Strongly sublinear separators and polynomial expansion

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Abstract

A result of Plotkin, Rao, and Smith implies that graphs with polynomial expansion have strongly sublinear separators. We prove a converse of this result showing that hereditary classes of graphs with strongly sublinear separators have polynomial expansion. This confirms a conjecture of the first author.

1   Introduction

The concept of graph classes with bounded expansion was introduced by Nešetřil and Ossona de Mendez [9] as a way of formalizing the notion of sparse graph classes. Let us give a few definitions.

For a graph G, a *k-minor of G* is any graph obtained from G by contracting pairwise vertex-disjoint subgraphs of radius at most k and removing vertices and edges. Thus, a 0-minor is just a subgraph of G. Let us define $\nabla_k(G)$ as

$$\max\left\{\frac{|E(G')|}{|V(G')|} : G' \text{ is a } k\text{-minor of } G\right\}.$$

For a function $f : \mathbf{Z}_0^+ \to \mathbf{R}_0^+$, we say that an expansion of a graph G is *bounded by f* if $\nabla_k(G) \leq f(k)$ for every $k \geq 0$. We say that a class $\mathcal{G}$ of graphs has *bounded expansion* if there exists a function $f : \mathbf{Z}_0^+ \to \mathbf{R}_0^+$ such that f bounds the expansion of every graph in $\mathcal{G}$. If such a function f is a polynomial, we say that $\mathcal{G}$ has *polynomial expansion.*

The definition is quite general—examples of classes of graphs with bounded expansion include proper minor-closed classes of graphs, classes of graphs with bounded maximum degree, classes of graphs excluding a subdivision of a fixed graph, classes of graphs that can be embedded on a fixed surface with bounded number of crossings per edge and many others, see [12]. On the other hand, bounded expansion implies a wide range of interesting structural and algorithmic properties, generalizing many results from proper minor-closed classes of graphs. For a more in-depth introduction to the topic, the reader is referred to the book of Nešetřil and Ossona de Mendez [11].

One of the useful properties of graph classes with bounded expansion is the existence of small balanced separators. A *separator* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and no edge joins a vertex of $A \setminus B$ with a vertex of $B \setminus A$. The *order* of the separator is $|A \cap B|$. A separator (A, B) is *balanced* if $|A \setminus B| \leq 2|V(G)|/3$ and $|B \setminus A| \leq 2|V(G)|/3$. Note that $(V(G), V(G))$ is a balanced separator. For $c \geq 1$ and $0 \leq \beta < 1$, we say that a graph G has $c \bullet^\beta$-*separators* if every subgraph H of G has a balanced separator of order at most $c|V(H)|^\beta$. For a graph class $\mathcal{C}$, let $s_\mathcal{C}(n)$ denote the smallest nonnegative integer such that every graph in $\mathcal{C}$ with at most n vertices has a balanced separator of order at most $s_\mathcal{C}(n)$. We say that $\mathcal{C}$ has *strongly sublinear separators* if there exist $c \geq 1$ and $0 < \delta \leq 1$ such that $s_\mathcal{C}(n) \leq cn^{1-\delta}$ for every $n \geq 0$. Note that if $\mathcal{C}$ is subgraph-closed, this implies that every graph in $\mathcal{C}$ has $c \bullet^{1-\delta}$-separators.

Lipton and Tarjan [7] proved that the class $\mathcal{P}$ of planar graphs satisfies $s_\mathcal{P}(n) = O(\sqrt{n})$, and demonstrated the importance of this fact in the design of algorithms [8]. This result was later generalized to graphs embedded on other surfaces [4] and all proper minor-closed graph classes [1, 5]. The following result by Plotkin, Rao, and Smith connects the expansion and separators.

Theorem 1. *Given a graph G with m edges and n vertices, and integers l and h, there is an $O(mn/l)$-time algorithm that finds either an $(l \log_2 n)$-minor of K_h in G, or a balanced separator of order at most $O(n/l + lh^2 \log n)$.*

Using this result, Nešetřil and Ossona de Mendez [10] observed that graphs with expansion bounded by subexponential function have separators of sublinear order. The bound on expansion is tight in the sense that 3-regular expanders (which have exponential expansion) do not have sublinear separators. In fact, polynomial expansion implies strongly sublinear separators, which qualitatively generalizes the results of [7, 4, 5].

Corollary 2. *For any $d \geq 0$ and $k \geq 1$, there exists $c \geq 1$ and $\delta = \frac{1}{4d+3}$ such that if the expansion of a graph G is bounded by $f(r) = k(r + 1)^d$, then G has $c\bullet^{1-\delta}$-separators. Furthermore, there exists an algorithm that returns a balanced separator of G of order at most $c|V(G)|^{1-\delta}$ in time $O(|V(G)|^\delta |E(G)|)$.*

Proof. For any integer $n \geq 1$, let $l(n) = \lceil n^\delta \rceil$ and $h(n) = \lceil n^{1/4-\delta/2} \rceil$. Since $f(l(n) \log_2 n) = O(n^{d\delta} \log^d n)$ and $1/4 - \delta/2 > d\delta$, there exists $n_0 \geq 1$ such that $f(l(n) \log_2 n) < \frac{h(n)-1}{2}$ for every $n \geq n_0$.

Consider any subgraph G' of G, and let $n = |V(G')|$. Since the expansion of G is bounded by $f(r)$, the expansion of G' is bounded by $f(r)$ as well. We aim to show that G' has a balanced separator of order $O(n^{1-\delta})$. Without loss of generality, we can assume that $n \geq n_0$. We apply Theorem 1 to G' with $l = l(n)$ and $h = h(n)$. Every $(l \log_2 n)$-minor of G' has edge density at most $f(l \log_2 n) < \frac{h-1}{2}$, and thus G' does not contain K_h as an $(l \log_2 n)$-minor. Consequently, the algorithm of Theorem 1 produces a balanced separator of order $O(n/l + lh^2 \log n) = O(n^{1-\delta} + n^{1/2} \log n) = O(n^{1-\delta})$. $\square$

Our main result is the converse: in subgraph-closed classes, strongly sublinear separators imply polynomial expansion.

Theorem 3. *For any $c \geq 1$ and $0 < \delta \leq 1$, there exists a function $f(r) = O\!\left(r^{5/\delta^2}\right)$ such that if a graph G has $c\bullet^{1-\delta}$-separators, then its expansion is bounded by f.*

Let us remark that a weaker variant of Theorem 3 was conjectured by Dvořák [2], who hypothesized that strongly sublinear separators imply subexponential expansion, and proved this weaker claim under the additional assumption that G has bounded maximum degree. Together with Corollary 2, Theorem 3 shows the equivalence between strongly sublinear separators and polynomial expansion.

Corollary 4. *Let $\mathcal{C}$ be a subgraph-closed class. Then $\mathcal{C}$ has strongly sublinear separators if and only if $\mathcal{C}$ has polynomial expansion.*

Note that to guarantee separators of order $O(n^{1-\delta})$, Corollary 2 only requires the expansion to be bounded by $r^{O(1/\delta)}$, while given separators of order $O(n^{1-\delta})$, Theorem 3 guarantees the expansion bounded by $r^{O(1/\delta^2)}$. For a cubic graph G, let G_δ denote the graph obtained from G by subdividing each edge exactly $\lfloor |V(G)|^{\delta/(1-\delta)} \rfloor$ times, and let

$$\mathcal{C}_\delta = \{H : H \subseteq G_\delta \text{ for a cubic graph } G\}.$$

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Then balanced separators in $\mathcal{C}_\delta$ have order $\Omega(n^{1-\delta})$ and the expansion of $\mathcal{C}_\delta$ is $r^{O(1/\delta)}$. Hence, the relationship between the exponents in Corollary 2 is tight up to constant multiplicative factors. On the other hand, we believe Theorem 3 can be improved.

Conjecture 1. *There exists $k > 0$ such that for any $c \geq 1$ and $0 < \delta \leq 1$, if a graph G has $c\bullet^{1-\delta}$-separators, then its expansion is bounded by $f(r) = O\big(r^{k/\delta}\big)$.*

The rest of the paper is devoted to the proof of Theorem 3. In Section 2, we recall some results relating separators with tree-width and expanders. In Section 3, we give partial results towards bounding the density of minors of graphs with strongly sublinear separators. In Section 4, we show that a bounded-depth minor of a graph with strongly sublinear separators still has strongly sublinear separators (for a somewhat worse bound on their order). Finally, in Section 5, we combine these results to give a proof of Theorem 3.

2 Separators, tree-width and expanders

For $\alpha > 0$, a graph G is an α-*expander* if for every $A \subseteq V(G)$ of size at most $|V(G)|/2$, there exist at least $\alpha|A|$ vertices of $V(G) \setminus A$ adjacent to a vertex of A. Random graphs are asymptotically almost surely expanders.

Lemma 5 (Kolesnik and Wormald [6]). *There exists an integer n_0 such that for every even $n \geq n_0$, there exists a 3-regular $\frac{1}{7}$-expander on n vertices.*

Let us recall a well-known fact on the relationship between tree-width and separators, see e.g. [13].

Lemma 6. *Any graph G has a balanced separator of order at most $tw(G)+1$.*

Corollary 7. *If H is an α-expander for some $\alpha > 0$, then $tw(H) \geq \frac{\alpha}{3(1+\alpha)}|V(H)| - 1$.*

Proof. Let (A, B) be a balanced separator of H of order at most $tw(H)+1$. Let $S = A \cap B$, let $A' = A \setminus B$ and let $B' = B \setminus A$. Without loss of generality, $|A'| \leq |B'|$, and thus $|A'| \leq |V(H)|/2$. Since H is an α-expander, we have $|S| \geq \alpha|A'|$, and thus $|A'| \leq \frac{1}{\alpha}|S|$. On the other hand, since the separator (A, B) is balanced, we have $|B'| \leq \frac{2}{3}|V(H)|$, and thus $|A'| + |S| \geq \frac{1}{3}|V(H)|$. Therefore,

$$\left(\frac{1}{\alpha} + 1\right)|S| \geq \frac{1}{3}|V(H)|$$

$$|S| \geq \frac{\alpha}{3(1 + \alpha)}|V(H)|.$$

The claim follows, since $|S| \leq \text{tw}(H) + 1$. $\quad\square$

For later use, let us remark that an approximate converse to Lemma 6 holds, as was proved by Dvořák and Norin [3].

Theorem 8. *If every subgraph of G has a balanced separator of order at most k, then G has tree-width at most $105k$.*

Corollary 9. *For $c \geq 1$ and $0 \leq \beta < 1$, if a graph G has $c \bullet^\beta$-separators, then every subgraph H of G has tree-width at most $105c|V(H)|^\beta$.*

The aim of this section is to argue that in a dense graph, we can always find a large expander of maximum degree 3 as a subgraph. To prove this, we first find a bounded-depth clique minor, using the following result of Dvořák [2].

Theorem 10. *Suppose that $0 < \varepsilon \leq 1$ and let $m = \left\lceil \frac{1}{2\varepsilon^2} \right\rceil$. If a graph on n vertices has at least $2 \cdot 32^m t^4 n^{1+\varepsilon}$ edges, then it contains K_t as a 4^m-minor.*

Next, we take a 3-regular expander subgraph of this clique which exists by Lemma 5, and we observe that it corresponds to a subgraph of the original graph of maximum degree 3, in which each path of vertices of degree two has bounded length. Such a subgraph is still a decent expander. However, we will only need the corresponding lower bound for the tree-width of such a graph.

Lemma 11. *Suppose that $0 < \varepsilon \leq 1$ and let $m = \left\lceil \frac{1}{2\varepsilon^2} \right\rceil$. Let n_0 satisfy Lemma 5 and let $t \geq \max(n_0, 600)$ be an even integer. If a graph on n vertices has at least $2 \cdot 32^m t^4 n^{1+\varepsilon}$ edges, then it contains a subgraph H of maximum degree 3 with $|V(H)| \leq 4^{m+1}t$ and with $\text{tw}(H) \geq \frac{t}{25}$.*

Proof. By Theorem 10, G contains K_t as a 4^m-minor, and by Lemma 5, G contains a 3-regular $\frac{1}{7}$-expander H_0 with t vertices as a 4^m-minor. Hence, G contains a subgraph H of maximum degree three such that $|V(H)| \leq (3 \cdot 4^m + 1)|V(H_0)| \leq 4^{m+1}t$ and H_0 is a minor of H. By Corollary 7, $\text{tw}(H) \geq \text{tw}(H_0) \geq \frac{1}{24}|V(H_0)| - 1 = \frac{t}{24} - 1 \geq \frac{t}{25}$. $\quad\square$

3 The densities of graphs with strongly sublinear separators and their minors

In this section, we give two bounds on edge densities of graphs. Firstly, we show that graphs with strongly sublinear separators have bounded edge density; in other words, they satisfy the condition from the definition of bounded expansion for ∇_0.

Lemma 12. *For any $c \geq 1$ and $0 < \delta \leq 1$, let $a_\delta(c) \geq 1$ be the unique real number satisfying $a_\delta(c)^\delta = 4c\log^2(ea_\delta(c))$. If G has $c\bullet^{1-\delta}$-separators, then G has at most $a_\delta(c)|V(G)|$ edges.*

Proof. Let $h(n) = \frac{n}{2\log en}$. By induction on the number of vertices of G, we prove a stronger claim: If G has $c\bullet^{1-\delta}$-separators, then G has at most $a_\delta(c)(|V(G)| - h(|V(G)|))$ edges. Note that $a_\delta(c)(|V(G)| - h(|V(G)|)) \geq a_\delta(c)|V(G)|/2$, and thus the claim trivially holds if $|V(G)| \leq a_\delta(c)$.

Suppose that $|V(G)| > a_\delta(c)$. Let (A, B) be a balanced separator of G of order at most $c|V(G)|^{1-\delta}$ and let $G_1 = G[A]$ and $G_2 = G[B]$. Let $n = |V(G)|$, $n_0 = |V(G_1 \cap G_2)|$, $n_1 = |V(G_1) \setminus V(G_2)|$ and $n_2 = |V(G_2) \setminus V(G_1)|$. Since $n > a_\delta(c)$, we have $n/3 > cn^{1-\delta} \geq n_0$. For $i \in \{1, 2\}$, we have $|V(G_i)| = n_0 + n_i < n/3 + 2n/3 < n$, and thus $|E(G_i)| \leq a_\delta(c)(n_0 + n_i - h(n_0 + n_i))$ by the induction hypothesis. It follows that

$$
\begin{aligned}
|E(G)| &\leq |E(G_1)| + |E(G_2)| \\
&\leq a_\delta(c)(n + n_0 - h(n_0 + n_1) - h(n_0 + n_2)) \\
&= a_\delta(c)(n - h(n) + [n_0 + h(n) - h(n_0 + n_1) - h(n_0 + n_2)]).
\end{aligned}
$$

Therefore, we need to prove that $n_0 \leq h(n_0 + n_1) + h(n_0 + n_2) - h(n)$. Recall that $n_0 \leq cn^{1-\delta}$, $n_1, n_2 \leq 2n/3$, and $n = n_0 + n_1 + n_2 > a_\delta(c)$. Without loss of generality, $n_1 \leq n_2$. Since h is increasing and concave for $n \geq 3$, and since $n_0 + n_1 = n - n_2 \geq n/3 \geq 3$, we have $h(n_0 + n_1) + h(n_0 + n_2) \geq h(n/3) + h(2n/3 + n_0) \geq h(n/3) + h(2n/3)$. We conclude that

$$
\begin{aligned}
h(n_0 &+ n_1) + h(n_0 + n_2) - h(n) \\
&\geq h(n/3) + h(2n/3) - h(n) \\
&= \frac{n}{6}\left(\frac{1}{\log(en) - \log(3)} + \frac{2}{\log(en) - \log(3/2)} - \frac{3}{\log(en)}\right) \\
&\geq \frac{n}{6\log^2 en}((\log(en) + \log(3)) + 2(\log(en) + \log(3/2)) - 3\log(en)) \\
&\geq \frac{n}{4\log^2 en} = \frac{n^\delta}{4\log^2 en}n^{1-\delta} \geq \frac{a_\delta(c)^\delta}{4\log^2(ea_\delta(c))}n^{1-\delta} \\
&= cn^{1-\delta} \geq n_0,
\end{aligned}
$$

as required. $\qquad\square$

Let us remark that for a fixed $\delta > 0$, we have $a_\delta(c) = O\big((c\log^3 c)^{1/\delta}\big)$.

Secondly, we aim to show that the density of bounded-depth minors of graphs with strongly sublinear separators grows only slowly with the

number of their vertices—slower than n^ε for every $\varepsilon > 0$. Of course, we will eventually show that it is actually bounded by a constant, but we will need this auxiliary result to do so.

Lemma 13. *For any $c \geq 1$, $0 < \delta \leq 1$, $0 < \varepsilon \leq 1$ and $r \geq 1$, let $m = \left\lceil \frac{1}{2\varepsilon^2} \right\rceil$, let n_0 satisfy Lemma 5, let t be the smallest even integer greater than $\max\left(n_0, (42000c4^m r)^{1/\delta}\right)$ and let $b_{c,\delta,\varepsilon}(r) = 2 \cdot 32^m t^4$. If G has $c\bullet^{1-\delta}$-separators, then every r-minor F of G has less than $b_{c,\delta,\varepsilon}(r)|V(F)|^{1+\varepsilon}$ edges.*

Proof. Suppose that F has at least $b_{c,\delta,\varepsilon}(r)|V(F)|^{1+\varepsilon}$ edges. By Lemma 11, F contains a subgraph H of maximum degree 3 with $|V(H)| \leq 4^{m+1}t$ and with $\mathrm{tw}(H) \geq \frac{t}{25}$. Hence, G contains a subgraph H' of maximum degree 3 with $|V(H')| \leq (3r+1)|V(H)| \leq 4^{m+2}rt$ such that H is a minor of H', and thus $\mathrm{tw}(H') \geq \mathrm{tw}(H) \geq \frac{t}{25}$. By Corollary 9, we have $\mathrm{tw}(H') \leq 105c|V(H')|^{1-\delta}$. Therefore,

$$\frac{t}{25} \leq 105c4^{m+2}rt^{1-\delta}$$
$$t^\delta \leq 42000c4^m r.$$

This contradicts the choice of t. $\qquad\qquad\square$

Let us remark that for fixed c, δ, and ε, we have $b_{c,\delta,\varepsilon}(r) = O\left(r^{4/\delta}\right)$.

4 Sublinear separators in bounded-depth minors

We now prove that if G has strongly sublinear separators, then any bounded-depth minor of G also has strongly sublinear separators. Together with Lemma 12, this will give the bound on their density.

Lemma 14. *For any $c \geq 1$, $0 < \delta \leq 1$ and $r \geq 1$, let $\varepsilon = \min\left(1, \frac{\delta}{6(1-\delta)}\right)$ and let $p_{c,\delta}(r) = 316c(b_{c,\delta,\varepsilon}(r)r)^{1-\delta}$. If G has $c\bullet^{1-\delta}$-separators, then every r-minor of G has $p_{c,\delta}(r)\bullet^{1-\frac{5}{6}\delta}$-separators.*

Proof. Let H be an r-minor of G. Since every subgraph of H is also an r-minor of G, it suffices to prove that H has a balanced separator of order at most $p_{c,\delta}(r)|V(H)|^{1-\frac{5}{6}\delta}$.

By Lemma 13, H has at most $b_{c,\delta,\varepsilon}(r)|V(H)|^{1+\varepsilon}$ edges. Hence, there exist a subgraph H' of G with at most $2b_{c,\delta,\varepsilon}(r)r|V(H)|^{1+\varepsilon} + |V(H)| \leq$

$3b_{c,\delta,\varepsilon}(r)r|V(H)|^{1+\varepsilon}$ vertices such that H is a minor of H'. By Corollary 9, we have

$$\mathrm{tw}(H') \leq 315c \left(b_{c,\delta,\varepsilon}(r)r|V(H)|^{1+\varepsilon}\right)^{1-\delta} \leq 315c(b_{c,\delta,\varepsilon}(r)r)^{1-\delta}|V(H)|^{1-\frac{5}{6}\delta}.$$

Since H is a minor of H', we have $\mathrm{tw}(H) \leq \mathrm{tw}(H')$. Therefore, by Lemma 6, H has a balanced separator of order at most

$$\mathrm{tw}(H') + 1 \leq 315c(b_{c,\delta,\varepsilon}(r)r)^{1-\delta}|V(H)|^{1-\frac{5}{6}\delta} + 1 \leq p_{c,\delta}(r)|V(H)|^{1-\frac{5}{6}\delta}$$

as required. $\qquad\square$

Let us remark that for fixed c and δ, we have $p_{c,\delta}(r) = O\big(r^{(4/\delta+1)(1-\delta)}\big) = O\big(r^{4/\delta}\big)$.

5   Polynomial expansion

Finally, we can prove our main result.

Proof of Theorem 3. For any $r \geq 1$, every r-minor of G has $p_{c,\delta}(r)\bullet^{1-\frac{5}{6}\delta}$-separators by Lemma 14, where $p_{c,\delta}(r) = O(r^{4/\delta})$. By Lemma 12, every r-minor of G has edge density at most $a_{\frac{5}{6}\delta}\big(p_{c,\delta}(r)\big) = O\big(p_{c,\delta}(r)^{\frac{5}{4\delta}}\big) = O\big(r^{5/\delta^2}\big)$. Therefore, $\nabla_r(G) \leq O\big(r^{5/\delta^2}\big)$. $\qquad\square$

References

[1] N. ALON, P. SEYMOUR, AND R. THOMAS, *A separator theorem for graphs with an excluded minor and its applications*, in Proceedings of the twenty-second annual ACM symposium on Theory of computing, ACM, 1990, pp. 293–299.

[2] Z. DVOŘÁK, *Sublinear separators, fragility and subexponential expansion*, ArXiv e-prints, 1404.7219 (2014).

[3] Z. DVOŘÁK AND S. NORIN, *Treewidth of graphs with balanced separations*, ArXiv e-prints, 1408.3869 (2014).

[4] J. GILBERT, J. HUTCHINSON, AND R. TARJAN, *A separator theorem for graphs of bounded genus*, Journal of Algorithms, 5 (1984), pp. 391–407.

[5] K. Kawarabayashi and B. Reed, *A separator theorem in minor-closed classes*, in Proc. 51st Annual IEEE Symposium on. Foundations of Computer Science, 2010.

[6] B. Kolesnik and N. Wormald, *Lower bounds for the isoperimetric numbers of random regular graphs*, SIAM Journal on Discrete Mathematics, 28 (2014), pp. 553–575.

[7] R. Lipton and R. Tarjan, *A separator theorem for planar graphs*, SIAM Journal on Applied Mathematics, 36 (1979), pp. 177–189.

[8] ———, *Applications of a planar separator theorem*, SIAM Journal on Computing, 9 (1980), pp. 615–627.

[9] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion I. Decomposition.*, European J. Combin., 29 (2008), pp. 760–776.

[10] ———, *Grad and classes with bounded expansion II. Algorithmic aspects.*, European J. Combin., 29 (2008), pp. 777–791.

[11] ———, *Sparsity – Graphs, Structures, and Algorithms*, Springer, 2012.

[12] J. Nešetřil, P. Ossona de Mendez, and D. Wood, *Characterisations and examples of graph classes with bounded expansion*, Eur. J. Comb., 33 (2012), pp. 350–373.

[13] N. Robertson and P. D. Seymour, *Graph minors. II. Algorithmic aspects of tree-width*, Journal of algorithms, 7 (1986), pp. 309–322.