TI-Lecture-12-5-2020

12 May, 2020 13:56

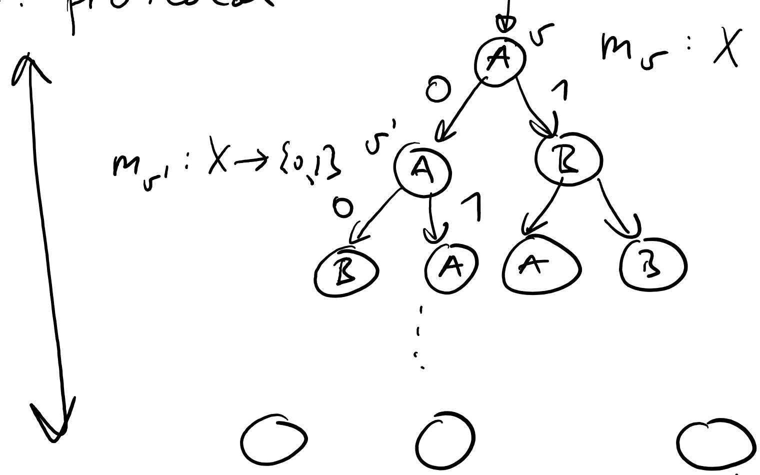
Alice Bob

$x \in \{0,1\}^n$ $y \in \{0,1\}^n$

$f(x,y)$

... protocol

Communication cost



both know the answer

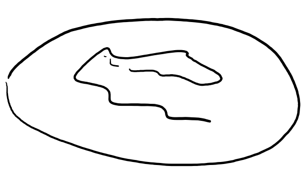
$F : \cancel{X} \times Y \to \{0,1\}$ $X = Y = \{0,1\}^n$

$D(f) = \min_P$ depth of P; P is a protocol for f

Ex: $EQ(x,y) = [x \stackrel{?}{=} y]$ $n+1$

$DISJ(x,y) = [\exists i \text{ s.t. } x_i = y_i = 1]$ $n+1$

$MED(x,y) = \dots$ $\dots$



Combinatorial rectangle $X = \{0,1\}^n$ $Y = \{0,1\}^n$

Combinatorial rectangle

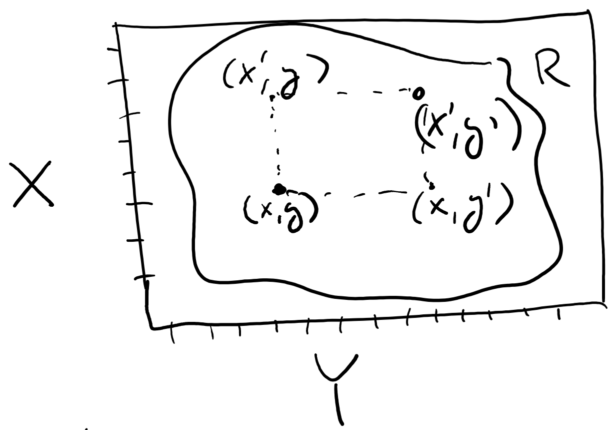
$X = \{0,1\}^n \quad Y = \{0,1\}^n$

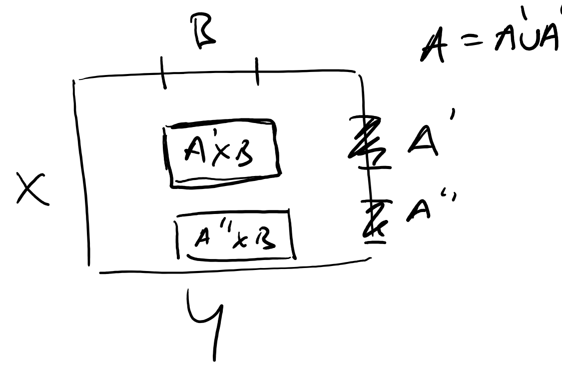
- $A \times B$ — combinatorial rectangle, $\quad A \subseteq X \quad B \subseteq Y$

Claim: $R \subseteq X \times Y$.

R is a combinatorial rectangle iff

$(*) \quad \forall (x,y), (x',y') \in R \; : \; (x,y') \in R, \; (x',y) \in R$ eq.





"$\Rightarrow$" trivial

"$\Leftarrow$" $\quad A = \{x \,;\, \exists y \; (x,y) \in R\} \quad B = \{y \,;\, \exists x \; (x,y) \in R\}$ ✓

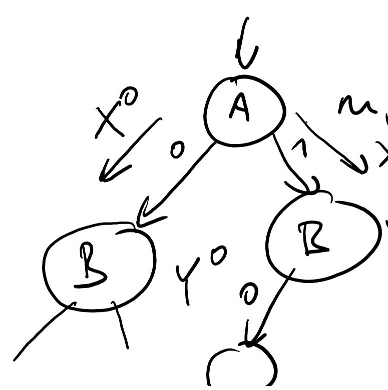
$A \times B = R \qquad R \subseteq A \times B$

$A \times B \subseteq R$?

$(x,y) \in A \times B \quad x \in A \Rightarrow \exists y' \; (x,y') \in R \quad \exists x' \; (x',y) \in R$

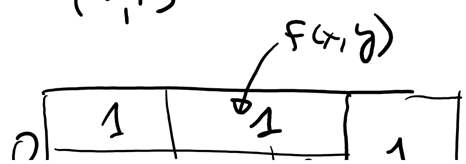
by $(*)$ $\; (x,y) \in R \quad \Rightarrow R = A \times B$ ∎

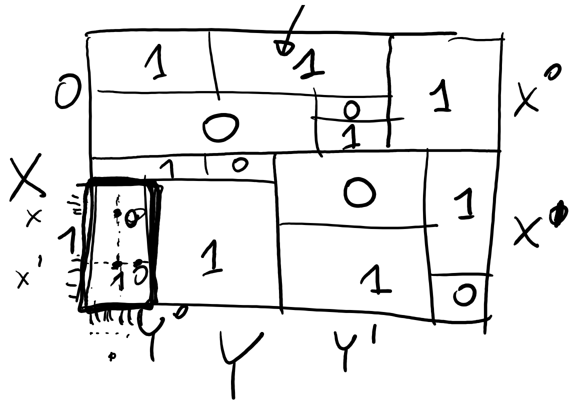
$X = X^0 \cup X^1$



$m_v : X \to \{0,1\}$

$m_{v'} : Y \to \{0,1\}$





the inputs that reach a particular node in the protocol form a combinatorial rectangle

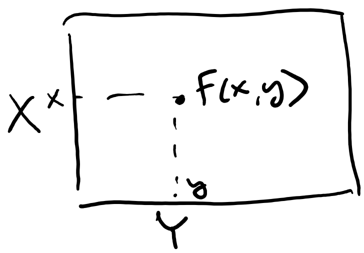
(x,y) (x',y') reach the same node

then (x',y) would also reach the same node
(x,y')

monochromatic rectangle ... rectangle of inputs (x,y) with the same $f(x,y)$ value.

0 - rectangle $f(x,y)=0$
1 - rectangle $f(x,y)=1$

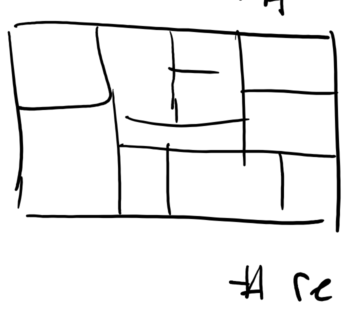
M_f



• protocol for f splits M_f into monochromatic rectangles.

M_f

$P \rightarrow$



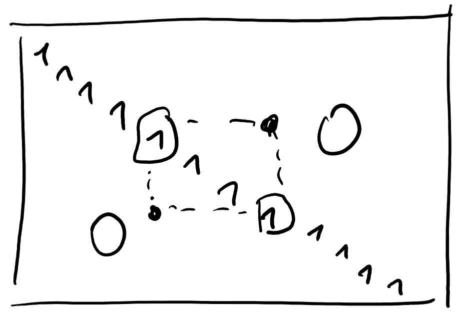
$d(P)$ # rectangles $\leq 2^{d(P)}$

• If f has com. cost. $D(f)$ then M_f can be partitioned into $\leq 2^{D(f)}$ monochromatic combinatorial rectangles.

Ex: $EQ[x=y]$

$X = Y = \{0,1\}^n$

X



M_{EQ}

Y

- How many mono. comb. rectangles to cover M_{EQ}?

→ each one (x,x) has to be in a separate comb. rect.

$2^{D(f)} \geq$ #mono rect. $\geq 2^n + 1$

$2^{D(f)} \geq 2^n + 1$

$f = EQ$

$2^n \cdot 2$

$D(f) > n$

$D(f) \leq n+1 \implies D(f) = n+1$ ☒

$A_i \times B_i$

$A_i' \times B_i'$

i..

$A_i = \{x ;\ x_i = 1\}$ $B_i = \{y ;\ y_i = 0\}$

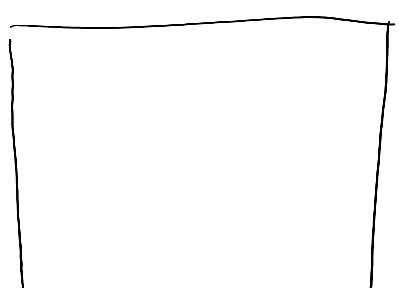
$A_i' = \{x ,\ x_i = 0\}$ $B_i' = \{y ;\ y_i = 1\}$

↳ overlapping

Ex: $DISJ(x,y) = [\exists i,\ x_i = y_i = 1]$

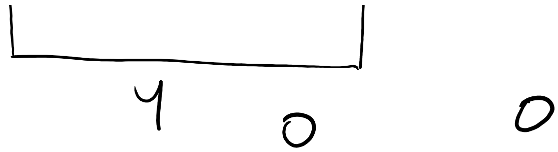
$X = Y = \{0,1\}^n$

X



M_{DISJ}

Y



$DISJ(x,y)=0 \qquad (x,\bar{x}), \quad (x',\bar{x'})$

$x \neq x' \Rightarrow x \cap \bar{x'} \neq \emptyset$ or $x' \cap \bar{x} \neq \emptyset$

1 or 1

$\geq 2^n$ 0-mono rectangles

$\Rightarrow \geq 2^n+1 \quad \Rightarrow D(DISJ) \geq n+1$

$\Rightarrow D(DISJ) = n+1$ $\boxtimes$

f ... How to cover M_f by monochromatic rectangles.

- If you need at least t of them then $D(f) \geq \lg t$.

Ex:

$|X| = |Y| = 3$

X \ Y				
	1	0	0	1
	1	1	1	2
	0	0	1	3

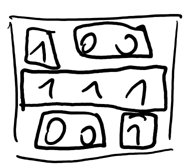
M_f

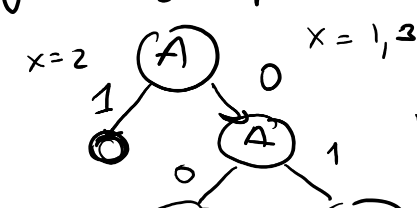
Bad example
Hm, Ooops

$C(f)$ ... # mono rectangles to cover M_f. 5

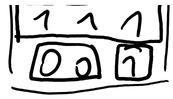
$C^D(f)$ ... # of rectangles partitioning M_f monochromatically

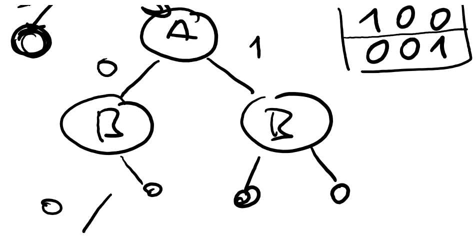
$C^P(f)$ ... # of rectangles given by a protocol for f 6

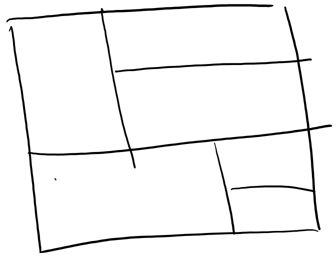




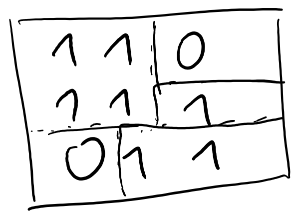
1	0	0
0	0	1





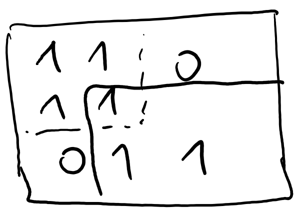


Ex:



non-overlapping $C^D(f) = 5$

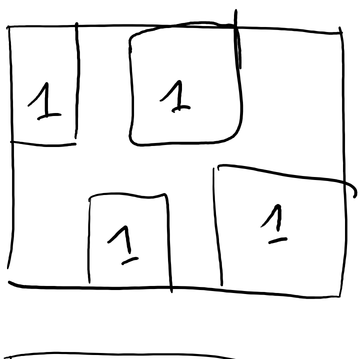
$\geq$



overlapping $C(f) = 4$

$$C(f) \le C^D(f) \le C^P(f) \le 2^{D(f)}.$$

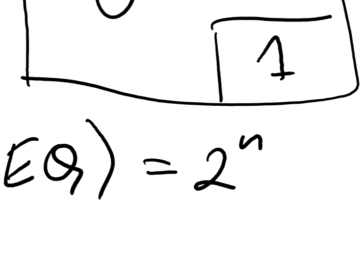
Ex: • M_f has rank $\geq r$ (over reals) then $D(f) \geq \lg r$.



M_f

$\mathrm{rank}(M_f) \le \overbrace{\#\text{ 1-mono rectangles}}^{m}$

$M_f = \sum$ sum of m rank 1 matrices



rank 1 matrix

· $\mathrm{rank}(M_{EQ}) = 2^n$

$D(f) \geq \lg \mathrm{rank}(M_f)$.

· $\operatorname{rank}(M_{EQ}) = 2^n$

$D(f) \geq \lg \operatorname{rank}(M_f)$.

· $IP : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

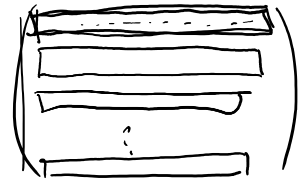
$IP(x,y) = \underline{\sum_{i=1}^{n} x_i \cdot y_i}$ mod 2

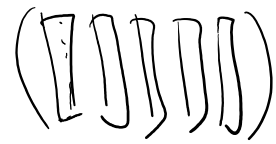
$\operatorname{rank}(IP) \geq 2^n - 1$

$\Rightarrow$ $\underline{D(IP) \geq n}$.

Hadamard code

M_{IP}





$$(M_{IP})^2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ & 2^{n-1} & & 2^{n-2} \\ & & \ddots & \\ & 2^{n-2} & & 2^{n-1} \end{pmatrix}$$

2^{n-1}

$\operatorname{rank}(M_{IP}^2) = 2^n - 1$

$(2^{n-2} \cdots 2^{n-2})$