NTIN 100 Intro to Info Transmission and Processing summer 2019/2020

1st homework assignment - Entropy and mutual information

turn in by March 17th, 2020.

Problem 1.

a) For which probability distributions $p_1, p_2, \ldots, p_n$, the entropy $H(p_1, p_2, \ldots, p_n) = 0$?

b) Let probability distributions $p_1, p_2, \ldots, p_n$ and $p'_1, \ldots, p'_n$ be such that for some indexes i and j, and $\epsilon \in (0,1)$, $p'_i = \epsilon p_i + (1-\epsilon)p_j$, $p'_j = \epsilon p_j + (1-\epsilon)p_i$, and for all other k, $p'_k = p_k$. What is the relationship between $H(p_1, p_2, \ldots, p_n)$ and $H(p'_1, p'_2, \ldots, p'_n)$? How can one generalize this?

c) Prove that if $H(Y|X) = 0$ for two random variables X and Y then Y is a function of X.

Problem 2. Let X, Y, Z and $g(X)$ be random variables, that may or may not be dependent. Prove that

a) $H(X) \geq H(g(X))$,

b) $H(X, Y|Z) \geq H(X|Z)$,

c) $I(X, Y : Z) \geq I(X : Z)$,

d) $I(X : Y|Z) \geq I(X : Z|Y) - I(X : Z) + I(X : Y)$.

and decide when equality occurs.

Problem 3. For random variables X, Y, Z define $I(X : Y : Z) = I(X : Y) - I(X : Y|Z)$.

a) Show that $I(X : Y : Z)$ is symmetric that is $I(X : Y : Z) = I(Z : X : Y) = I(Y : Z : X)$.

b) Find X, Y, Z such that $I(X : Y : Z) < 0$.

c) Show that $I(X : Y : Z) = H(X, Y, Z) - H(X, Y) - H(Y, Z) - H(Z, X) + H(X) + H(Y) + H(Z)$.

Problem 4. For random variables X, Y, Z define $\rho(X, Y) = H(X|Y) + H(Y|X)$.

a) Show that $\rho(X, Y) \geq 0$, $\rho(X, Y) = \rho(Y, X)$, and $\rho(X, Y) + \rho(Y, Z) \geq \rho(X, Z)$.

b) If there is a bijection g such that $X = g(Y)$ we say that $X \approx Y$. Show that $\rho(X, Y) = 0$ if and only if $X \approx Y$.