APSP

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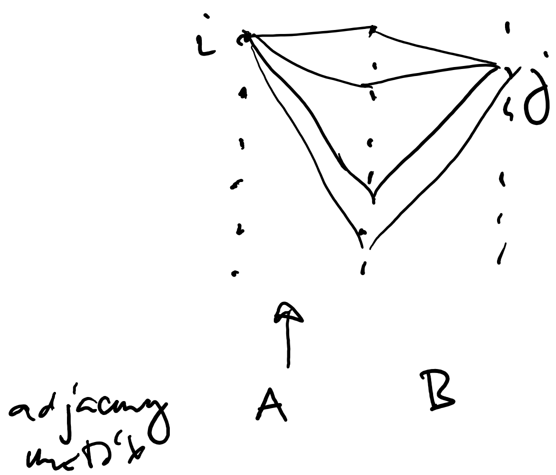
APSP: a weighted adjacency matrix of a graph G with no negative cycle, find the cost of the shortest path between any two vertices.

n ..., # of vertices in G $\longrightarrow$ $O(n^3)$ alg.

$(\min,+)$-product $\otimes_{\min}$ of $n \times n$ matrices A & B:

$$(A \otimes_{\min} B)_{ij} = \min_{k=1}^{n} A_{ik} + B_{kj}$$

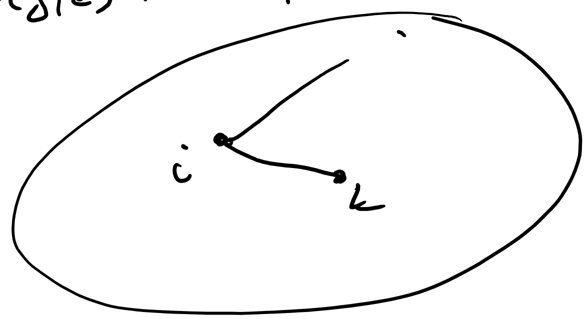


$A \otimes_{\min} B$ = cost of going from left to right.

Observation:
- If $\otimes_{\min}$ on $n \times n$ matrices can be done in time $T(n)$ $\Longrightarrow$ APSP can be done in $T(n) \log n$.
- If APSP can be done in time $T(n)$ then $\otimes_{\min}$ on $n \times n$ matrices can be done in time $T(n)$.

Pf: straightforward ✓ $\boxed{\boxtimes}$

• **Min-weight Triangle**: Given a graph G on n vertices with edge cost $w: E \to \mathbb{Z}$, Find $i, j, k \in V(G)$ minimizing $w(i,j) + w(j,k) + w(k,i)$



• Min-weight Triangle $\leq \otimes_{min}$ on $n \times n$ matrices

• **Negative Triangle**: Given G as for min-weight triangle, is the cost of the min-weight triangle negative?

• Negative Triangle $\leq$ Min-weight Triangle

Thm: $\otimes_{min} \leq$ negative triangle.

Pf:

All Pairs Min Triangle (APMT): Given G & $w: E \to \mathbb{Z}$
find for each $i, j \in V(G)$ $\min_k w(i,j) + (j,k) + w(k,i)$

All Pairs Negative Triangle (APNT): — // —
determine for each $i, j \in V(G)$: $[w(i,j) + w(j,k) + w(k,i) \underset{?}{<} 0]$

- APSP ≤ APNT ✓
- APNT in time $T(n) \Rightarrow$ APNT in time $T(n)\lg M$.

all entries $\in \{-M, \dots M\}$.

$\Rightarrow$ binary search for each i,j for the value D_{ij}

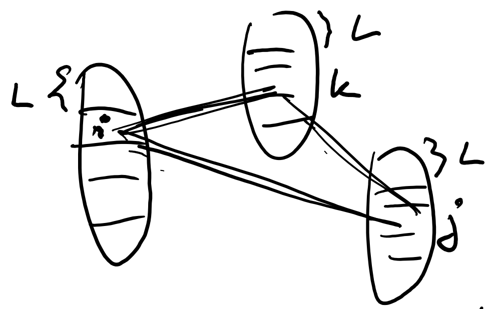
so that $w(i,j)+w(j,k)+w(k,i)+D_{ij}=0$.

- Finding a negative triangle $\equiv$ detecting negative triangle.
- Finding a negative triangle can be done in time $T(n)$
- If finding a negative triangle can be done in time $T(L)\left(n^2+\left(\frac{n}{L}\right)^3\right)$

then APNT can be done in time

for any $L \in \{1,\dots,n\}$.

Pf:



Say G is tripartite, divide each part into blocks of size L and for each triple of blocks find all triangles by repeated calls to find a triangle on that triple. Whenever a triangle i,j,k with negative weight is found, adjust weight $w(i,j)=6M$ & record the existence of negative $\triangle$ for i & j.

calls to finding negative $\triangle$:

$$T(L)\cdot n^2 + T(L)\cdot\left(\frac{n}{L}\right)^3$$

$$T(L)\cdot n^2 + T(L)\cdot\left(\frac{\dots}{L}\right)$$

successful call — 1 unsuccessful call for each triple of blocks

$\square$

$\Rightarrow$ If negative Δ can be found in time $O(n^{3-\varepsilon})$ then APSP can be solved in $\tilde{O}(n^{3-\varepsilon/3})$

Pf: set $L = n^{1/3}$.

$\square$

Graph radius

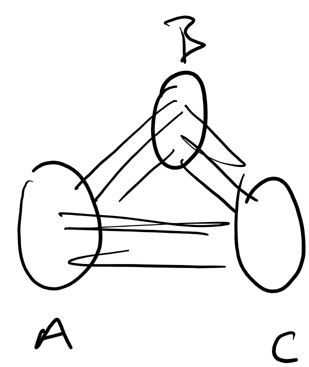
$$R(G) = \min_u \max_v d(u,v)$$ ... radius of a graph G

- $R(G) \le$ APSP ✓
- APSP $\le R(G)$

Pf: negative $\Delta \le R(G)$

$\hookrightarrow$ tripartite graph G

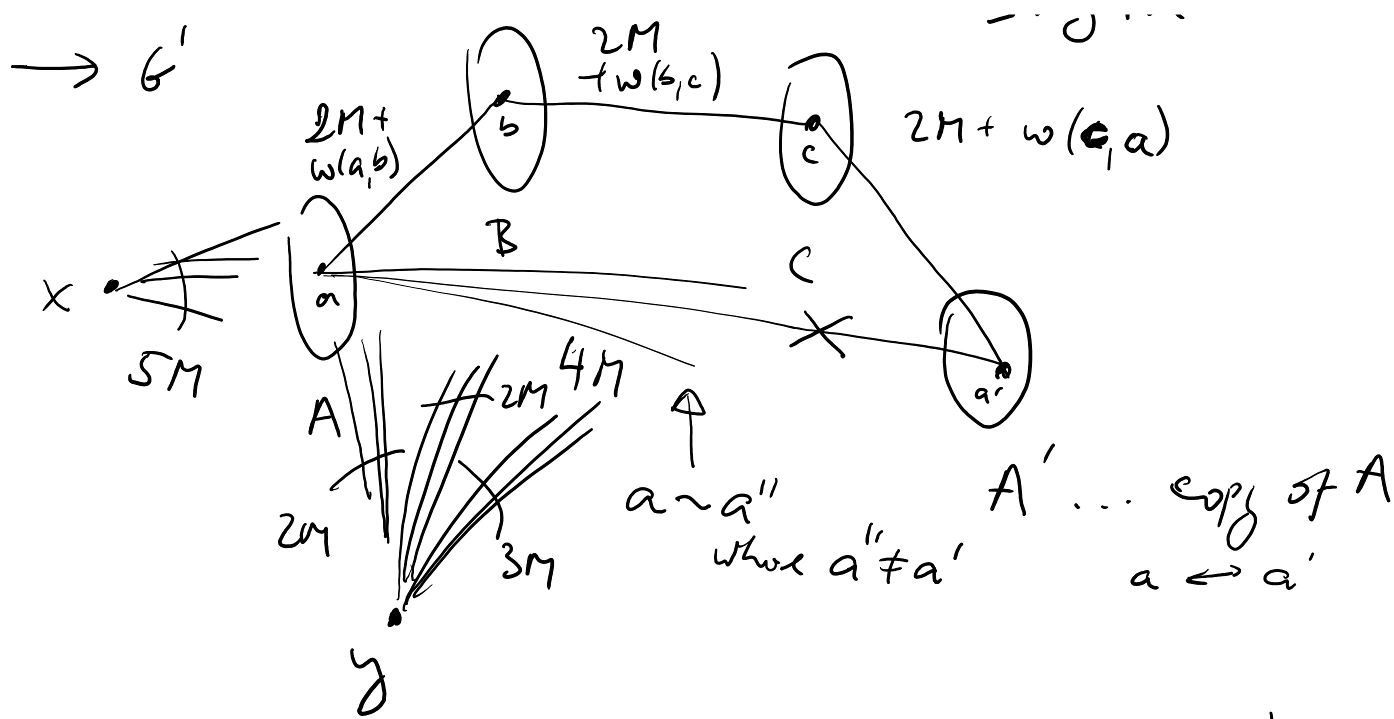
weights $\in \{-M, \dots, M\}$



$\exists$ negative Δ ?

$\to G'$

$2M + w(b,c)$



- negative $\triangle$ in G iff $\exists a \in A(G')$ $d(a,a') < 6M$

$\forall a \in A \quad \forall v \in \{x,y\} \cup \{B\} \cup \{C\} \cup \{A\}$

$$d(a,v) \leq 5M$$

$\forall a \in A \quad \forall a'' \in A \quad a \neq a'' \Rightarrow d(a,a'') \leq 4M$

$$d(a,a') = \begin{cases} < 6M & \text{if } \exists \triangle \text{ in } G \text{ passing through } a \\ \geq 6M & \text{otherwise} \end{cases}$$

- if radius is $< 6M$ then the center must be in A.