Algorithms as Lower Bounds

Lecture 3: NEXP vs ACC

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Definition: ACC Circuits

An **ACC** circuit family **{ C_n }** has the properties:

- Every C_n takes n bits of input and outputs a bit
- There is a fixed ***d*** such that every C_n has depth at most ***d***
- There is a fixed ***m*** such that the gates of C_n are

AND, OR, NOT, MODm (unbounded fan-in)

MODm($x_1,\ldots,x_t$) = 1 iff $\sum_i x_i$ is divisible by m

Remarks

1. The default size of C_n is **polynomial in n**
2. ***Strength:*** this is a ***non-uniform*** model of computation (can compute some undecidable languages)
3. ***Weakness:*** ACC circuits can be efficiently simulated by ***constant-layer neural networks***

Proof Strategy for ACC Lower Bounds

1. **Show that faster ACC-SAT algorithms imply lower bounds against ACC**

Theorem **(Example)**

If **ACC-SAT** with n inputs and $\mathbf{2^{n^{o(1)}}}$ size is in **$O(2^n/n^{10})$ time** (for all constant depths and moduli), then **EXP^{NP}** doesn't have $\mathbf{2^{n^{o(1)}}}$ size ACC circuits.

2. **Design faster ACC-SAT algorithms!**

Theorem For all **d, m** there's an **$\varepsilon > 0$** such that **ACC-SAT** on circuits with n inputs, depth **d**, **MODm** gates, and $\mathbf{2^{n^{\varepsilon}}}$ size can be solved in **$2^{n - \Omega(n^{\varepsilon})}$ time**

Detailed Proof

Theorem If ACC-SAT on circuits with n inputs and $2^{n^{o(1)}}$ size is in $O(2^n/n^{10})$ time, then
EXP^{NP} doesn't have $2^{n^{o(1)}}$ size ACC circuits.

Proof Idea Show that if both:

- **ACC-SAT with n inputs and $2^{n^{o(1)}}$ size is in $O(2^n/n^{10})$ time**
- **EXP^{NP} has $2^{n^{o(1)}}$ size ACC circuits**

then **$NTIME[2^n] \subseteq NTIME[o(2^n)]$** **(a contradiction)**

Work with a "compressed" version of the 3SAT problem:

Exponentially long formulas are encoded with polynomial-size circuits

Theorem If ACC-SAT on circuits with n inputs and $2^{n^{o(1)}}$ size is in $O(2^n/n^{10})$ time, then $\mathbf{EXP}^{\mathbf{NP}}$ isn't in $2^{n^{o(1)}}$ size ACC.

For a circuit $\mathbf{C} : \{0,1\}^n \to \{0,1\}$, let **tt(C)** be its truth table:

the output of **C** on all 2^n assignments, in lex. order

Succinct 3SAT: ***Given a circuit C, is tt(C) a satisfiable 3CNF?***

Theorem [GW, PY '80s] Succinct 3SAT is **NEXP**-complete.

Succinct 3SAT is in NEXP: evaluate circuit C on all possible assignments, and solve the resulting 3SAT instance

Succinct 3SAT is NEXP-hard. Follows from:

"For all L ∈ NP, there's a TIME[poly(log n)] reduction from L to 3SAT"

Padding ⇒ "For all L ∈ NEXP, there is a TIME[poly(n)] reduction from L to exponentially-long 3SAT"

The TIME[poly(n)] reduction can be described with a circuit!

Theorem If ACC-SAT on circuits with n inputs and $2^{n^{o(1)}}$ size is in $O(2^n/n^{10})$ time, then $\mathbf{EXP^{NP}}$ isn't in $2^{n^{o(1)}}$ size ACC.

For a circuit $\mathbf{C : \{0,1\}^n \to \{0,1\}}$, let **tt(C)** be its truth table: the output of **C** on all 2^n assignments, in lex. order

Succinct 3SAT: ***Given a circuit C, is tt(C) a satisfiable 3CNF?***

Lemma 1 [..., JMV'15] For all $L \in \mathbf{NTIME[2^n]}$, there is a polytime reduction R_L from L to Succinct 3SAT such that:

- $x \in L \Leftrightarrow R_L(x) = C_x$ encodes a satisfiable 3CNF formula
- C_x is **ACC**, has size $\mathbf{n^{10}}$, and **n + 4 log n** inputs, where n = |x|

Corollary **Succinct 3SAT for ACC circuits of n inputs & $\mathbf{n^{10}}$ size is in nondet $2^n\, poly(n)$ time but not in nondet $\frac{2^n}{n^5}$ time.**

(Otherwise, we'd contradict the nondet. time hierarchy!)

Theorem If ACC-SAT on circuits with n inputs and $2^{n^{o(1)}}$ size is in $O(2^n/n^{10})$ time, then $\mathbf{EXP}^{\mathbf{NP}}$ isn't in $2^{n^{o(1)}}$ size ACC.

Succinct 3SAT: ***Given a circuit C, is tt(C) a satisfiable 3CNF?***

Say that **Succinct 3SAT has ACC satisfying assignments** if for **every C** such that **tt(C)** is a satisfiable 3CNF, there is an ACC circuit **D** of $\mathbf{2^{|C|^{o(1)}}}$ **size** such that **tt(D)** is a variable assignment that satisfies **tt(C)**.

Succinct 3SAT has ACC satisfying assignments

$\equiv$ *"All satisfiable formulas which are compressible have a satisfying assignment which is somewhat compressible"*

Lemma 2 If $\mathbf{EXP}^{\mathbf{NP}}$ has $2^{n^{o(1)}}$ size ACC circuits then **Succinct 3SAT has ACC satisfying assignments**

Theorem If ACC-SAT on circuits with n inputs and $2^{n^{o(1)}}$ size is in $O(2^n/n^{10})$ time, then $\mathbf{EXP^{NP}}$ isn't in $2^{n^{o(1)}}$ size ACC.

Succinct 3SAT: ***Given a circuit C, is tt(C) a satisfiable 3CNF?***

Lemma 2 If $\mathbf{EXP^{NP}}$ has $2^{n^{o(1)}}$ size ACC circuits then **Succinct 3SAT has ACC satisfying assignments**

Proof The following can be computed in $\mathbf{EXP^{NP}}$:

On input (C, i), use an NP oracle and binary search to find the lexicographically first satisfying assignment to tt(C). Output the i-th bit of this assignment.

By assumption: there is a $\mathbf{2^{|C|^{o(1)}}}$ size ACC circuit **D(C, i)** which outputs the **i**-th bit of a satisfying assignment to **tt(C)**.

Now for any circuit **C'**, define the circuit **E(i) := D(C', i)** Then **E** has $\mathbf{2^{|C|^{o(1)}}}$ size, and **tt(E)** satisfies **tt(C')**

Theorem If ACC-SAT on circuits with n inputs and $2^{n^{o(1)}}$ size is in $O(2^n/n^{10})$ time, then $\mathbf{EXP^{NP}}$ isn't in $2^{n^{o(1)}}$ size ACC.

An overview:

Assume "fast" ACC-SAT and small ACC circuits for EXP^{NP}

Use to solve Succinct3SAT in NTIME[$2^n/n^5$]

(contradiction!)

Outline of Succinct3SAT algorithm:

Given a Succinct3SAT instance C (an ACC circuit)

1. **Guess a small ACC circuit Y encoding a satisfying assignment for the exponentially-long 3CNF tt(C)**

   (which exists, by Lemma 2 and small circuits for EXP^{NP})

2. **Use "fast" Circuit-SAT algorithm to check that tt(D) satisfies tt(C) in $O(2^n/n^5)$ time**

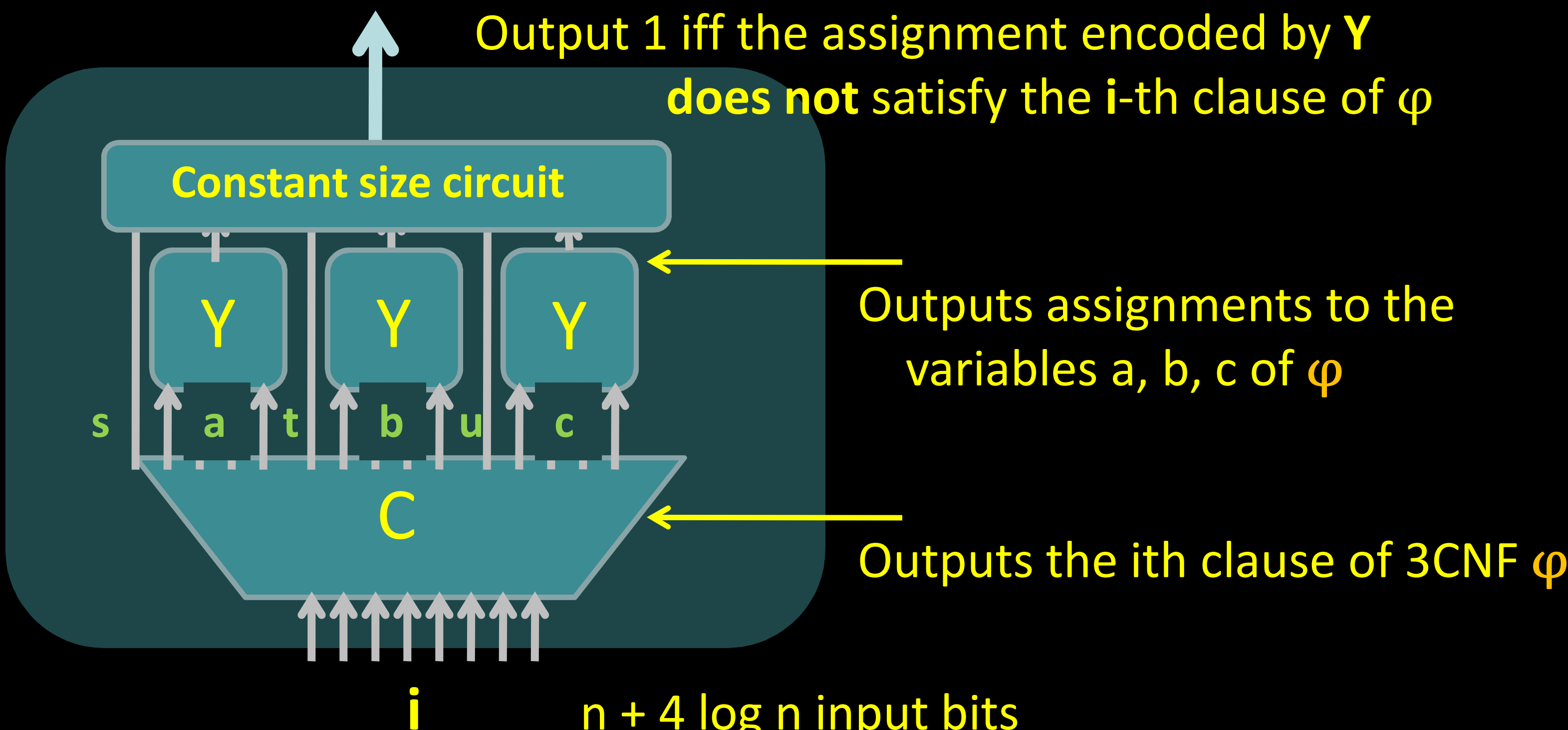
Fast Algorithm for Succinct3SAT

Given Succinct3SAT instance **C** **(an ACC circuit of n inputs)**

Nondeterministically guess ACC circuit Y of $2^{n^{o(1)}}$ **size**

Y(j) is intended to output the j-th bit of a satisfying assignment for φ

Construct the following circuit **D** of $2^{n^{o(1)}}$ size:



Using ACC-SAT algorithm: determine satisfiability of **D** in **o(2^n) time!**

Proof Strategy for ACC Lower Bounds

1. **Show that faster ACC-SAT algorithms imply lower bounds against ACC**

Theorem (Example)
If **ACC-SAT** with n inputs and $\mathbf{2^{n^{o(1)}}}$ size is in
$O(2^n/n^{10})$ time (for all constant depths and moduli), then
EXP^{NP} doesn't have $\mathbf{2^{n^{o(1)}}}$ size ACC circuits.

2. **Design faster ACC-SAT algorithms!**

Theorem For all **d, m** there's an **$\varepsilon > 0$** such that
ACC-SAT on circuits with n inputs, depth **d**, **MODm** gates,
and $\mathbf{2^{n^{\varepsilon}}}$ size can be solved in **$2^{n - \Omega(n^{\varepsilon})}$ time**

Ingredients for Solving ACC SAT

Ingredients:

1. **A known representation of ACC**

   [Yao '90, Beigel-Tarui'94] Every ACC function **$f : \{0,1\}^n \to \{0,1\}$** can be expressed in the form

   $$f(x_1,\ldots,x_n) = g(h(x_1,\ldots,x_n))$$

   - **h** is a multilinear polynomial with **K** monomials, **$h(x_1,\ldots,x_n) \in \{0,\ldots,K\}$ for all $(x_1,\ldots,x_n) \in \{0,1\}^n$**
   - **K** is not "too large" *(quasipolynomial in circuit size)*
   - **$g : \{0,\ldots,K\} \to \{0,1\}$** can be an arbitrary function

2. **"Fast Fourier Transform" for multilinear polynomials:**

   Given a multilinear polynomial h in its coefficient representation, the value h(x) can be computed over all points $x \in \{0,1\}^n$ in **2^n poly(n)** time.

1. Polynomials Representing ACC

Very special cases:

1. **Writing OR(x_1, …, x_n) as a g of h:**
   g(y) = 1 iff y > 0, h = x_1 + … + x_n
2. **Writing AND(x_1, …, x_n) as a g of h**
   g(y) = 1 iff y = n, h = x_1 + … + x_n
3. **Writing MODm(x_1, …, x_n) as a g of h…**

Slightly less special case:

[Razborov-Smolensky, Aspnes et al., Tarui]
AC0 can be represented using a *distribution* of polylog-degree polynomials over the integers.

In fact can use a "small" number **S** of polynomials **($S = n^{poly(\log n)}$)**

Can take MAJORITY value of all **S** different polynomials.

Let g(y) = 1 iff y ≥ S/2, let h be the sum of all S polynomials

2. Fast Multipoint Evaluation

Theorem: Given the **2^n** coefficients of a multilinear polynomial **h** in **n** variables, the value **h(x)** can be computed on all points **x** $\in$ **$\{0,1\}^n$** in **2^n poly(n)** time.

Can write **$h(x_1, \ldots, x_n) = x_1\, h_1(x_2, \ldots, x_n) + h_2(x_2, \ldots, x_n)$**

Want a 2^n table T that contains the value of h on all 2^n points.

Algorithm: If n = 1 then return T = [h(0), h(1)]

Recursively compute the 2^{n-1} table T_1 for the values of h_1,
and the 2^{n-1} table T_2 for the values of h_2

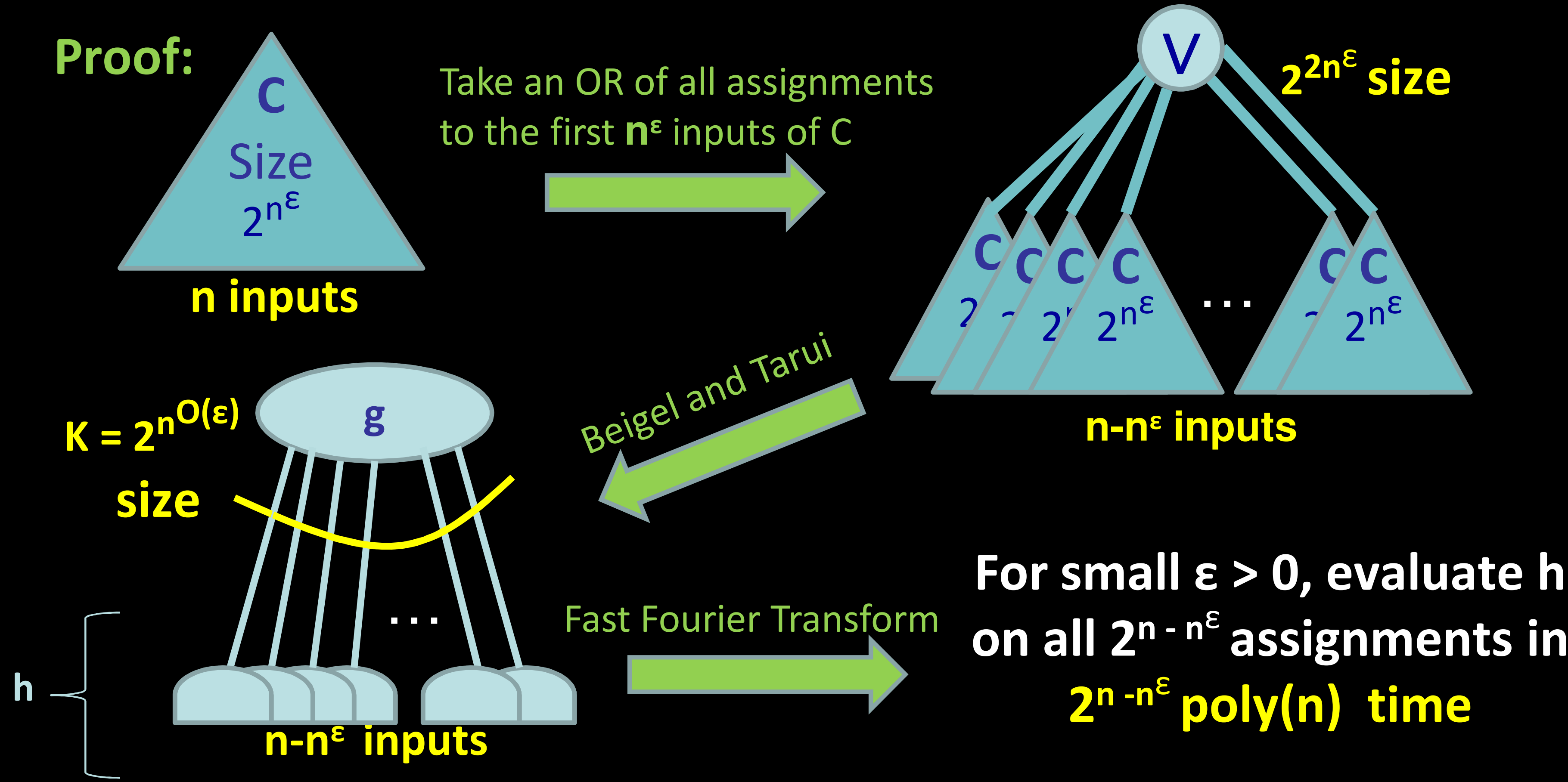
Return the table T = $(T_2)(T_1 + T_2)$ of 2^n entries

Running time has the recurrence **$R(2^n) \leq 2\, R(2^{n-1}) + 2^n\, \mathrm{poly}(n)$**

Corollary: We can compute g of h on all x $\in$ $\{0,1\}^n$ in only 2^n poly(n) time

ACC Satisfiability Algorithm

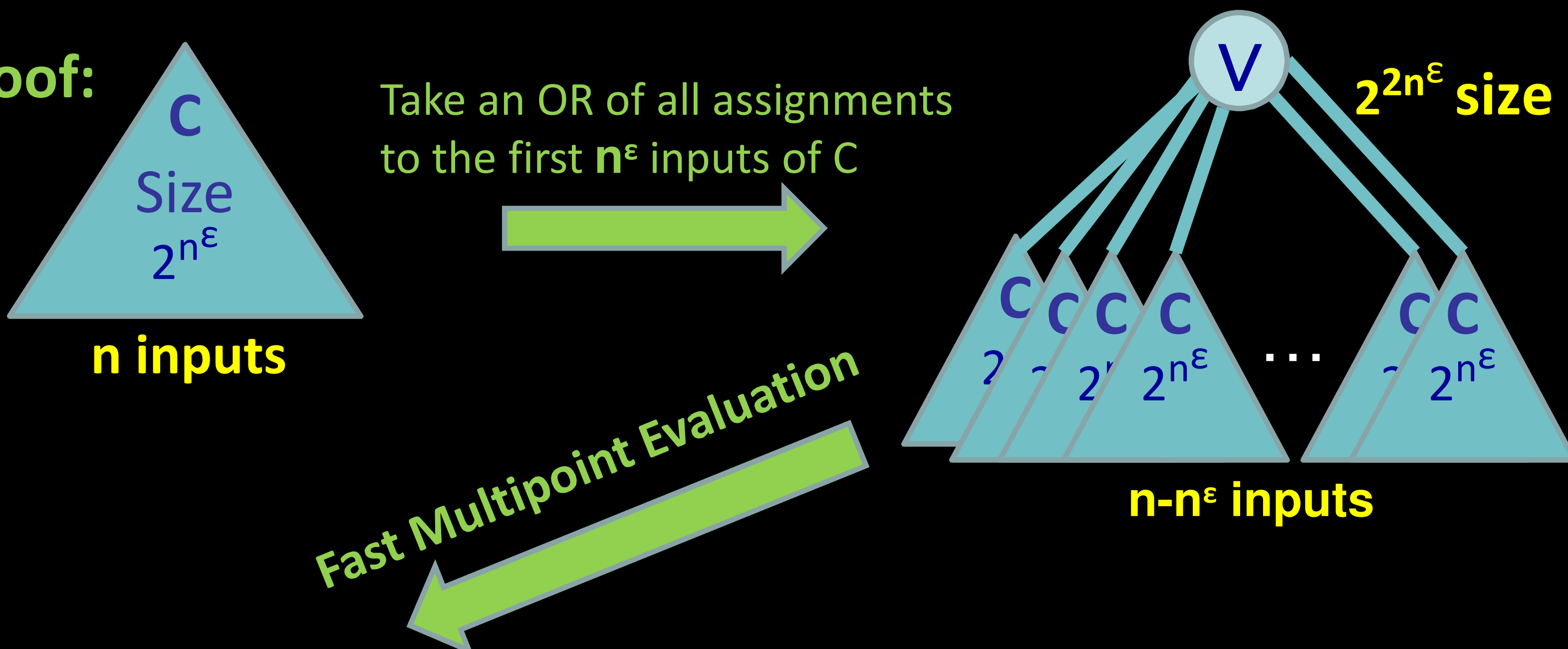
Theorem For all d, m there's an ε > 0 such that ACC[m] SAT with depth d, n inputs, $2^{n^{\varepsilon}}$ size can be solved in $2^{n - \Omega(n^{\varepsilon})}$ time



Fast Multipoint Circuit Evaluation $\Rightarrow$ Circuit Lower Bounds

Theorem If we can **evaluate a circuit of size s on all 2^n inputs** in **2^n poly(n) + poly(s)** time, then **Circuit-SAT** is in **$2^{n-n^{\varepsilon}}$** time

Proof:



For small $\varepsilon > 0$, can evaluate on all $2^{n-n^{\varepsilon}}$ assignments in $2^{n-n^{\varepsilon}}$ poly(n) + poly($2^{2n^{\varepsilon}}$) time

Theorem If ACC SAT with n inputs, $n^{O(1)}$ size is in $O(2^n/n^{10})$ time, then NEXP doesn't have $n^{O(1)}$ size ACC circuits.

Proceed just as with EXP^{NP}, but use the following lemma:

Lemma [IKW'02] If **NEXP ⊂ P/poly** then Succinct 3SAT has poly-size circuits encoding satisfying assignments.

The proof applies work on "hardness versus randomness"

1. If **EXP ⊂ P/poly** then **EXP = MA** [BFNW93]
2. If Succinct 3SAT does *not* have polysize SAT assignment circuits, then in **i.o.-NTIME[2^n]/n** we can *guess a function with high circuit complexity and verify it – just guess a satisfying assignment to a hard Succinct3SAT instance!*

Can derandomize MA infinitely often with n bits of advice:

EXP = MA ⊆ io-NTIME[2^n]/n ⊆ io-SIZE(n^k)

(this is a contradiction)

Theorem If ACC SAT with n inputs, $n^{O(1)}$ size is in $O(2^n/n^{10})$ time, then NEXP doesn't have $n^{O(1)}$ size ACC circuits.

Proceed just as with EXP^{NP}, but use the following lemma:

Lemma [IKW'02] If **NEXP** $\subset$ **P/poly** then Succinct 3SAT has poly-size circuits encoding satisfying assignments.

Lemma If **P** $\subset$ **ACC** then all poly-size *unrestricted* circuit families have equivalent poly-size ACC circuit families.

Corollary If **NEXP** $\subset$ **ACC** then Succinct 3SAT has poly-size ACC circuits encoding satisfying assignments.

This is all we need for the previous proof to go through. Also works for quasipolynomial size circuits.

Weak Derandomization Suffices

Theorem 2 **Suppose we are given a circuit C with n inputs, and are promised that it is either *unsatisfiable,* or at least ½ of its assignments are satisfying. Determine which.**
If this is in $O(2^n/n^{\log n})$ time then NEXP $\not\subseteq$ P/poly.

Proof Idea: Same as before, but **replace** the succinct reduction $\mathbf{R_L}$ from L to 3SAT with a **succinct PCP reduction**

Lemma 3 [BGHSV'05] For all $L \in NTIME(2^n)$,

there is a reduction S_L from L to **MAX CSP** such that:

$x \in L \Rightarrow$ All constraints of $S_L(x)$ are satisfiable

$x \notin L \Rightarrow$ At most ½ of the constraints are satisfiable

1. $|S_L(x)| = 2^n$ **poly(n)**
2. **The i-th constraint of $S_L(x)$ is computable in poly(n) time.**

Remark on a Nice Property of ACC

Thm: **Given an ACC circuit C of size S and n inputs, the truth table of C can be produced in 2^n poly(n) + $2^{\text{poly}(\log S)}$ time.**

The main result of this lecture is that this property suffices to separate NEXP from ACC.

Morally, this property *should* be enough to get EXP $\not\subset$ ACC

Observation: Let **L** $\in$ **TIME[4^n] \ TIME[3^n]**. Then the truth table of **L** $\cap$ **$\{0,1\}^n$** cannot be produced in **o(3^n)** time.

The non-uniformity of ACC prevents us from directly proving EXP lower bounds. But perhaps **NP $\neq$ uniform-ACC**

Q:* *Is there L $\in$ TIME[3^n] such that generating the 2^n-length truth table of L on n-bit inputs requires $\omega(3^n)$ time?

Future Progress

- **Replace NEXP with simpler complexity classes**
  May need to improve on exhaustive search for more complex problems

Open Problem ***Does faster COUNTING of satisfying assignments for circuits imply stronger lower bounds?***

- **Replace ACC with stronger circuits**
  Design SAT algorithms for stronger circuits!
  Using PCP Theorem: can weaken the hypotheses

Open Problem ***Can Boolean formulas of size s be evaluated on all n-variable assignments in*** $poly(s) + 2^n poly(n)$ ***time?***

- **Find more connections between algorithms and lower bounds!**