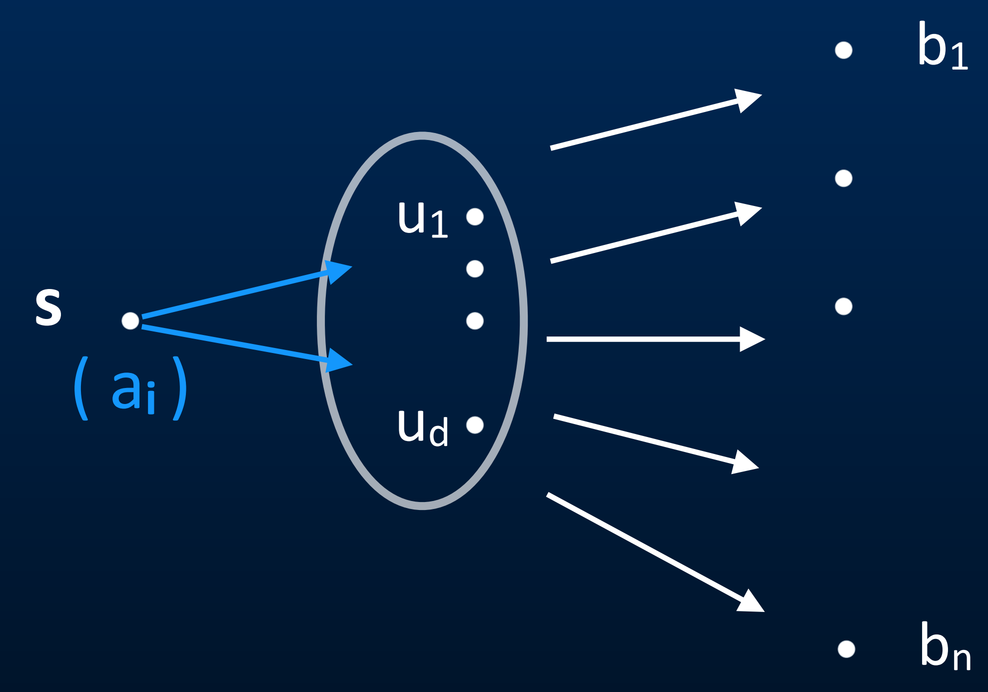
Theorem: If **dynamic #SSR** can be solved with $O(m^{0.99})$ update and query times, then **OV** can be solved in $\tilde{O}(n^{1.99})$ time (and SETH is false).

Proof: **Orthogonal Vectors** → **dynamic #SSR**

a_1 (0,0,1,…,1)
(0,1,1,…,1)
…
a_n (1,0,1,…,0)

b_1 (0,0,1,…,1)
(1,0,1,…,1)
…
b_n (1,1,1,…,1)

(1,0,1,…,0)
(0,0,1,…,1)



O(nd) updates,
m = O(nd) edges

~Ω(m) per update!

For each a_i:

1. add edges s → u_j iff $a_i[j]=1$
2. ask #SSR(s),
   and if ***< n + (1s in a_i)***, output "yes".
3. remove edges and move on to next a_i

Theorem: If **dynamic #SSR** can be solved with $O(m^{0.99})$ update and query times, then **OV** can be solved in $\tilde{O}(n^{1.99})$ time (and SETH is false).

Observation: The LB holds even if preprocessing time is $O(n^{100})$.

Why?

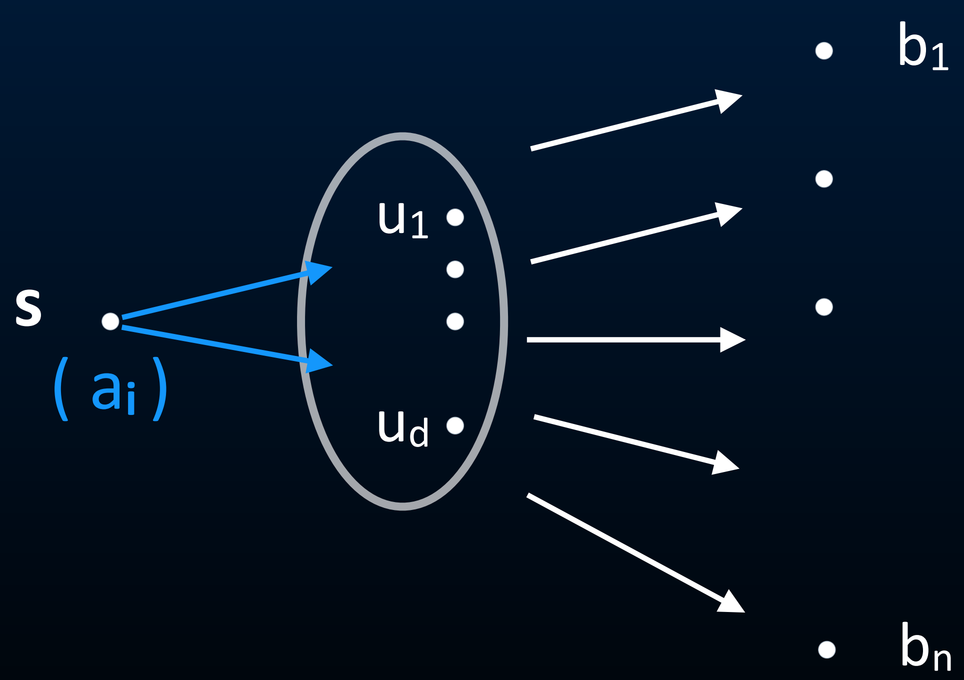
Lemma: OV with $|A| = n^a, |B| = n^b$ requires $\Omega(n^{a+b-\varepsilon})$ time.

Let $|A| = n, |B| = n^{1/100}$

LB is $\Omega(n^{1/100})$

but also $m = \tilde{O}(n^{1/100})$

(preprocessing time is negligible)



Conditional Lower Bounds?

Connectivity

Minimum Spanning Tree (MST)

Maximal Matching

Reachability

SETH

Strongly Connected Components (SCC)

s,t-shortest-path

(Bipartite) Maximum Matching

s,t-Max Flow

Diameter

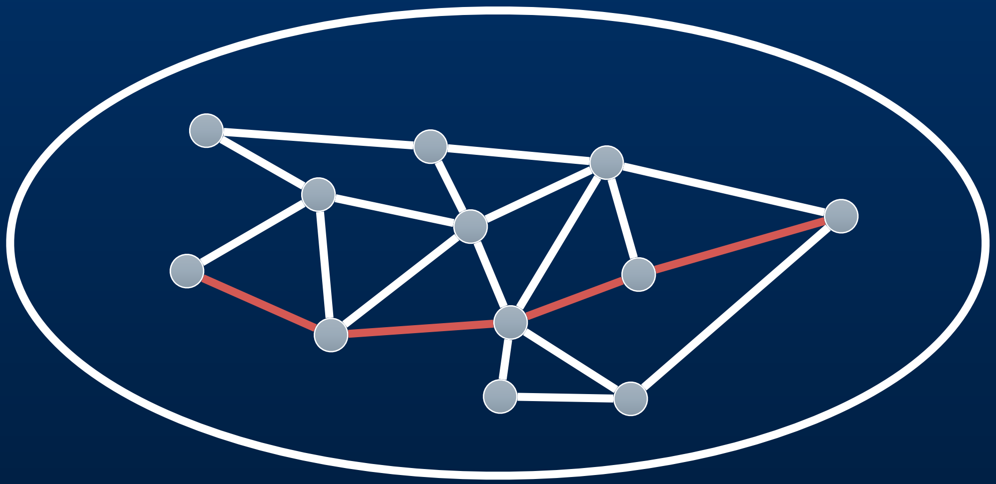
Next: an even higher SETH lower bound.

Dynamic Diameter

Input: an undirected graph G

Updates: Add or remove edges.

Query: What is the diameter of G?



Upper bounds for dynamic All-Pairs-Shortest-Paths:
Naive: ~*O(mn)* per update.
[Demetrescu-Italiano 03', Thorup 04']: amortized ~*O(n²)*.

Theorem: 1.3-approximation for the diameter of a sparse graph under edge updates with amortized $O(m^{1.99})$ updates refutes SETH!

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Proof outline:

Three Orthogonal Vectors

(0,0,…,1)	(1,0,…,1)	(1,0,…,1)
(0,1,…,1)	(0,1,…,0)	(0,0,…,1)
…	…	…
(1,0,…,0)	(1,0,…,1)	(1,1,…,0)

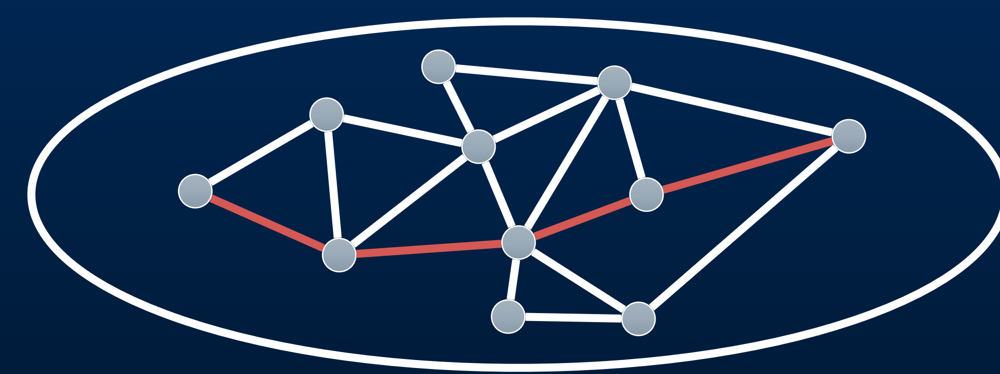
(1,0,1,…,0)

(0,1,1,…,0)

(1,1,1,…,0)

Given three lists of n vectors in $\{0,1\}^d$ is there an "orthogonal" triple?

dynamic Diameter



d=polylog(n)

Lemma: **3-OV** in $\tilde{O}(n^{3-\varepsilon})$ time refutes **SETH**

Theorem: 1.3-approximation for the diameter of a sparse graph under edge updates with amortized $O(m^{1.99})$ updates refutes SETH!

Proof outline:

Three Orthogonal Vectors

(0,0,…,1)	(1,0,…,1)	(1,0,…,1)
(0,1,…,1)	(0,1,…,0)	(0,0,…,1)
…	…	…
(1,0,…,0)	(1,0,…,1)	(1,1,…,0)

(1,0,1,…,0)

(0,1,1,…,0)

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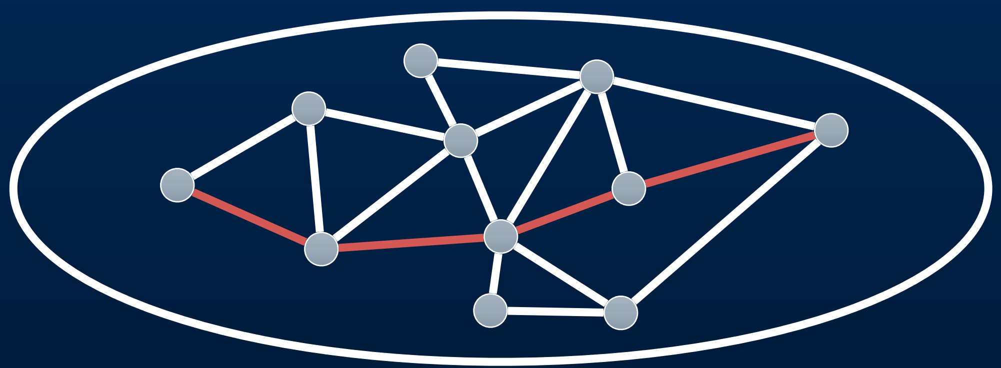
Given three lists of n vectors in $\{0,1\}^d$ is there an "orthogonal" triple?

dynamic Diameter



is the diameter 3 or more?

Graph G on ***m=O(nd)*** *nodes and edges,* **O(nd)** updates and queries

3-OVP in ***~O(n^2.9)*** time

(refutes SETH)

O(nd) updates/queries in ***~O(n^2.9)*** time

Amortized ***O(m^1.9)*** update/query time

d=polylog(n), m=~O(n)

Theorem: 1.3-approximation for the diameter of a sparse graph under edge updates with amortized $O(m^{1.99})$ updates refutes SETH!

Proof:

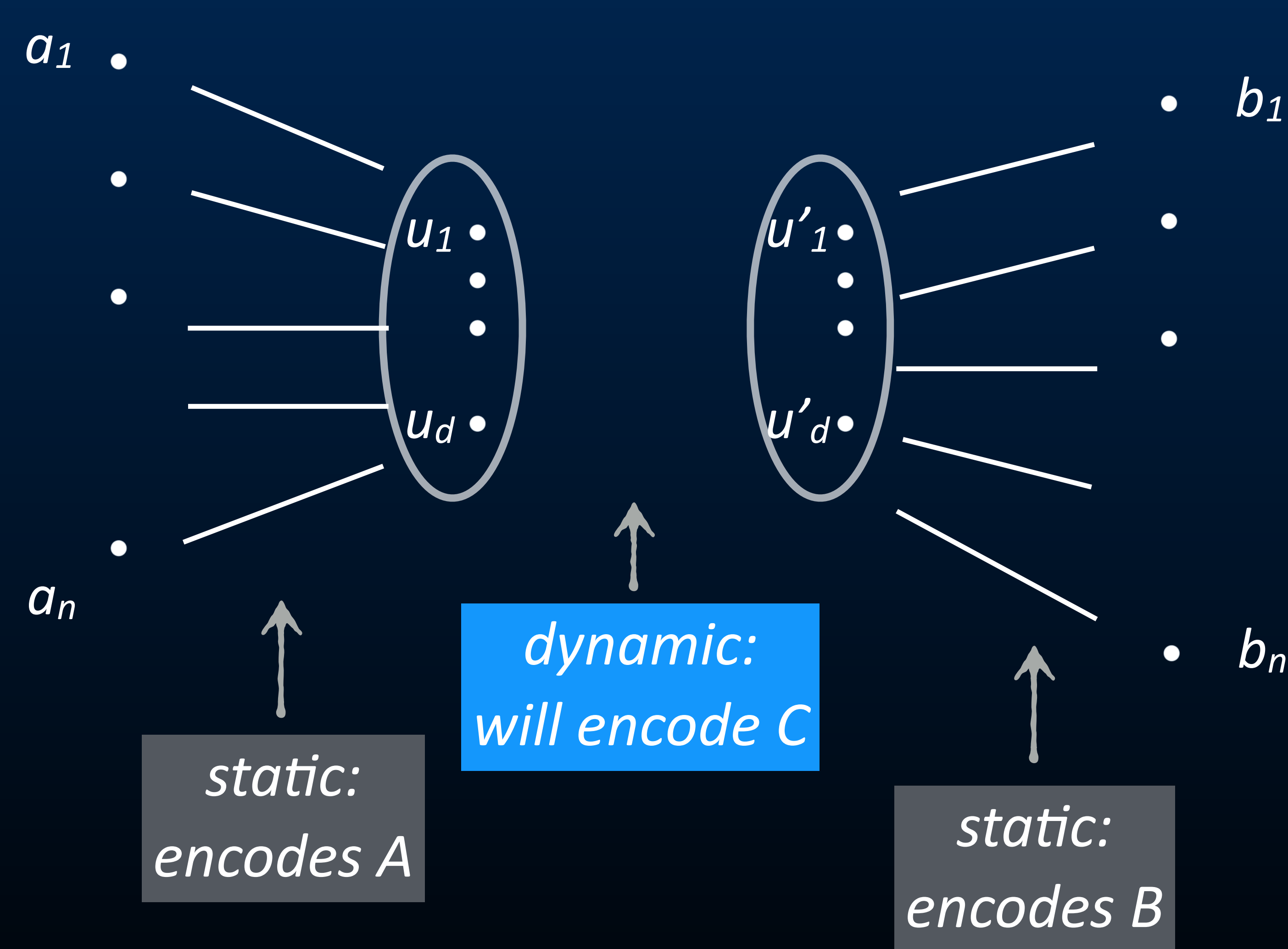
Three Orthogonal Vectors → **dynamic Diameter**

A	B	C
(0,0,…,1)	(1,0,…,1)	(1,0,…,1)
(0,1,…,1)	(0,1,…,0)	(0,0,…,1)
…	…	…
(1,0,…,0)	(1,0,…,1)	(1,1,…,0)

(1,0,1,…,0)

(0,1,1,…,0)

(1,1,1,…,0)



add edge

u'_j ——— b_i

iff $b_i[j]=1$

Theorem: 1.3-approximation for the diameter of a sparse graph under edge updates with amortized $O(m^{1.99})$ updates refutes SETH!

Proof:

Three Orthogonal Vectors → **dynamic Diameter**

A	B	C
(0,0,…,1)	(1,0,…,1)	(1,0,…,1)
(0,1,…,1)	(0,1,…,0)	(0,0,…,1)
…	…	…
(1,0,…,0)	(1,0,…,1)	(1,1,…,0)

(1,0,1,…,0)

(0,1,1,…,0)

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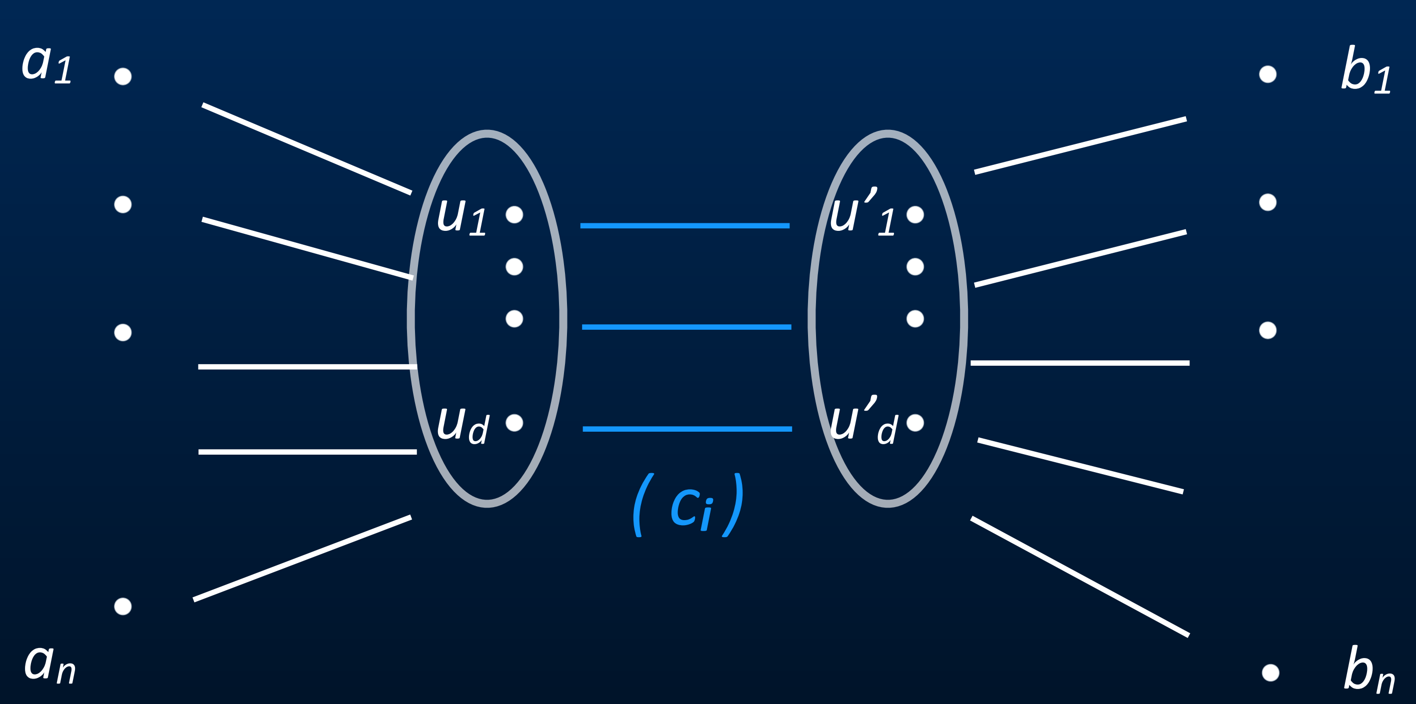


add edge

u'_j •——• b_i

iff $b_i[j]=1$

For each c_i:

1. add edges u_j •——• u'_j iff $c_i[j]=1$
2. ask Diameter query.

Theorem: 1.3-approximation for the diameter of a sparse graph under edge updates with amortized $O(m^{1.99})$ updates refutes SETH!

Proof:

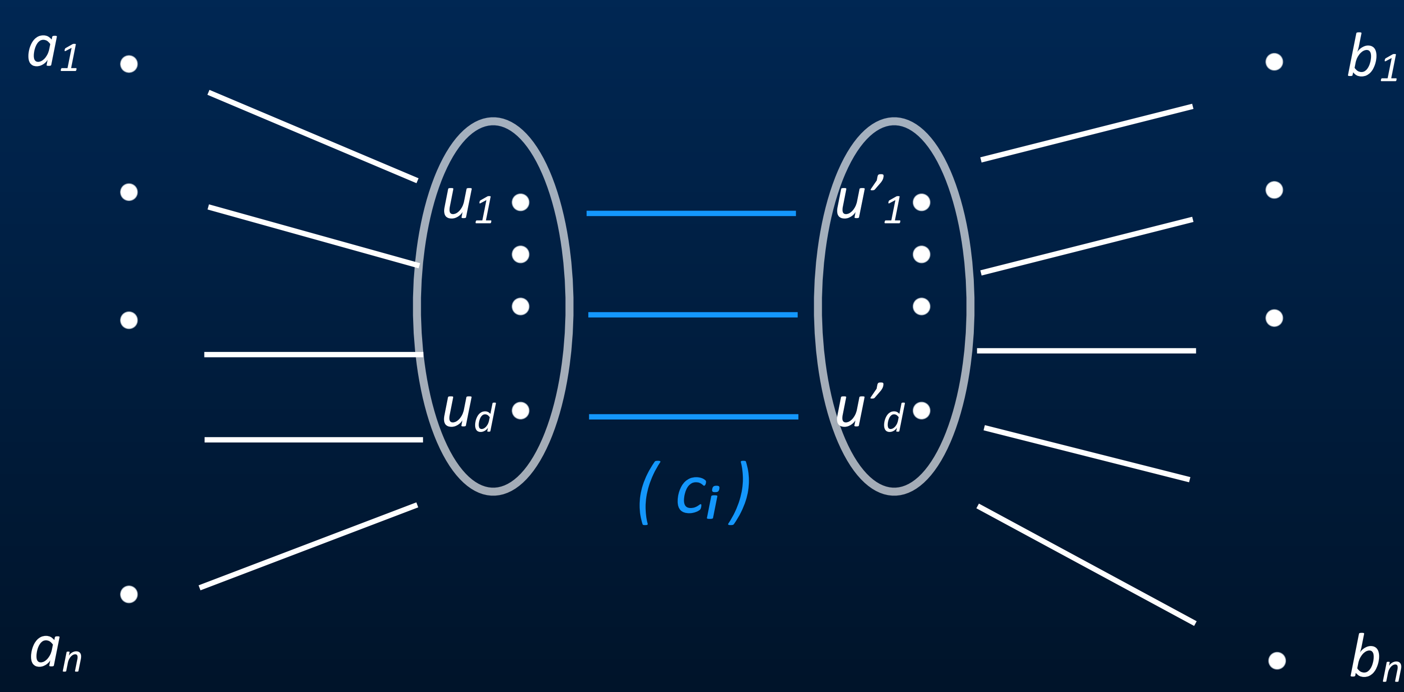
Three Orthogonal Vectors → **dynamic Diameter**

A	B	C
(0,0,…,1)	(1,0,…,1)	(1,0,…,1)
(0,1,…,1)	(0,1,…,0)	(0,0,…,1)
…	…	…
(1,0,…,0)	(1,0,…,1)	(1,1,…,0)

(1,0,1,…,0)

(0,1,1,…,0)

(1,1,1,…,0)



add edge

u'_j •———• b_i

iff $b_i[j]=1$

Observation:
The distance from *a* to *b* is more than ***3*** iff a,b,c_i are an orthogonal triple.

(no coordinate with all three 1's)

Theorem: 1.3-approximation for the diameter of a sparse graph under edge updates with amortized $O(m^{1.99})$ updates refutes SETH!

Proof:

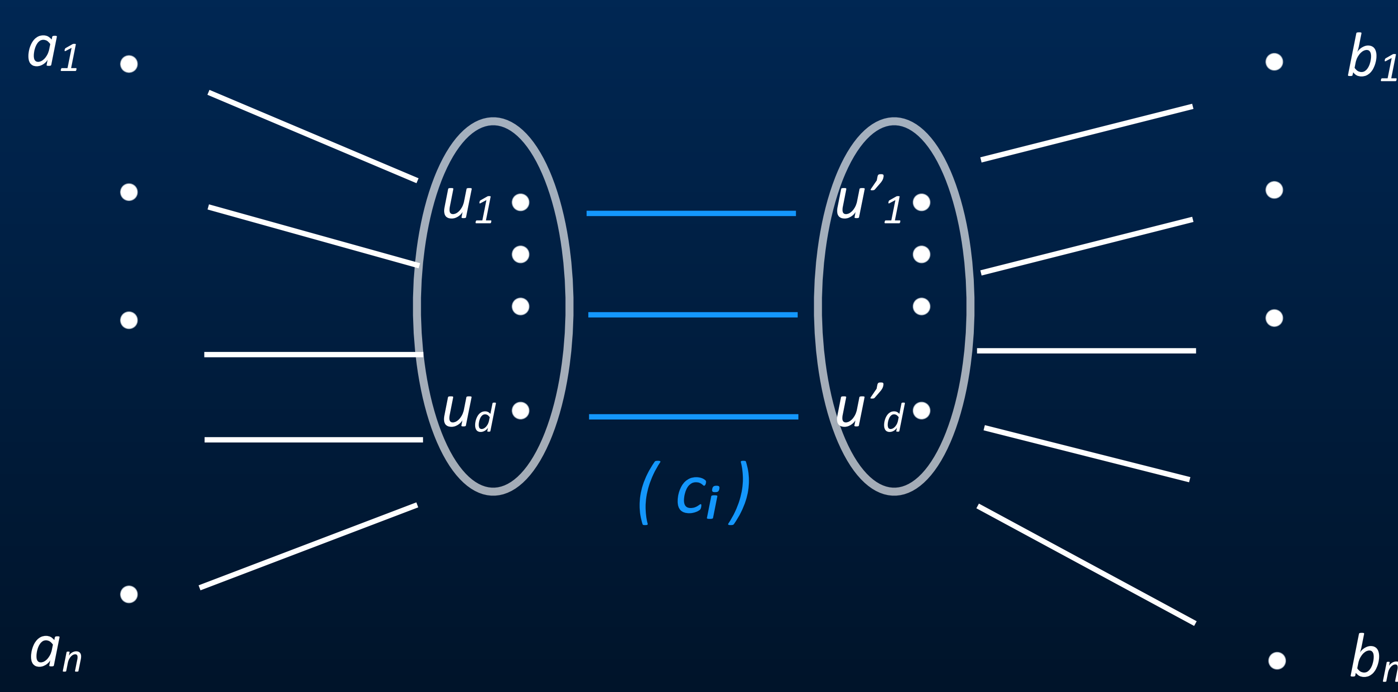
Three Orthogonal Vectors → dynamic Diameter

A	B	C
(0,0,…,1)	(1,0,…,1)	(1,0,…,1)
(0,1,…,1)	(0,1,…,0)	(0,0,…,1)
…	…	…
(1,0,…,0)	(1,0,…,1)	(1,1,…,0)

(1,0,1,…,0)

(0,1,1,…,0)

(1,1,1,…,0)



O(nd) updates,
m = *O(nd)* edges

~Ω(n²) per update!

For each c_i:

1. add edges u_j •——• u'_j iff $c_i[j]=1$
2. Query. If Diameter > 3, output "yes".
3. remove edges and move on to next c_i

Single Pair Problems

Theorem: s,t-shortest path with amortized $O(n^{1.99})$ updates refutes APSP.

APSP

↓

Negative Triangle ⟶ n-Pair-SP ⟶ dynamic s,t-SP

O(n) updates/queries

What about unweighted graphs?

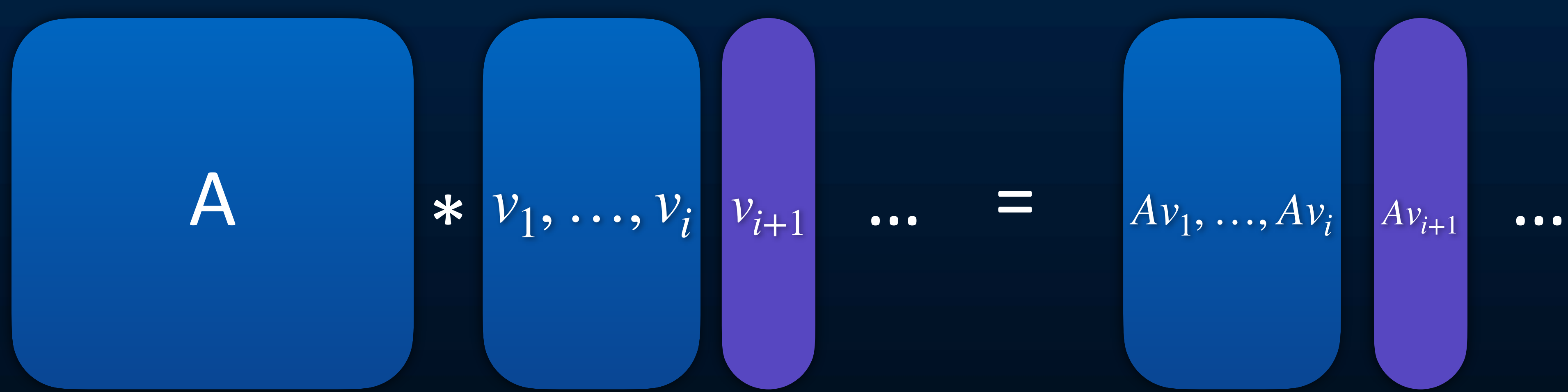
Can we assume that BMM/Triangle requires cubic time?

On Friday: "Combinatorial lower bounds"

OMv Lower Bounds

[Henzinger - Krinninger - Nanongkai - Saranurak STOC ‘15]
Most BMM lower bounds hold for non-combinatorial algorithms as well, under the Online Matrix Vector Multiplication Conjecture.

OMv problem: Given n x n Boolean matrix A and n Boolean vectors $v_1,\ldots,v_n$, given online, return each $A \cdot v_i$ right after v_i has been given.



<u>Theorem</u>: s,t-reachability with amortized $O(n^{0.99})$ updates refutes OMv.

Same for Maximum Bipartite Matching.

Fine-Grained Lower Bounds

Connectivity

Minimum Spanning Tree (MST)

Maximal Matching

Reachability

Strongly Connected Components (SCC)

s,t-shortest-path

(Bipartite) Maximum Matching

s,t-Max Flow

Diameter