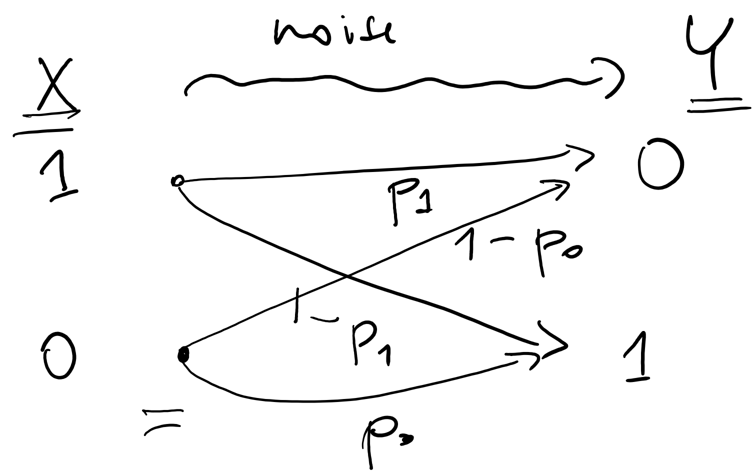
TI-Lecture-31-3-2020

31 March, 2020 12:58

Sending Data Over Unreliable channel

- relevant chapters from T&C.



Y – noisy variant of X

p_1 prob. of error when sending 1

p_0 prob. of error for 0.

- binary channel ... 0/1
- symmetric $p_0 = p_1 := p$

$$I(X:Y) = H(Y) - \underline{H(Y|X)} = H(Y) - \sum_{x \in \{0,1\}} p(x) \cdot \underbrace{H(Y|X=x)}_{H(p)}$$

$$= H(Y) - H(p) \leq \underline{1 - H(p)}.$$

- channel capacity : $C = \max_X I(X:Y) \leq 1 - H(p)$.

- k bits

$$\underbrace{X_1 \dots X_k}_{X} \rightsquigarrow \underbrace{Y_1 \dots Y_k}_{Y}$$

$p = 0{,}01.$ $k = 100$

$\Pr[\text{correct transmission i.e. } X = Y] = (1-p)^k$

$= 0.99^{100} = 0.37$

$k = 1000$ $\leq (0.37)^{10} \leq \frac{1}{1000}$

Ex 2:

$$\begin{cases} 0 \to 000 \\ 1 \to 111 \end{cases}$$

$$\underbrace{X}_{k \text{ bits}} \xrightarrow{\text{encoding}} \underbrace{C(X)}_{\cancel{k}\, n \text{ bits}} \rightsquigarrow Y' \xrightarrow{\text{decode}} X'$$

hope $X = X'$

(edit errors - insertion/deletions/flips)
only later

rate $R = \frac{k}{n}$ ... rate of transmission

$0 \leq \quad \leq 1$

$r = \frac{1}{3} = \frac{100}{300}$ $k = 100$ $n = 300$

$p = 0.01$

decode:

000	$\to$	0
010	$\to$	0
100	$\to$	0
001	$\to$	0
101	$\to$	1
011	$\to$	1
110	$\to$	1
111	$\to$	1

$\boxed{p < \frac{1}{2}}$

decoding to closest codeword

$$\Pr[\text{error in a given triplet}] = p^3 + \binom{3}{2} p^2 (1-p) \overset{(*)}{=}$$

$\binom{3}{2} = 3$

$$= \underline{0.000298}$$

k triplets

$$\Pr[\text{decoding correctly}] = (1-(*))^{1000} \geq \underline{0.97}$$

Q: How to encode & decode as to minimize the error prob. for a given channel.

- tradeoff between rate & error prob'ty?

close $\approx \underline{1 - H(p)}$ error ≈ 0 in a avg.

- $C \subseteq \{0,1\}^n$ ... code

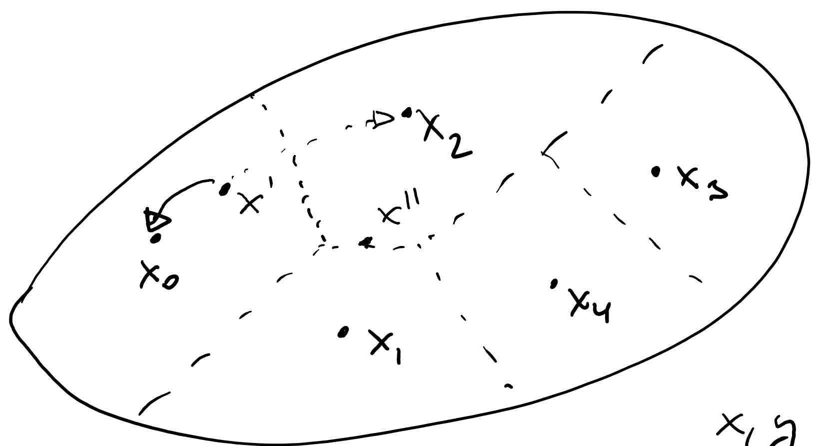
... codeword

- $C \subseteq \{0,1\}^n$ ... code

$x \in C$ $\qquad$ x ... codeword

[• $\{0,1\}^k \to C$ $\qquad$ irrelevant for quality of C when msg's are picked uniformly]

- encoding to the closest codeword



$\{0,1\}^n$

(Voronoi diagram)

Haming distance

$x, y \in \{0,1\}^n$

$\Delta_{Ham}(x,y) = |\{i;\ x_i \neq y_i\}|$

$$\cancel{k} < \cancel{k}'$$

$$p^{\ell}(1-p)^{n-\ell} < p^{\ell'}(1-p)^{n-\ell'} \Leftarrow p < \frac{1}{2}$$

- decoding to the closest codeword

<u>Thm:</u> (Shannon) Let $p < \frac{1}{2}$ be error pblity for binary symmetric channel. ~~the~~ Let $0 < R < 1 - H(p)$

$\forall \varepsilon > 0 \ \forall n$ large enough $\exists C \subseteq \{0,1\}$

s.t. $|C| = 2^{Rn} = 2^k$ & the probability of

$R = \frac{k}{n}$ Incorrect decoding of arbitrary msg $\in \{0,1\}^k$

sent over the channel is $\leq \varepsilon$.

~~x~~ msg ~~m~~ $\rightarrow \ x = C(m) \rightsquigarrow m' \rightarrow x' = Dec(m')$

$\Pr[x = x'] \geq 1 - \varepsilon$

$C_1, C_2, \dots \quad C_n \subseteq \{0,1\}^n \qquad$ error prob. $\varepsilon_n = 2^{-\Theta(n)} \xrightarrow[n\to\infty]{} 0$

· n bits ... how many errors do you expect to see?

$E[\#\text{errors}] = np$

... #errors is concentrated

= actual # of errors is close to np with high probability

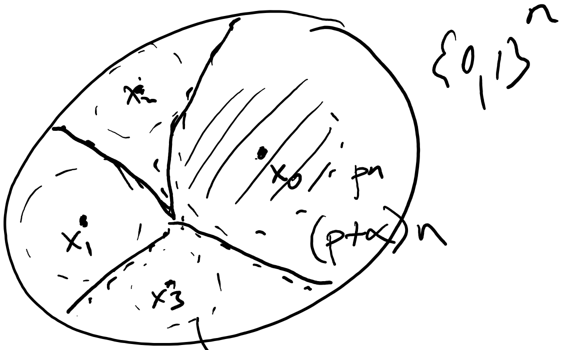
<u>Fact</u> (Chernoff inequality): $\forall 0<p<1$, $\forall 0<\alpha<1$.

$\exists$ constant $c_{p,\alpha}$. $X_1 \dots X_n$ r.var's independent ~~all~~ each distributed according to $X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$

$$\Pr\left[\left|\sum_{i=1}^n X_i - np\right| \geq \alpha n\right] \leq \underbrace{2 \cdot e^{-c_{p,\alpha} n}}.$$

$$\Pr\left[\left|\sum_{i=1}^{n} X_i - np\right| \geq \alpha n\right] \leq \alpha \cdot e^{\dots}$$

$$n \to \infty \qquad \to 0$$



$$Vol(n,\underline{r}) = \left|\left\{ y \in \{0,1\}^n ;\ \Delta_{\text{Hamm}}(y, 0^n) \leq r \right\}\right|$$

independ of 0^n

$$Ball_{n,r}(x_3) = \{ y \in \{0,1\}^n ;\ D_{\text{Ham}}(x_3, y) \leq r \}.$$

radius

Lemma: $\forall n, r \geq 1$, $r \leq \frac{n}{2}$: $\frac{2^{nH(\frac{r}{n})}}{n} \leq Vol(n,r) \leq 2^{nH(\frac{r}{n})}$.

$$\left[\frac{r}{n} \approx p \qquad 2^{n\underline{H(p)}}\right]$$

Pf: $Vol(n,r) \leq 2^{nH(\frac{r}{n})}$

$$n^n = (\underbrace{r}_{a} + \underbrace{(n-r)}_{b})^n = \sum_{i=0}^{n} \binom{n}{i} r^i (n-r)^{n-i} \geq \sum_{i=0}^{r} \binom{n}{i} r^r (n-r)^{n-r}$$

$r \leq \frac{n}{2}$ immediate

$(a+b)^n$

$$\frac{n^n}{r^r (n-r)^{n-r}} \geq \sum_{i=0}^{r} \binom{n}{i} = Vol(n,r)$$

$$= \left(\frac{n}{r}\right)^r \cdot \left(\frac{n}{n-r}\right)^{n-r} = 2^{r \lg \frac{n}{r} + (n-r) \lg \frac{n}{n-r}}$$

$$\left[r \lg \frac{n}{r} \ldots n-r \lg \frac{n}{n-r} \right]$$

$$\left(\frac{n}{r}\right)^{\cdot} \cdot \left(\frac{n}{n-r}\right) = 2$$

$$\sim 2^{r \cdot (\lg \frac{n}{r})}$$

$$= 2^{n \cdot \underbrace{\left[\frac{r}{n} \cdot \lg \frac{n}{r} + \frac{n-r}{n} \lg \frac{n}{n-r}\right]}_{H\left(\frac{r}{n}\right)}}$$

$$1 - \frac{r}{n} = \frac{n-r}{n}$$

$$\Rightarrow \quad 2^{n H\left(\frac{r}{n}\right)} \geq Vol(n, r). \qquad \blacksquare$$

$\boxed{\text{l-book}}$