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Faculty of Mathematics and Physics

MASTER THESIS

Bc. Filip Hlásek

Bounds on existence of odd and unique expanders

Computer Science Institute of Charles University

Supervisor: doc. Mgr. Michal Koucký, Ph.D.

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Abstract: We study the existence of expander graphs with a focus on odd and unique expanders. The main goal is to describe configurations of arguments for which there is no infinite family of expanders. The most imporant result is that for every graph there is a nonempty subset of at most half of its vertices, such that every other vertex is connected at least twice to the subset or not connected to the subset at all. It follows that certain classes of unique expanders cannot exist. On the other hand we present some configurations for which there are families of expanders.

Keywords: expander graphs, odd expansion, unique expansion, upper bounds

Dedication

I dedicate this thesis to my beloved grandparents, mother and father, for their never ending support and all their care.

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Contents

Introduction

Expander graphs (or expanders) are finite undirected multigraphs, which have strong connectivity properties. Besides, only relatively few edges are allowed. Connectivity of a graph is often defined as the least number of edges (or vertices) that has to be removed from the graph to disconnect the remaining vertices from each other. Unfortunately, it does not say much about the structure of the graph because in a sparse graph the least number of edges will be at most the smallest vertex degree (and equal to that value in many cases).

Expander graphs are finite sparse undirected multigraphs, in which every not too large subset of vertices has a lot of neighbors (i.e. it *expands*). The condition that the subset must not be too large is there mainly to ensure, that there are enough vertices left, which might be neighbors of the subset. More interesting part of the informal description is “a lot of neighbors”. One way to define the expansion of a subset of vertices is the number of edges leaving the subset. However, there are at least k edges leaving every nonempty subset of at most half of the vertices, if and only if the graph is k-edge-connected. Instead, we compare the number of neighbors with the size of the subset (this number is sometimes called the relative size of the cut). For example, if a subset of vertices S has at least $|S|/2$ neighbors, we say that the subset S expands by a factor of $1/2$. If every not too large subset of graph vertices expands by a factor of at least $1/2$, we say the graph expansion is at least $1/2$. “A lot of neighbors” does not mean a huge number, but rather a linear fraction of the size of the subset. For example, an expander can only expand by a factor of $1/100$, which is quite a small number, but still every reasonable subset of vertices expands by a constant fraction.

Expander graphs are finite sparse undirected multigraphs, which from the perspective of connectivity *resemble* complete graphs while having fewer edges. Moreover, only regular graphs (having all vertices of the same degree) are considered almost exclusively. The original motivation to study expander graphs was to design robust, but economical networks – ideally every pair of computers would be connected by a short path, while not using too many wires.

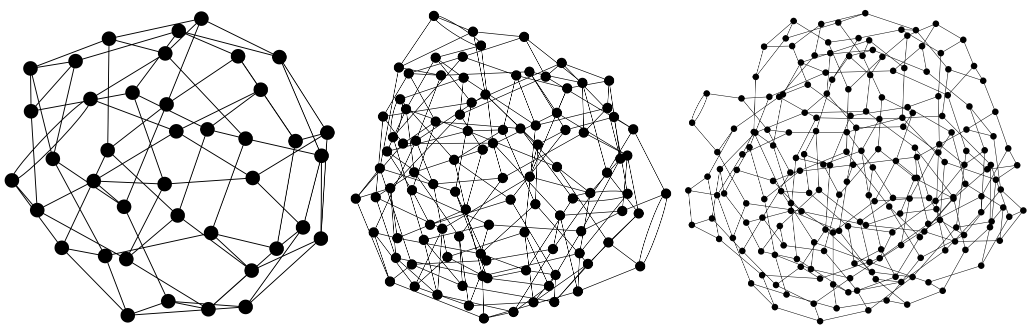

Figure 1: Expander graphs are sparse, yet highly connected. Random 4-regular graphs with 40 and 100 vertices and a random 3-regular graph with 200 vertices are displayed. As we show later, random regular graphs are almost surely good expanders.

A stronger requirement is to demand a lot of *unique neighbors* or *odd neighbors*. A vertex is called a unique neighbor of a subset of vertices if there is exactly one edge between the vertex and the subset. Similarly it is an odd neighbor if there is an odd number of edges between the vertex and the subset. We speak about a *unique-expander* if every not too large subset of its vertices uniquely expands, i.e. it has a constant fraction of unique neighbors. Odd expanders are defined analogously.

Every expander graph can be characterized by three basic parameters: the degree of regularity, the expansion property and a number α saying what "not too large" means when speaking about small subsets of vertices. For instance, if α is $1/3$, then only subsets of at most one-third vertices are considered when calculating the expansion. Inherently there is a trade-off between these parameters.

Expanders are not obvious to exist by any means. However, they have been studied a lot during past several decades and many constructions have been found. Every such method produces infinitely many d-regular expander graphs for a predefined value of d. The main goal is always to achieve the highest expansion while having the value of α a fixed constant. Every construction of expanders shows that for a particular combination of parameters there is an infinite family of expanders while not saying anything about better expanders. Some of the methods are randomized and they are using the fact that a random d-regular graph is an expander with high probability.

We investigate the relations between described parameters and state some restrictions on them under which no expander graphs can exist. Especially we would like to describe combinations of these parameters, for which there are no unique or odd expanders, while for just a slightly weaker restrictions families of expanders can be constructed. As far as we know, nobody has studied this particular topic extensively yet.

Theorem 11 can serve as an example of such restriction, it says that no infinite family of unique-expanders can exist for the value of parameter α higher than or equal to $1/2$. We consider this particular theorem together with partial results on Conjecture 1 (which restricts the existence of odd-expanders) the main contribution of this work. We are not aware of any result similar to our Theorem 10. It says, that in every graph there is a subset of at most half of the vertices, which does not have any unique neighbors, in other words, every neighbor of the subset is connected to it at least twice.

This work is divided into three main sections. In Chapter 1 basic terms are defined, related works are surveyed and the entire problem is specified in detail. It serves as a more technical introduction and motivation of this work. Chapter 2 presents some standard methods to prove expander existence and uses them to show what are the strictest conditions under which expander graphs exist. It serves as a prelude and complement of the final Chapter 3. In there we reveal and prove the upper bounds on the existence of unique and odd expanders. Some of the problems are left open and possible future work is discussed at the end of that chapter. The thesis is concluded by a brief summary.

1. What is an expander

1.1 Notations and definitions

In this section, we shall formally define basic notions and agree on notation. The list of abbreviations at the end of this work complements basic math terms.

We start by defining a graph, which we always consider undirected and we permit self-loops and multiple edges. Such a structure is more often called a multigraph or pseudograph, but we stick with the shorter term.

Intuitively a graph is an ordered pair (V, E), where V is the vertex set and E contains edges. Every edge is an unordered pair of vertices (possibly both vertices are the same in the case of a self-loop) and represents a connection between the respective vertices. Although E is often described as a multiset of edges, we are going to use a slightly different model, which is usually called an adjacency matrix.

Definition 1. *Let $n \in \mathbb{N}$. We say that an ordered pair $G = (V, E)$ is* ***a graph*** *with n vertices if V is the set of vertices such that $|V| = n$ (usually without loss of generality it can be assumed that $V = \{1, 2, \ldots, n\}$) and $E : V \times V \to \mathbb{N} \cup \{0\}$ is a function describing how many edges are between pairs of vertices. Because we are only interested in undirected graphs, the edge function must be symmetrical, i.e. $\forall v_1, v_2 \in V : E(v_1, v_2) = E(v_2, v_1)$.*
We say that ***the graph G is simple*** *if for every $1 \leq i, j \leq n$*

- *$E(i, i) = 0$ (contains no self loops) and*
- *$E(i, j) \leq 1$ (contains no multiedges).*

In this text, we will mostly be interested in larger subsets of vertices rather than individual connections. It brings us to the following definition:

Definition 2. *Let $G = (V, E)$ be a graph and $S, T \subseteq V$ two subsets of its vertices. We say that* ***the number of edges between S and T*** *is*

$$\sum_{v \in S} \sum_{w \in T} E(v, w)$$

and we denote that number $E(S, T)$.

Note that we are overloading the notation here, since symbol $E(\cdot, \cdot)$ is used in two meanings. It should always be clear from the context whether the input objects are vertices or sets. Moreover, the former definition can be viewed as for single element sets, in which case both definitions coincide.

The next notion that we use in usual meaning is the degree of a vertex. It is always a non-negative integer, which expresses the number of edges meeting in the vertex.

Definition 3. *Let $G = (V, E)$ be a graph and $v \in V$. We say that* ***the degree of the vertex v*** *in the graph G is the number of incident edges. In the case* *of a self-loop, the edge is considered to be incident twice to the same vertex and therefore contributes twice to the vertex degree. More formally*

$$\deg_G(v) = E(v,v) + \sum_{w \in V} E(v,w)$$

In general, the degree of a vertex can be any number between 0 and twice the number of edges in the graph, but of course, there are some obvious restrictions. Regardless, we will mostly be interested in graphs, where all of the vertices have the same degree. Moreover, the degree of all vertices will often be quite small compared to the number of vertices in the graph.

Definition 4. *The graph $G = (V, E)$ is called **d-regular** if all vertices $v \in V$ have degree equal to d.*

The sum of degrees of all vertices in d-regular graph is equal to $d|V|$ while every edge adds exactly two to that sum (it is sometimes called the handshaking lemma). It follows that $d|V|$ must always be an even number. That simple fact might bring technical difficulties when discussing "large regular graphs" because they may not exist at all for some configurations. For that reason, we always consider only such values, that $d|V|$ is divisible by two, even if it is not explicitly mentioned.

Two different vertices in a graph are called *neighbors* if there is an edge connecting them. For our purposes, we generalize that rather standard definition at the neighborhood of subsets of vertices.

Definition 5. *Let $G = (V, E)$ be a graph. For every subset S of its vertices V we say, that vertex $v \in V \setminus S$ is*

a neighbor of S *if $E(S, \{v\}) \geq 1$,*
an odd neighbor of S *if $E(S, \{v\})$ is odd,*
a unique neighbor of S *if $E(S, \{v\}) = 1$.*

Further we denote

$N(S) = \{v \mid v \text{ is a neighbor of } S\}$,
$N_{odd}(S) = \{v \mid v \text{ is an odd neighbor of } S\}$,
$N_{unique}(S) = \{v \mid v \text{ is a unique neighbor of } S\}$.

Now we are ready to define expander graphs formally. There are several different types, some of them are more important for our purposes, but we describe all of them carefully, just to be able to compare them.

Definition 6. *Let $\alpha, \epsilon > 0$ and $G = (V, E)$ be a d-regular graph on n vertices. We call G*

- ***(n, d, α, ϵ)-vertex-expander***
  if $\forall S \subseteq V$, such that $|S| \leq \alpha \cdot n$, $|N(S)| \geq |S| \cdot \epsilon$,
- ***(n, d, α, ϵ)-edge-expander***
  if $\forall S \subseteq V$, such that $|S| \leq \alpha \cdot n$, $E(S, V \setminus S) \geq |S| \cdot d \cdot \epsilon$,
- ***(n, d, α, ϵ)-odd-(neighbor)-expander***
  if $\forall S \subseteq V$, such that $|S| \leq \alpha \cdot n$, $|N_{odd}(S)| \geq |S| \cdot \epsilon$,
- ***(n, d, α, ϵ)-unique-(neighbor)-expander***
  if $\forall S \subseteq V$, such that $|S| \leq \alpha \cdot n$, $|N_{unique}(S)| \geq |S| \cdot \epsilon$.

In fact, according to this definition every d-regular graph on n vertices is an expander[1] for a suitable choice of ϵ and α. For example choosing $\alpha < 1/n$ puts no restriction on the graph structure and the notion would be pointless. The number ϵ can be viewed as an *expansion* property of the graph, which we always want to maximize (having n, d, $\alpha \geq 1/n$ fixed for a particular graph, the largest value of ϵ fulfilling the definition can be found). But still, every connected graph would be a vertex expander with the expansion at least $1/n$. Because of that, when speaking about expanders we usually fix the values d, α, ϵ and think about what graphs are (n, d, α, ϵ)-expanders for this specific triplet. Of course, some small graphs fulfill this definition trivially, but the definition gets more interesting for larger graphs. More formally, we can use the following definition, which considers infinite sets of graphs rather than individual objects.

Definition 7. *Let $d \in \mathbb{N}$, $\alpha, \epsilon > 0$ and $\{G_i\}_{i=1}^{\infty} = \{(V_i, E_i)\}_{i=1}^{\infty}$ be an infinite sequence of d-regular graphs, such that $\forall i \in \mathbb{N} : |V_i| < |V_{i+1}|$. We say that $\{G_i\}_{i=1}^{\infty}$ is* ***a (infinite) family of (d, α, ϵ)-vertex-expanders*** *if for every $i \in \mathbb{N}$ graph G_i is $(|V_i|, d, \alpha, \epsilon)$-vertex-expander. Families of edge-expanders, odd-expanders and unique-expanders are defined in the same fashion.*

Note that this definition fixes the regularity degree d, which implies that most of the graphs in the family are sparse, and the number of edges is linear to the number of vertices. Obviously, the complete graph is the best possible expander, but it is not that interesting for us because it is dense.

We conclude this section by remarking, that the theory of expander graphs is often studied for bipartite graphs only, which are easier to analyze and sufficient for most of the applications. Sometimes the results from bipartite expanders can be easily generalized for all graphs, but it is not always the case. Therefore, we investigate the more complex and general family of all graphs.

1.2 Trade-off between α and ϵ

Many authors just fix the value of α to be $1/2$ for good for vertex-expanders and edge-expanders, while keeping it much lower for unique-expanders and odd-expanders. We investigate whether there is something magical about the value $1/2$ or if it can be any other constant less than 1.

In the only known explicit construction of unique-expanders by Alon and Capalbo [2002] the authors choose a rather small positive constant α. It is not clear if this choice is only to keep the proof simple or if there is a fundamental problem with higher values of α.

Consider a d-regular graph $G = (V, E)$ with n vertices. It has parameters n and d fixed and the expansion property ϵ can be viewed as a function of α:

$$\boldsymbol{\epsilon_{max}(G, \alpha)} = \max_{\epsilon} \{\epsilon \,|\, G \text{ is } (n, d, \alpha, \epsilon)\text{-vertex-expander}\} = \min_{S \subseteq V,\, 1 \leq |S| \leq \alpha |V|} \frac{|N(S)|}{|S|}.$$

This way, we do not have two values to investigate, but rather one relation between them, which is a non-increasing function of α. For a small value of α the expansion can be as large as d, while as the value of α tends to 1 the expansion

[1] We use the term *expander* freely to represent any type of expander graph.

property is decreasing to 0. Analogous definitions can be introduced for unique and odd expansion property. The function ϵ_{max} as a function of α is usually decreasing much faster for unique and odd expansion than for the vertex expansion, because only some of the neighbors are unique or odd. Actually in Chapter 3 we show that it drops to zero for $\alpha < 1/2$ and in many cases much earlier.

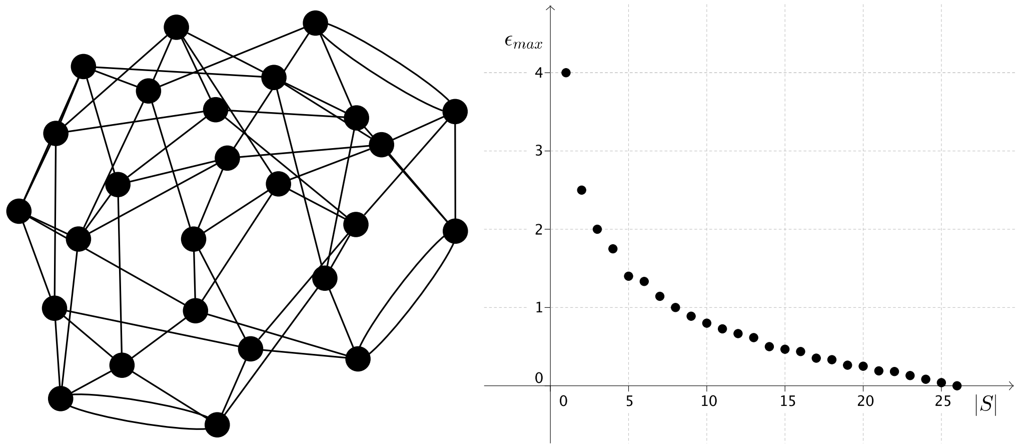

Figure 1.1: A 5-regular expander graph with 26 vertices and the graph of the corresponding ϵ_{max} function. For clarity it is displayed as a dependency on the size of the subset rather than the exact value of α.

1.3 Related topics

Loosely speaking, expander graphs are combinatorial structures, which feature various properties of complete graphs, while having fewer edges. One of the most important property mentioned earlier is the strong connectivity. Another closely related property of expander graphs is that every random walk converges to stationary distribution quite fast (they are rapidly mixing), i.e. their mixing time is $\log^{\mathcal{O}(1)} n$ (this has been known for quite a long time, but a nice explanation can be found in Motwani and Raghavan [1995]).

Surprisingly, the expander graphs theory has many applications in the complexity theory, where relations between complexity classes can sometimes be described using the structure of expanders (Valiant [1977], Sipser [1988]). More natural application of expander graphs is derandomization of algorithms, where the properties of expanders can be used to approximate larger probabilistic spaces by smaller ones (Naor and Naor [1993], Impagliazzo et al. [1994]). The last topic we only briefly mention is designing robust computer network, where good connectivity and a little number of edges is an obvious requirement and it is one of the first motivations for studying expanders (Song et al. [2002], Pippenger [1987]).

Unique-neighbor-expanders can be used to design networks, which simplify distributed algorithms in Arora et al. [1990] and Pippenger [1996] as noted by Alon and Capalbo [2002].

Finally, odd-neighbor-expanders are useful in the construction of error-correcting codes as originally proposed by Gallager [1963] and further elaborated by Sipser and Spielman [1996]. The odd number of neighbors is important to avoid ties and the error can always be uniquely classified.

A concept seemingly related to our work is the theory of *dominating sets* studied by Goldwasser and Klostermeyer [2007]. A subset S of vertices of the given graph is called an odd dominating set if every vertex v is connected to the subset by an odd number of edges, i.e. $E(\{v\}, S)$ is odd. The most significant difference from the subset odd-expansion is that vertices in the subset S must be odd neighbors as well. The original motivation of dominating sets study was *Lights Out Puzzle*[2]. However, mostly existence of dominating sets and related enumeration problems are studied, while we are interested in the size of the subset.

Another related topic is the theory of *trapping sets*. Some authors define an even-trapping-set to be a subset of vertices without odd neighbors. However, they only consider them for bipartite graphs, study related complexity problems and their usage in low-density parity-check codes (Karimi and Banihashemi [2011], Karimi and Banihashemi [2013]).

[2] `http://mathworld.wolfram.com/LightsOutPuzzle.html`

2. Existence of expanders

In this chapter, we show the existence of some expander families. Either a deterministic algorithm can be designed to prove the existence of such a combinatorial object, or a counting argument can be used to show that there are just too many graphs, and therefore some of them must be expanders. Most of the claims here are based a mathematical folklore. Unlike other related works, we focus on the trade-off between values of α and ϵ. Results stated in this chapter are not surprising, but still interesting. The main intention of this chapter is to complement upper bounds and provide an extensive theory.

2.1 Basic relations between expander types

We start this section by listing two simple lemmas relating vertex-expanders with edge-expanders. Both expansion types imply that the graph is an expander in the other sense as well, but the expansion characteristic declines.

Lemma 1. *Let $n, d \in \mathbb{N}, \alpha, \epsilon > 0$.*
Every (n, d, α, ϵ)-vertex-expander is $(n, d, \alpha, \frac{\epsilon}{d})$-edge-expander as well.

Proof. Let S be a subset of at most αn vertices of the given graph. It has at least $\epsilon|S|$ neighbors, which means there are at least $\epsilon|S|$ edges between S and the rest of the graph. The required edge expansion is achieved by definition. □

Lemma 2. *Let $n, d \in \mathbb{N}, \alpha, \epsilon > 0$.*
Every (n, d, α, ϵ)-edge-expander is (n, d, α, ϵ)-vertex-expander as well.

Proof. Consider a graph $G = (V, E)$, which is (n, d, α, ϵ)-edge-expander and $S \subseteq V$, such that $1 \leq |S| \leq \alpha n$. By definition there are at least $|S| \cdot d \cdot \epsilon$ edges between S and $N(S)$. It follows that $|N(S)| \geq \frac{|S| \cdot d \cdot \epsilon}{d} = |S| \cdot \epsilon$. □

However, the terms are qualitatively equivalent, because every vertex-expander is an edge-expander and vice versa, it might be easier to think about them in one way or another in some cases. Even more interesting is the expansion property ϵ may significantly differ and it must be strictly distinguished between the two expander types whenever we study the expansion value more precisely.

The first of the following two lemmas is merely a trivial observation, but it might be a useful connection between different expander types. The other one is a quite surprising consequence of very large vertex expansion and it can be actually used to create unique expanders.

Lemma 3. *Let $n, d \in \mathbb{N}, \alpha, \epsilon > 0$.*
Every (n, d, α, ϵ)-unique-expander is (n, d, α, ϵ)-odd-expander as well and every (n, d, α, ϵ)-odd-expander is (n, d, α, ϵ)-vertex-expander as well.

Proof. Every unique neighbor is an odd neighbor by definition, which is always a neighbor. Therefore for every $S \subseteq V$ it is $N_{unique}(S) \subseteq N_{odd}(S) \subseteq N(S)$, which implies $|N_{unique}(S)| \leq |N_{odd}(S)| \leq |N(S)|$. □

Lemma 4. *Let $n, d \in \mathbb{N}, \alpha, \epsilon > 0$. Every $\left(n, d, \alpha, \left(\frac{d}{2} + \epsilon\right)\right)$-vertex-expander is $(n, d, \alpha, 2\epsilon)$-unique-expander as well.*

Proof. Suppose there is a graph $G = (V, E)$, which is $\left(n, d, \alpha, \left(\frac{d}{2} + \epsilon\right)\right)$-vertex-expander and $S \subseteq V$, such that $1 \leq |S| \leq \alpha n$ and $|N_{unique}(S)| < 2\epsilon|S|$. Subset S has at least $\left(\frac{d}{2} + \epsilon\right)|S|$ neighbors and most of them are not unique. In particular there are less than $2\epsilon|S|$ unique neighbors and consequently more than $\left(\frac{d}{2} - \epsilon\right)|S|$ neighbors of S that are not unique. There must be at least two edges between S and every neighbor that is not unique and at least one edge between a unique neighbor and S. In total

$$E(S, N(S)) \geq |N_{unique}(S)| + 2\left(|N(S)| - |N_{unique}(S)|\right) = 2|N(S)| - |N_{unique}(S)| >$$

$$> (d + 2\epsilon)|S| - 2\epsilon|S| = d|S|,$$

which contradicts d-regularity of the graph. □

Note that edge expansion implies vertex expansion and it might imply unique expansion. Unfortunately the vertex expansion guaranteed by Lemma 2 is at most 1, while Lemma 4 requires vertex expansion more than $d/2$ in order to provide a positive unique expansion. It is not only that these lemmas are not strong enough, but no such relation can be obtained in general. Even in the unlikely situation when all the edges expand there still might be no unique neighbors at all. Graphs with *vertex expansion* strictly over $d/2$ are difficult to construct and no deterministic algorithm is known to the authors.

On the other hand Alon and Capalbo [2002] are using 8-regular *edge-expanders* with expansion over $1/2$ to create a family of $(4, \alpha, 1/40)$-unique-neighbor-expanders for a strictly positive constant α. Good edge-expanders are also difficult to construct explicitly, but there is an algorithm proposed by Lubotzky et al. [1988] and carefully analyzed by Kahale [1992], which gives an infinite family of $(8, 4\alpha, 0.525)$-edge-expanders, that can be used for the above construction of unique-expanders.

2.2 Random regular graphs

A classical result of Pinsker [1973] is that a random d-regular graph is almost surely a good expander. In this section we briefly discuss, how important is randomness in expander theory.

It is not obvious, how a random d-regular graph with n vertices can be generated and what distribution would be the best for our purposes. The best possible random graph generator would only generate expander graphs and all other graphs with zero probability. However, it is not exactly what we are looking for and what we imagine behind randomness. Intuitively, we want a fairly simple generator, which gives us such distribution, which is easy to analyze and most of the produced graphs are expanders.

Since there is only finitely many d-regular graphs with n vertices, the first obvious option is the uniform distribution, which generates every such graph with the same probability. Bollobás [1980] tried to analyze this distribution and it turned out that it is not even that simple to calculate how many different d-regular graphs exist. Another way might be to take d independent random matchings, which are much easier to generate and analyze. The downside of that approach is that it never generates self-loops and only works for an even number of vertices. Instead Bollobás [1988] and Bollobás [2001] proposed a slightly different construction, which is uniform on d-regular graphs with n vertices with labeled edges and has been proven much easier to analyze and powerful enough to produce expander graphs. We are now going to briefly describe the original *Bollobás' configuration model*.

We start by taking n isolated vertices and consider d *half-edges* leaving each of them such that their endpoints are distinct. See an example for $n = 3$ and $d = 4$ on Figure 2.1.

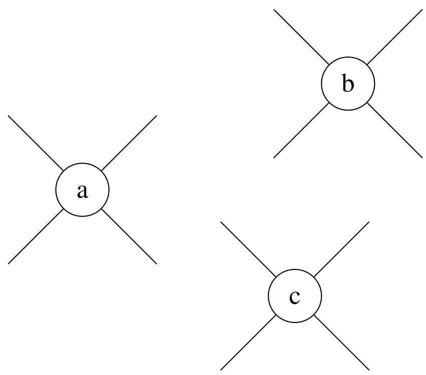

Figure 2.1: An example of the initial stage of Bollobás' random regular graph construction.

As the next step we take a random matching of these half-edges and merge every pair of matched half-edges from vertices v_1 and v_2 to an edge from vertex v_1 to vertex v_2 (possibly v_1 and v_2 being the same vertex, which leads to a loop).

This process can be equivalently described as fixing an arbitrary ordering of the half-edges and taking them in that order. Every time we find an unmatched half-edge, we merge it with another still unmatched half-edge uniformly at random. Note that every ordering of the half-edges leads to the same distribution of graphs. One possible matching can be seen on Figure 2.2 and the corresponding graph.

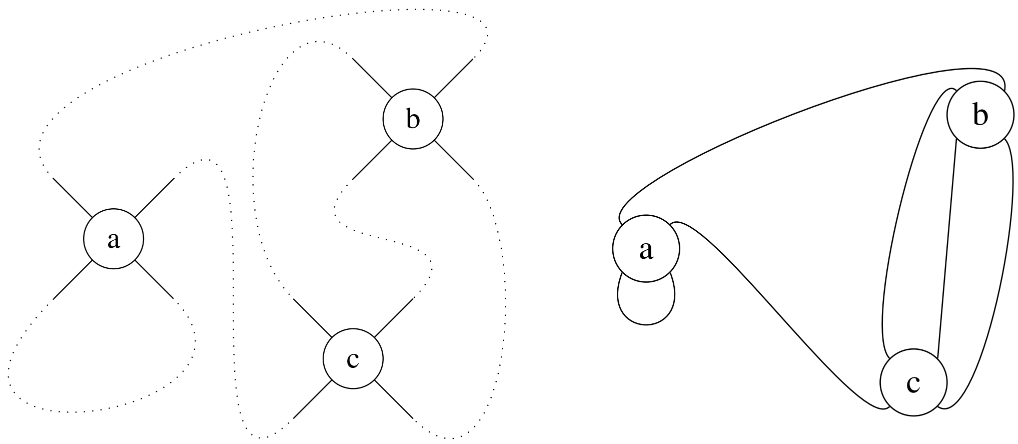

Figure 2.2: An example of the matching stage of Bollobás' random regular graph construction and the resulting graph.

It is clear that every graph can be generated by the described construction. However, some graphs can be generated multiple ways. Therefore, it leads to a non-uniform distribution of the generated graphs. As an example, we can consider 2-regular graphs with 2 vertices. There are only two such graphs – two self-loops are generated with probability 1/3 while 2 multiedges between the vertices are generated with probability 2/3 as can be seen in Figure 2.3.

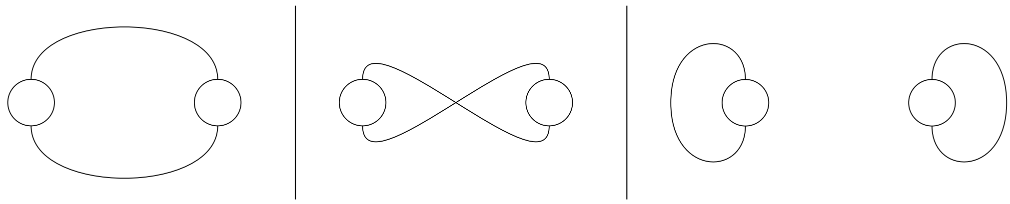

Figure 2.3: The result of generating random 2-regular graphs with 2 vertices. There are 3 equally probably matchings, each leading to one of the displayed graphs. In the first and the second options the helf-edges are matched differently, however leading to the same graph.

Lastly, we want to remark that Ellis [2011] studied Bollobás' construction closely and showed that a simple graph is generated with surprisingly high probability that depends only on d. In particular

Theorem 5. *Let $d \in \mathbb{N}$. For $n \in \mathbb{N}$ denote $G^\star(n)$ a random d-regular graph with n vertices generated by Bollobás' construction. Then*

$$\lim_{n\to\infty} \Pr\left[\ G^\star(n) \text{ is simple}\ \right] = e^{-\left(d^2-1\right)/4}.$$

2.3 Probabilistic proofs

So-called *probabilistic method* is a powerful tool, which is often used to show that complex graph structures exist. Imagine we take a random graph as described in the previous chapter and analyze it using standard methods in probability theory. Using some known inequalities, it might be possible to prove that with a positive probability our random graph poses the required property. We do not have to actually run the generator, it is rather an abstract model, which can be useful to prove an existence of an object, which we cannot explicitly describe.

We use fairly standard probabilistic method (Bollobás [1988]) to prove the following theorem.

Theorem 6. *Let $d \geq 140$ be a natural number. There exists $\epsilon > 0$ and an infinite family of $(d, 1/2, \epsilon)$-edge-expanders and $(d, 1/2, \epsilon)$-vertex-expanders.*

Proof. Consider a small fixed value of $0 < \epsilon < 1/8$, which will be specified later. Let $n \in \mathbb{N}$ be such that $n \cdot d$ is even. Consider a random d-regular graph $G = (V, E)$ with n vertices and let $S \subseteq V$ be a subset of its vertices, such that $0 < |S| \leq n/2$. Denote $s = |S|$ the size of the set S and $E_S = E(S, V \setminus S)$ the number of edges leaving the set S.

Let $0 \leq k \leq \epsilon s d \leq \epsilon n d/2$ be an integer. By the handshaking lemma it can be easily seen that if $sd - k$ is an odd number, then $\Pr\left[\ E_S = k\ \right] = 0$. Otherwise the probability that k edges are leaving the set S can be precisely calculated by selecting which edges are those k, matching them with any of the $nd - sd$ edge endpoints outside of the set S and then taking all the remaining $sd - k$ endpoints in predefined order and match them among themselves.

$$\begin{aligned}
\Pr[\ E_S = k\ ] = &\binom{sd}{k} \cdot \overbrace{\left(\frac{nd-sd}{nd-1} \cdot \frac{nd-sd-1}{nd-3} \cdots \frac{nd-sd-k+1}{nd-2k+1}\right)}^{\leq 1} \cdot \\
&\cdot \left(\frac{sd-k-1}{nd-2k-1} \cdot \frac{sd-k-3}{nd-2k-3} \cdots \frac{1}{nd-sd-k+1}\right) \\
&\leq \left(\frac{sde}{\epsilon sd}\right)^{\epsilon sd} \cdot \left(\frac{sd-k-1}{nd-2k-1}\right)^{\frac{sd-k}{2}} && \text{Usual bound } \tbinom{a}{b} \leq \left(\tfrac{ae}{b}\right)^b. \\
&\leq \left(\frac{e}{\epsilon}\right)^{\epsilon sd} \cdot \left(\frac{sd}{nd-k}\right)^{\frac{sd-k}{2}} \\
&\leq \left(\frac{e}{\epsilon}\right)^{\epsilon sd} \cdot \left(\frac{sd}{nd(1-\epsilon/2)}\right)^{\frac{sd}{4}} && \text{Using } k \leq \epsilon sd \leq \epsilon nd/2. \\
&\leq \left(\frac{e}{\epsilon}\right)^{\epsilon sd} \cdot \left(\frac{s}{n \cdot 3/4}\right)^{\frac{sd}{4}} && \text{Since } \lim_{\epsilon \to 0} \left(\tfrac{e}{\epsilon}\right)^{\epsilon} = 1, \text{ we can choose } \epsilon, \text{ such that } \left(\tfrac{e}{\epsilon}\right)^{\epsilon} \leq \sqrt[4]{\tfrac{9}{8}}. \\
&\leq \left(\frac{9}{8}\right)^{\frac{sd}{4}} \cdot \left(\frac{4s}{3n}\right)^{\frac{sd}{4}} \\
&\leq \left(\frac{3s}{2n}\right)^{\frac{sd}{4}}
\end{aligned}$$

The intuition behind this part is that every edge with one endpoint in the subset S has probability roughly s/n that the other endpoint is contained in S as well. In total there are about sd of such edges.

$$\begin{aligned}
\Pr[\ E_S < \epsilon sd\ ] &= \sum_{k=0}^{\lceil \epsilon sd \rceil - 1} \Pr[\ E_S = k\ ] \\
&\leq \epsilon sd \cdot \left(\frac{3s}{2n}\right)^{\frac{sd}{4}} \\
&\leq \frac{sd}{4} \cdot \left(\frac{3s}{2n}\right)^{\frac{sd}{4}} && \text{For } sd \geq 140 \text{ it holds } \tfrac{sd}{4} \leq \left(\tfrac{10}{9}\right)^{\frac{sd}{4}}. \\
&\leq \left(\frac{5s}{3n}\right)^{\frac{sd}{4}}
\end{aligned}$$

Using Boole's inequality (also known as the *union bound*) we obtain

$$\Pr[\ \exists S \subseteq V, 0 < |S| \leq n/2 : E_S < \epsilon |S| d\ ] \leq \sum_{\substack{S \subseteq V \\ 0 < |S| \leq n/2}} \Pr[\ E_S < \epsilon |S| d\ ]$$

$$
\begin{aligned}
&\le \sum_{s=1}^{\lfloor n/2 \rfloor} \binom{n}{s} \left(\frac{5s}{3n}\right)^{\frac{sd}{4}} \\
&\le \sum_{s=1}^{\lfloor n/2 \rfloor} \left(\left(\frac{ne}{s}\right)^{s} \left(\frac{5s}{3n}\right)^{s\frac{d}{8}} \right) \left(\frac{5n}{6n}\right)^{\frac{sd}{8}} && \text{Using } d \geq 8. \\
&\le \sum_{s=1}^{\lfloor n/2 \rfloor} \left(\frac{5e}{3}\right)^{s} \left(\frac{5}{6}\right)^{\frac{sd}{8}} && \text{For } d \geq 100 \text{ it holds } \left(\tfrac{5}{6}\right)^{\frac{d}{8}} < \tfrac{3}{10e}. \\
&\le \sum_{s=1}^{\lfloor n/2 \rfloor} \left(\frac{1}{2}\right)^{s} < 1,
\end{aligned}
$$

which means, there must be a $(n, d, 1/2, \epsilon)$-edge-expander. Repeating the whole process for more values n gives the infinite family. Note that the value $\epsilon < 1/220$ fulfills the requirements. Lemma 2 implies, that the same set of graphs is a family of $(d, 1/2, \epsilon)$-vertex-expanders.

□

The following theorem shows that the constructed graphs are actually expanders for almost arbitrary positive value of α.

Theorem 7. *Let $n \in \mathbb{N}$ and $1/2 < \alpha \leq \frac{n-1}{n}$ and $G = (V, E)$ be $(n, d, 1/2, \epsilon)$-edge-expander. Then G is $(n, d, \alpha, (1-\alpha)\epsilon)$-edge-expander.*

Proof. Let $S \subseteq V$ be a nonempty subset of vertices of G. If $|S| \leq n/2$ then by assumption the subset S expands by a factor of at least ϵ and therefore by $(1-\alpha)\epsilon$ as well. Otherwise $n/2 < |S| \leq \alpha n \leq n-1$ and denote $S' = V \setminus S$ a nonempty subset of at most $(1-\alpha)n$ vertices. By assumption

$$E(S, S') \geq |S'| \cdot d \cdot \epsilon \geq (1-\alpha)n \cdot d \cdot \epsilon \geq |S| \cdot d \cdot (1-\alpha)\epsilon.$$

The required edge expansion of G is achieved by definition. □

This theorem answers our main question about the value of α for vertex and edge-expanders. For a positive value of $\alpha < 1$ and an appropriate choice of d and ϵ there is an infinite family of (d, α, ϵ)-edge-expanders. Values found in our proof are not optimal in any way and are chosen to keep the proof as simple as possible. The general trade-off between these values is not known to the authors and it might be an interesting field to study, however, it is beyond the scope of this work.

The following theorem shows that vertex-expanders can have α linear to $1/d$ and ϵ linear to d.

Theorem 8. *Let $d \geq 100$ and $n \in \mathbb{N}$ such that dn is even, then*

$$\Pr\left[\text{ random } d\text{-regular graph is } \left(n, d, \frac{1}{100d}, \frac{d}{6}\right)\text{-vertex-expander } \right] \geq \frac{1}{2}.$$

Proof. Consider a random d-regular graph $G = (V, E)$ with n vertices. Let S be a nonempty subset of $s \leq \frac{n}{100d}$ its vertices and T be a subset of $t = \left\lfloor s \cdot \frac{d}{6} \right\rfloor$ its vertices, such that S and T are disjoint. If $N(S) \subseteq S \cup T$, then subset S would not expand enough, however, we will upper bound the probability that such an event occurs and show that it is actually quite unlikely.

We begin generating the graph by matching half-edges starting at vertices in the subset S. Let us fix an arbitrary ordering of all half-edges starting at vertices in S and while there is any unmatched half-edge, take the one with the lowest number in our ordering and match it with an unmatched half-edge uniformly at random. This process can be repeated at least $\left\lfloor \frac{sd}{2} \right\rfloor$ times (there might be no unmatched half-edges left after this number of steps). For $1 \leq i \leq \left\lfloor \frac{sd}{2} \right\rfloor$ denote Q_i the event that i-th matched half-edge has both endpoints in $S \cup T$ (one of them is in S by definition). The following probability is always the same for $1 \leq i \leq \left\lfloor \frac{sd}{2} \right\rfloor$.

$$\Pr\left[\, Q_i \;\middle|\; Q_1 \cap \cdots \cap Q_{i-1} \,\right] = \frac{sd + td - 2(i-1) - 1}{nd - 2(i-1) - 1}.$$

Originally, there were $sd + td$ half-edges starting at vertices in $S \cup T$, but $2(i-1)$ of them have already been matched and the current half-edge can not be matched with itself. Similarly we calculate that there are still $nd - 2(i-1) - 1$ options in total.

Note that the probability is independent on exact choices for other half-edges. As long as all previously matched half-edges have both endpoints in $S \cup T$ the probability for the current half-edge is precisely evaluated above. Next, we want to upper bound the probability that all edges with vertices from S have both endpoints in $S \cup T$.

$$\begin{aligned}
\Pr\left[\, N(S) \subseteq S \cup T \,\right] \leq \Pr\left[\, Q_1 \cap \cdots \cap Q_{\left\lfloor \frac{sd}{2} \right\rfloor} \,\right] &= \prod_{i=1}^{\left\lfloor \frac{sd}{2} \right\rfloor} \frac{sd + td - 2i + 1}{nd - 2i + 1} \\
&\leq \prod_{i=1}^{\left\lfloor \frac{sd}{2} \right\rfloor} \frac{sd + td}{nd} \\
&\leq \left(\frac{s + \frac{sd}{6}}{n} \right)^{\left\lfloor \frac{sd}{2} \right\rfloor} \\
&\leq \left(\frac{sd}{n} \right)^{\left\lfloor \frac{sd}{2} \right\rfloor}
\end{aligned}$$

Since $\frac{sd}{n} \leq \frac{1}{100}$ we can see that this probability is fairly low. Now we iterate through all possible subsets T and show that it is quite unlikely that there exists such a small subset T for which all the edges with an endpoint in S would fit into $S \cup T$. By the union bound

$$\begin{aligned}
\Pr[\,|N(S)| < \epsilon|S|\,] &\le \binom{n}{t}\left(\frac{sd}{n}\right)^{\lfloor \frac{sd}{2} \rfloor} \\
&\le \left(\frac{ne}{\lfloor \frac{sd}{6} \rfloor}\right)^{\lfloor \frac{sd}{6} \rfloor}\left(\frac{sd}{n}\right)^{\lfloor \frac{sd}{2} \rfloor} \\
&\le \left(\frac{6ne}{sd}\right)^{\frac{sd}{6}}\left(\frac{sd}{n}\right)^{\lfloor \frac{sd}{2} \rfloor} \\
&\le \left(\frac{6ne}{sd}\right)^{\frac{sd}{6}}\left(\frac{sd}{n}\right)^{\frac{sd}{3}} = \left(\frac{6esd}{n}\right)^{\frac{sd}{6}}.
\end{aligned}$$

For the given $s \le \frac{n}{100d}$ and by another union bound

$$\Pr[\,\exists S \subseteq V, |S| = s : |N(S)| < \epsilon|S|\,] \le \binom{n}{s}\left(\frac{6esd}{n}\right)^{\frac{sd}{6}}$$

$$\begin{aligned}
&\le \left(\frac{ne}{s}\right)^{s}\left(\frac{6esd}{n}\right)^{\frac{sd}{6}} \\
&\le \left(\frac{s}{n}\right)^{\frac{sd}{6}-s}\left(6e^2d\right)^{\frac{sd}{6}} \\
&\le \left(\frac{1}{100d}\right)^{\frac{sd}{6}-s}\left(6e^2d\right)^{\frac{sd}{6}} \\
&\le (100d)^{s}\left(\frac{6e^2}{100}\right)^{\frac{sd}{6}} \\
&\le \left(100d\left(\frac{6e^2}{100}\right)^{\frac{d}{6}}\right)^{s} && (6e^2 \sim 44.33) \\
&\le \left(100d\left(\frac{1}{2}\right)^{\frac{d}{6}}\right)^{s} && \text{Using the assumption } d \ge 100 \text{ it can be shown by induction that } \tfrac{100d}{2^{d/6}} \le \tfrac{1}{3}. \\
&\le \left(\frac{1}{3}\right)^{s}.
\end{aligned}$$

Finally by the last union bound

$$\Pr\left[\begin{array}{c}\text{random } d\text{-regular graph is} \\ \left(n, d, \frac{1}{100d}, \frac{d}{6}\right)\text{-vertex-expander}\end{array}\right] \ge 1 - \sum_{s=1}^{\lfloor \frac{nd}{100} \rfloor}\left(\frac{1}{3}\right)^{s} \ge \frac{1}{2}.$$

□

This theorem can be improved to show that vertex-expanders exist for any value of $\epsilon < d/2$.

2.4 Explicit constructions

Although a random graph is a good expander with high probability, according to Blum et al. [1981] it is a coNP-complete problem to test the expansion property in general. Therefore there always was a desire for a deterministic (or explicit) constructions of expander graphs. Several such constructions have been described in Reingold et al. [2001], Margulis [1973]. Most of the constructions are using $\alpha = 1/2$ as a constant. Already mentioned work of Alon and Capalbo [2002] is the only explicit construction of unique-expanders known to us. The authors only mention, that there exists a suitable positive value α, but do not discuss its value at all.

3. Upper bounds for the value α

The goal of this chapter is to describe values of α for which there is no infinite family of (d, α, ϵ)-unique-neighbor-expanders and no infinite family of (d, α, ϵ)-odd-neighbor-expanders. It is not obvious that any such value of α exists. However, we state several upper bounds, which might not be optimal, but are not too far away.

Firstly, note that the argument used in Theorem 7 cannot be reproduced here. If a subset of at most half of the vertices edge-expands, then its complement edge-expands as well (although the expansion property is reduced). This is not true for unique or odd expansion. Note that large odd-expansion of a subset of vertices does not guarantee any unique neighbors of its complement.

We start this chapter with an observation, which is using the fundamental fact that a subset of vertices cannot have more neighbors than the number of vertices left in the graph. This bound is an asymptotically tight description of the trade-off between α and ϵ for vertex-expanders (see Theorem 8).

Lemma 9. *Let $d \in \mathbb{N}$ and $\epsilon > 0$ and $\alpha > 1/(1+\epsilon)$. There is no infinite family of (d, α, ϵ)-vertex-expanders.*

Proof. Let $G = (V, E)$ be a d-regular graph with n vertices. For simplicity, we only consider such n, that αn is an integer (in general it is only more technical). Let $S \subseteq V$ be any subset of its αn vertices. Because sets S and $N(S)$ are by definition disjoint, it follows that $|S| + |N(S)| \leq n$. Therefore

$$|N(S)| \leq n - |S| = \frac{1}{\alpha}|S| - |S| < \epsilon|S|,$$

what means that G cannot be (n, d, α, ϵ)-vertex-expander. □

3.1 Unique neighbor expanders

Because of Theorem 7 there is no general bound on the value of α for vertex-expanders. Let us try to use Lemma 4, which claims that large vertex expansion implies unique expansion, to show that there is no restriction on α for unique-neighbor-expander graphs as well. However, it is not possible to achieve vertex expansion more than $d/2$ with an arbitrarily large value of α. In fact Lemma 9 implies that α must be at most $\frac{2}{2+d}$ in that case. This does not prove that unique-expanders cannot exist for higher values of α, but only that vertex-expanders cannot straightforwardly guarantee them.

The following result is a surprisingly general theorem about all graphs with no restrictions.

Theorem 10. *Let $G = (V, E)$ be a graph. There exists a nonempty subset of its vertices $S \subseteq V$ such that $|S| \leq \frac{|V|}{2} + 1$, which does not have any unique neighbors ($|N_{unique}(S)| = 0$).*

Remark. The stated bound is tight, because of an even length path graph with.

Proof. We describe an algorithm, which finds the required subset of vertices. We discuss the following two cases:

1. If the graph G has a perfect matching, then fix any such matching. We start with an empty set S and gradually construct it by adding vertices. First, choose any matching edge and put both of its endpoints to the set S. Next, as long as there is a unique neighbor w of the set S, we add its matching neighbor u to the set S. After the vertex u is added to S, the vertex w already has two neighbors in S and it can never become a unique neighbor again. Therefore the vertex u is never added to the set S twice. Note that we never add both vertices of a matching edge to the set S except for the starting one. After the algorithm finishes, at most one vertex from every matching edge is contained in the set S plus an extra vertex for the edge we started with. In total $|S| \leq \frac{|V|}{2} + 1$, as required.

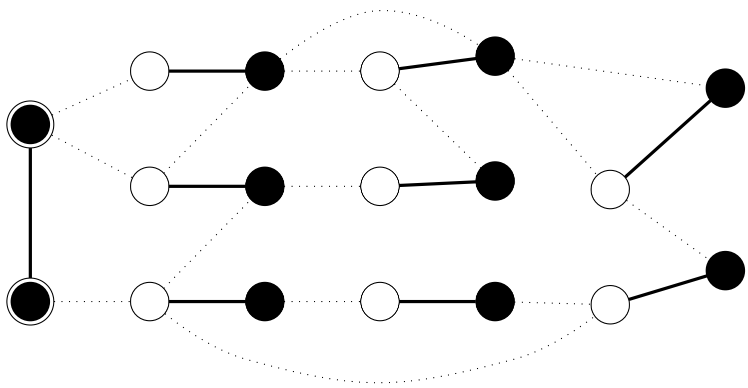

Figure 3.1: An example of the graph which has a perfect matching. Vertices that belong to the set S are filled dark, while these that does not belong to the set S are empty.

2. If the graph G does not have a perfect matching, then consider any maximum matching and denote an unmatched vertex v. We start with the set $S = \{v\}$ and run the same algorithm as in the previous case. As long as there is a unique neighbor w of the set S, we add its matching neighbor u to the set S. The described procedure does not specify what happens if the vertex w is unmatched. However, we show that no unmatched vertex can ever become a unique neighbor of the set S. Suppose it has happened for the first time and let v_0 be the unmatched unique neighbor of the set S. Look at its only neighbor in the set S and call it v_1. The vertex v_1 was added to the set S, because its matching neighbor v_2 was a unique neighbor by that time. Take the original only neighbor of v_2 and denote it v_3. Continue further for $i > 1$ let v_{2i} be the matching neighbor of v_{2i-1} and let v_{2i+1} be the original only neighbor of v_{2i}, which caused that v_{2v-1} was added to the set S. The vertex v_{2i+1} was added to the set S earlier than the vertex v_{2i}. The process ends when the vertex $v = v_k$ is reached. Observe that $v_0, v_1, v_2, v_3, \dots, v_k$ is an augmenting path, which contradicts the maximality of the matching. After the algorithm finishes at most one vertex from every matching edge is contained in the set S plus the vertex v. In total $|S| \leq \frac{|V|-1}{2} + 1$, as required.

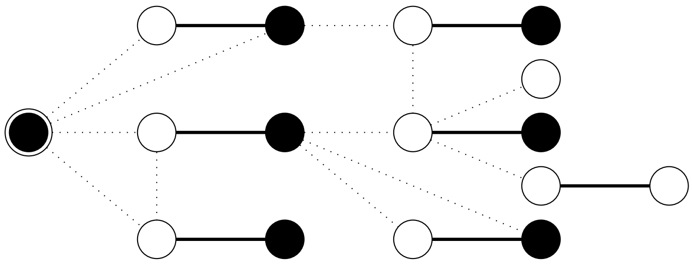

Figure 3.2: An example of the graph which does not have a perfect matching. Vertices that belong to the set S are filled dark, while these that does not belong to the set S are empty.

□

Self-loops and multiedges are allowed in the theorem, however for clarity, we can first remove them from the graph, then find the required subset S of vertices in the remaining simple graph and finally return them back in the graph. Self-loops do not create any unique neighbors. If there is a multiedge between two vertices, which are both in S or both are not present in S, it can not add unique neighbors. The last case is when one of the endpoints v is contained in the set S while the other endpoint w is not. In that case, there are at least two edges between S and the vertex w, therefore it is a unique neighbor.

The proof of the above theorem may look unnecessarily complicated, and a very natural question is whether it is possible to avoid using the maximum matching. One way to simplify the proof might be a greedy approach, which in every step adds an arbitrary neighbor of the unique neighbors to make it a non-unique neighbor of the set S. Unfortunately, no such neighbor has to exists. We can also add the unique neighbor to the set S, but it does not work as well. Consider the star graph displayed on Figure 3.3. If the greedy algorithm starts by the central vertex, all other vertices are added to the set.

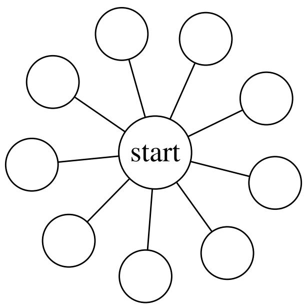

Figure 3.3: An example of the star graph, for which the greedy algorithm will find too large subset of vertices without unique neighbors if it starts by the central vertex.

Theorem 11. *Let $d \geq 3$ and $0 < \epsilon < 1$. There is no family of $(d, \frac{1}{2} - \frac{\epsilon}{4d}, \epsilon)$-unique-expanders.*

Proof. Suppose $\{(V_i, E_i)\}_{i=1}^{\infty}$ is such an infinite family of unique expanders. We take one of them, for which $n = |V_i| > 24d/\epsilon$ and find the smallest nonempty subset of its vertices $S \subseteq V_i$ for which $|N_{unique}(S)| = 0$. If $|S| \leq \left(\frac{1}{2} - \frac{\epsilon}{4d}\right) n$ then the subset S would not uniquely expand. If we remove a single vertex from a set of vertices, the number of its unique neighbors increases by at most d. Let us construct S' from S by removing some vertices such that

$$\frac{n}{3} \leq \left(\frac{1}{2} - \frac{\epsilon}{4d}\right) n - 1 < |S'| \leq \left(\frac{1}{2} - \frac{\epsilon}{4d}\right) n.$$

(Note that he first inequality holds, because $\epsilon < 1$ and $d \geq 3$ and $n \geq 12$.)
By Theorem 10 and using $n > 24d/\epsilon$ we obtain

$$|N_{unique}(S')| \leq (|S| - |S'|)d \leq \left(\frac{\epsilon n}{4d} + 2\right) d = \frac{\epsilon n}{4} + 2d < \frac{\epsilon n}{3} < \epsilon |S'|.$$

The set S' has too lttle unique neighbors, which contradicts that (V_i, E_i) is a $(n, d, \frac{1}{2} - \frac{\epsilon}{4d}, \epsilon)$-unique-expander. □

Despite the lower bound stated by Theorem 10 is tight in general, for higher degree graphs it might be possible to state even stronger results. If we managed to bound α linearly by $\frac{1}{d}$ it would mean that it might be possible to prove the other bound by vertex-expanders method mentioned in Lemma 4. The following theorem is getting closer to that goal.

Theorem 12. *Let $d \geq 3$, $n \geq \frac{d}{\log d}$ and $G = (V, E)$ be a d-regular graph with n vertices, with no self-loops and multiedges (it is a simple graph). There exists a nonempty subset of its vertices $S \subseteq V$ such that $|S| \leq \frac{16n \log d}{d}$, for which all other vertices have at least 2 neighbors in S. In other words*

$$\forall v \in V \setminus S : E(S, \{v\}) \geq 2.$$

Proof. Start by choosing a random set S' of vertices where every vertex from V is added independently at random with the probability $p = \frac{\log d}{d} < 0.5$. The random variable $|S'|$ is binomially distributed and by Markov's inequality

$$\Pr\left[\ |S'| \geq \frac{4n \log d}{d}\ \right] \leq \frac{1}{4}.$$

On the other hand probability that S' does not contain any element is fairly low:

$$\Pr[\ |S'| = 0\ ] = (1-p)^n = \left(1 - \frac{\log d}{d}\right)^n \leq \left(1 - \frac{\log d}{d}\right)^{\frac{d}{\log d}} \leq e^{-1}$$

We can also straightforwardly evaluate the probability that a vertex, which is not in the set S', has a small number of neighbors in S' (using the fact that there are no self-loops and multiedges):

$$\begin{aligned}
\Pr[\ E(S', \{v\}) < 2\ ] &= \Pr[\ E(S', \{v\}) = 0\ ] + \Pr[\ E(S', \{v\}) = 1\ ] = \\
&= (1-p)^d + d \cdot p \cdot (1-p)^{d-1} \\
&= \left(1 - \frac{1}{q}\right)^{q \log d} + \frac{d \cdot p}{1-p} \cdot \left(1 - \frac{1}{q}\right)^{q \log d} && \text{Denote } q = \tfrac{d}{\log d}. \\
&\leq d^{-1} + \frac{d \log d}{(1-p)d} \cdot d^{-1} \\
&\leq \frac{1 + 2 \log d}{d} && \text{Using that } p \leq 0.5. \\
&\leq \frac{3 \log d}{d}
\end{aligned}$$

Let C be the set of all vertices $v \in V \setminus S'$ which have at most one neighbor in S'. The size of C is a random variable with the expected value at most $\frac{3n \log d}{d}$ (using the linearity of expectation). By Markov's inequality

$$\Pr\left[\ |C| \geq \frac{12n \log d}{d}\ \right] \leq \frac{1}{4}.$$

Now we are ready to define the set $S = S' \cup C$. Obviously all remaining vertices in $V \setminus S$ have at least two neighbors in set S, because they all have at least two neighbors in S'. Since S' and C are disjoint, the size of their union is the sum of their sizes. Using union bound inequality we obtain

$$\Pr\left[\ |S| \geq \frac{16n \log d}{d}\ \right] \leq \Pr\left[\ |S'| \geq \frac{4n \log d}{d} \vee |C| \geq \frac{12n \log d}{d}\ \right] \leq \frac{1}{4} + \frac{1}{4} = 0.5.$$

Finally, putting it all together we get

$$\Pr\left[\ 0 < |S| \leq \frac{16n \log d}{d}\ \right] \geq 1 - e^{-1} - 0.5 > 0,$$

which implies that such a set S exists. □

Straightforwardly it follows that no infinite family of *simple* $(d, \frac{16 \log d}{d}, \epsilon)$-unique-neighbor-expanders exists. In general the value of α for unique-expanders can not be restricted by any $o(1)$ function of d. Imagine a family of (d, α, ϵ)-unique-neighbor expanders and add a self-loop to every vertex in every graph of the family. The expansion property is not affected while the degree increases arbitrarily.

In fact, self-loops and multiedges do never increase the unique expansion of the graph. They can only be used to increase the degree (or to make the graph regular). Therefore, it makes some sense to study unique-neighbor-expanders for simple graphs only.

Theorem 12 can be slightly generalized and we state two separate extensions. Since they are not that important for our work, we do not provide a full formal proof.

Theorem 13. *Let $d \geq 3$, $n \geq \frac{d}{\log d}$ and $G = (V, E)$ be a simple graph with n vertices with the smallest degree equal to d. There exists a nonempty subset of its vertices $S \subseteq V$ such that $|S| \leq \frac{16n \log d}{d}$, for which all other vertices have at least 2 neighbors in S.*

Proof. The only difference here is the probability $\Pr[\ E(S, \{v\}) = 1\ ]$, which is equal to

$$\deg_G(v) \cdot p \cdot (1 - p)^{\deg_G(v) - 1},$$

however, it can be shown (using derivations for example) that the highest value of this expression is attained at d. □

Theorem 14. *Let $k \geq 2$, there is $d_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $d \geq d_0$ and every simple simple d-regular graph with n vertices there exists a nonempty subset S of $\mathcal{O}\left(\frac{n \log d}{d}\right)$ its vertices, for which all other vertices have at least k neighbors in it.*

Proof. The key part is to bound the following probability:

$$\Pr[\ E(S', \{v\}) < k\ ] = \sum_{m=0}^{k-1} \binom{d}{m} p^m (1-p)^{d-m}$$

By an appropriate choice of probability p it can be estimated to be $\mathcal{O}\left(\frac{\log d}{d}\right)$. $\square$

Note that the bound stated in the Theorem 12 might not be optimal. We believe that a subset of $\mathcal{O}(n/d)$ vertices without unique neighbors can always be found. We can try to adjust the proof presented to achieve this bound by choosing the probability to be a constant fraction of $1/d$ instead. Then, the size of the set S' has the required size with a high probability. However, there are likely too many unique neighbors C – a constant fraction of all vertices in the graph. We can repeat the process with the remaining unique neighbors, but it must be done roughly $\log d$ times before its size is decreased to $\mathcal{O}(n/d)$. Consequently, it leads to the same result as stated above.

Let us present some experimental results. It takes a lot of computational time to test all simple d-regular graphs with n vertices and therefore we only managed to run the algorithm for quite small values of n. The results clue that the optimal bound might be around $2/d$, but it is only our guess because it is almost impossible to guess the behavior of the bound at such a small data. Nevertheless, there are always at most two vertices of the complete graph ($(n-1)$-regular graph) without unique neighbors (actually, any two will do). Another interesting part of the table is the row describing $(n/2)$-regular graphs. Theorem 12 shows that the size of the subset is always bounded by a constant while the table suggests that it might be quite a small constant. It can be an interesting challenge to investigate this combinatorial subproblem more deeply, however we did not study it any further.

number of vertices n	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
all graphs	3	4	5	6	7	8	9	10	11
2-regular graphs	2	3	4	5	6	7	8	9	10
3-regular graphs	2	3	4	4	6	6	7	8	8
4-regular graphs	-	2	4	4	5	6			
$\left(\frac{n}{2}\right)$-regular graphs	2	3	4	4	4	4			
$(n-1)$-regular graph	2	2	2	2	2	2	2	2	2

Table 3.1: The maximum size of the minimal nonempty set of vertices without unique neighbors among all simple graphs in the given category.

3.2 Odd neighbor expanders

The odd neighbor expander is a less restricted version of the unique neighbor expander studied in the previous section. Any bounds on the value of α deduced for odd expanders can be directly applied to unique expanders. However, it is much more difficult to find a subset of vertices without any odd neighbors. For unique neighbors we exclusively used the fact, that if we have a unique neighbor of any subset of vertices and add one of its neighbors to the set, it already has at least two neighbors in the set and it can never become a unique neighbor again even if we add more vertices to the set. It is not the case in odd expanders since an even neighbor can easily become an odd neighbor if we add one of its neighbor to the set. Nevertheless, we claim a very similar theorem as for unique expanders.

Definition 8. *Let $G = (V, E)$ be a graph. A nonempty subset of its vertices $S \subseteq V$ such that $|S| \leq \frac{|V|}{2} + 1$, which does not have any odd neighbors (i.e. $|N_{odd}(S)| = 0$) is called **a small even subset of G**.*

Note that adding or removing self-loops does not change odd neighbors of any subset of vertices. Likewise adding or removing two multiedges between a pair of vertices does not change odd neighbors of any subset of vertices. It means that we can only consider simple graphs.

Conjecture 1. *Every graph has a small even subset.*

This is a more general version of Theorem 10, because not only unique neighbors are forbidden, but also no odd neighbors are allowed. As a consequence there would be no infinite family of $(d, \frac{1}{2}, \epsilon)$-odd-neighbor-expanders.

The conjecture can be easily shown for disconnected graphs:

Lemma 15. *Let $G(V, E)$ be a graph and a subset of its vertices $A \subsetneq V$, such that $0 < |A| < |V|$ and $E(A, V \setminus A) = 0$. Then there is a small even subset of G.*

Proof. Both A and $V \setminus A$ are nonempty subsets of V with no odd neighbors. The smaller of them is a small even subset of G. $\square$

We do not know, how to prove Conjecture 1, nevertheless we can at least prove it for most bipartite graphs.

Theorem 16. *Let $G = (V, E)$ be a graph, such that $V = A \cup B$, $A \cap B = \emptyset$, $E(A, A) = 0$, $E(B, B) = 0$ and $|A| > |B|$. There exists a small even subset of G.*

Proof. Denote $a = |A|$, $A = \{v_1, \ldots, v_a\}$, $b = |B|$ and $B = \{w_1, \ldots w_b\}$. Imagine, that $v_1, \ldots, v_a$ are $\{0, 1\}$ variables and $w_1, \ldots, w_b$ are linear equations saying that the sum of the neighboring vertices must be an even number. More precisely, we define the following system of equations in the field $\mathbb{Z}_2$:

$$\begin{array}{lccccccccc}
w_1: & E(v_1, w_1)x_1 & + & E(v_2, w_1)x_2 & + & \ldots & + & E(v_a, w_1)x_a & = & 0 \\
w_2: & E(v_1, w_2)x_1 & + & E(v_2, w_2)x_2 & + & & + & E(v_a, w_2)x_a & = & 0 \\
\vdots & \vdots & & & & \ddots & & & \vdots & \\
w_b: & E(v_1, w_b)x_1 & + & E(v_2, w_b)x_2 & + & \ldots & + & E(v_a, w_b)x_a & = & 0
\end{array}$$

Consider a solution $(x_1, \ldots, x_a)$ of the above equation and define a set of vertices

$$S = \{v_a \,|\, x_a = 1\}.$$

No vertex from the set B can be an odd neighbor of S, because it would contradict the corresponding equation. No vertex from the set A is a neighbor of S at all. Therefore $|N_{odd}(S)| = 0$ as required. It remains to show that we can pick such a solution $(x_1, \ldots, x_a)$, that S is nonempty and contains at most $\frac{V}{2} + 1$ vertices.

We fix $x_{b+2}, x_{b+3}, \ldots, x_a$ to value 0 and study the resulting system of b linear equations of $b+1$ variables. Since all zeros is a trivial solution, the dimension of solution space must be at least two and there is another solution $(x_1, x_2, \ldots, x_{b+1})$, which is not consisted of all zeros. At last, we have found a solution of the original system of equations with at most $b+1$ ones, which implies that the corresponding set S is nonempty and contains at most $b+1 \leq \frac{b}{2} + \frac{a}{2} + 1 \leq \frac{V}{2} + 1$ vertices. $\square$

Let us provide some intuition behind the proof of the above theorem. Imagine Gaussian elimination reducing the system of equations by elementary row operations. These operations have a nice analogy in the graph theory:

1. **Row switching** only relabels the vertices and does not change the structure of the graph.

2. **Row multiplication** can be only performed for a nonzero constant. Multiplying an equation by 1 does not change it at all.

3. **Row addition** is the only operation, which does something non-trivial. Without loss of generality (row switching) consider we replace equation corresponding to the vertex w_1 by the sum of equations corresponding to vertices w_1 and w_2. It can be seen as combining edges leading from vertex w_1 with edges leading from w_2 and removing those sharing the other endpoint. There is no odd neighbor for any subset of A in the new graph if and only if there was no odd neighbor for the same subset in the original graph (vertex w_1 had an even number of neighbors in the set originally, and we added an even number of other neighbors, possibly removing some pairs of duplicitous edges).

Gaussian elimination chooses one variable x (a vertex $v \in A$ in our analogy) and an equation w in which it is present (corresponding to a neighbor of x). Next, the vertex w is "added" to all other neighbors of v leaving the vertex v a leaf (degree equal to 1) in the resulting graph.

Lemma 17. *If Conjecture 1 is true of all graphs containing no vertices with degree equal to 1, then it is true for all graphs.*

Proof. Suppose it is not and consider any such graph with the least number of vertices $G = (V, E)$ and a vertex $v \in V$, such that $\deg_G(v) = 1$. If the graph contains only two vertices, then V is the small even subset itself. Otherwise, if we remove vertex v, its only neighbor w and all incident edges from the graph, we obtain a smaller graph with $|V| - 2$ vertices. By assumption it contains a nonempty subset of its vertices S' of size at most $|V|/2$ with no odd neighbors. The only vertex, which can be an odd neighbor of S' in the original graph is w. In such a case we take $S = S' \cup \{v\}$ and $S = S'$ otherwise. In both cases S is a small even subset of the original graph. $\square$

Gaussian elimination creates a leaf, then it temporarily removes the leaf from the graph with its only neighbor and solve the smaller instance of the same problem. The solution of the original problem can be easily obtained from the solution of the subproblem.

Note that we used the fact, that parts A and B have a different number of vertices. If they have the same size, we can drop one vertex and run the same algorithm for the remaining vertices. When we return that one vertex to the graph, it can be the only odd neighbor of the set, which still contradicts the existence of any family of bipartite $(d, 1/2, \epsilon)$-odd-neighbor-expanders.

The following theorem shows how the problem of finding a small even subset can be reduced to biconnected graphs.

Definition 9. *Let $G(V, E)$ be a graph. We call $v \in V$* ***an articulation*** *if the remaining vertices can be decomposed in two disjoint nonempty subsets with no edges between them. More formally there exist $A, B \subseteq V$, such that $v \notin A$ and $v \notin B$ and $A \cap B = \emptyset$ and $A \cup B \cup \{v\} = V$ and $|A| \neq 0$ and $|B| \neq 0$ and $E(A, B) = 0$. A graph with no articulation is called* ***biconnected****.*

Theorem 18. *Let $0 < \beta < 1$. Suppose that for every biconnected graph $G' = (V', E')$ there is a nonempty subset $S' \subseteq V'$ of its vertices, such that $|S'| \leq \beta|V'|$ and $|N_{odd}(S')| = 0$. Then for every graph $G = (V, E)$ there is a nonempty subset $S \subseteq V$ of its vertices, such that $|S| \leq \beta|V|$ and $|N_{odd}(S)| = 0$.*

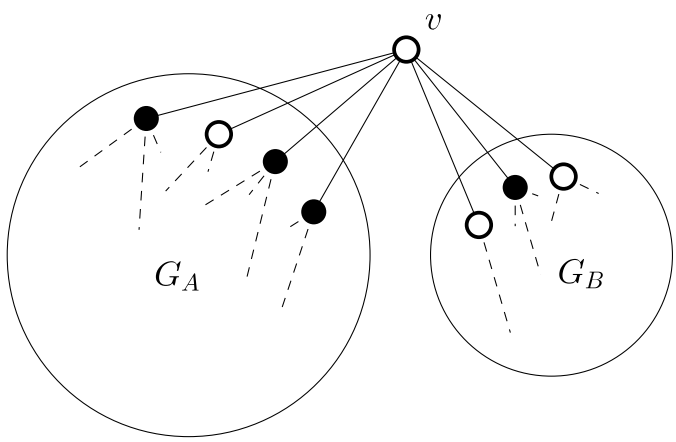

Figure 3.4: A graph, which has an articulation v and corresponding subsets A and B. Vertices that belong to sets S_A and S_B are filled dark.

Proof. For contrary suppose that there is a graph $G = (V, E)$ which does not have a subset of vertices with the required property. Take the counterexample with the smallest number of vertices. It cannot be biconnected and therefore there is an articulation $v \in V$ and the corresponding decomposition A, B from the previous definition. Let G_A and G_B be subgraphs of G induced by A and B respectively (we keep the vertices and all edges between them). These graphs have fewer vertices than the graph G and by assumption there are corresponding subsets S_A and S_B of the required sizes ($0 \leq |S_A| \leq \beta|A|$ and $0 \leq |S_B| \leq \beta|B|$) without odd neighbors. Both of the subsets have almost no odd neighbors in the original graph G, in fact, only the vertex v can be an odd neighbor of S_A and the same holds for S_B. If either $N_{odd}(S_A) = \emptyset$ or $N_{odd}(S_B) = \emptyset$ then the corresponding subset meets all the conditions from the statement. Otherwise a nonempty subset $S = S_A \cup S_B$ has no odd neighbors in the graph G (no vertex from A and B can be an odd neighbor because then it would be an odd neighbor in G_A or in G_B and v

has an odd number of neighbors both in S_A and S_B, which gives an even number of neighbors in total) and it consists of at most $\beta|A| + \beta|B| \leq \beta|V|$ vertices. In both cases we found a subset of with required properties. $\square$

For comparison we also attach a table displaying how large is the minimal even subset in certain groups of graphs. This experiment supports Conjecture 1.

number of vertices n	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
all graphs	3	4	5	6					
2-regular graphs	2	3	4	5	6	7	8	9	10
3-regular graphs	2	4	4	6	6	6	8	8	
4-regular graphs	-	2	4	4	5	6			
$\binom{n}{2}$-regular graphs	2	3	4	4	5	6			
$(n-1)$-regular graph	2	2	2	2	2	2	2	2	2

Table 3.2: The maximum size of the minimal even subset of among all simple graphs in the given category.

3.3 Further research and open problems

The purpose of this section is to summarize a possible future work and to remind still open problems.

Problem 1. What is the general trade-off between values d, α and ϵ for vertex and edge expanders? For what combination of these values there are infinite families of expanders? Although it has been mentioned in Vadhan [2012], that a random bipartite multigraph is a bipartite (n, d, α, ϵ)-vertex-expander with high probability for sufficiently large n if

$$d > \frac{H(\alpha) + H(\alpha\epsilon)}{H(\alpha) - \alpha\epsilon H(1/\epsilon)},$$

where $H(p) = p\log(1/p) + (1-p)\log(1/(1-p))$ is the binary entropy function. Unfortunately the claim is not proven there and no reference is provided. Moreover only bipartite expanders are considered.

Problem 2. Is there a nonempty subset of $\mathcal{O}(1/d)$ vertices without unique neighbors in every simple d-regular graph? Note that even if the answer to the question is negative, it does not necessarily mean, that simple unique-neighbor-expanders with this high value of α exist. There still might be a subset with a fairly low number of unique neighbors with this size.

Problem 3. Is there a subset of 4 vertices in every simple n-regular graph with $2n$ vertices without unique neighbors? If not what is the least number of vertices needed? The existence of such constant follows from Theorem 12.

Problem 4. As we did not prove Conjecture 1 it remains an open problem. However, it is enough to proof it for biconnected graphs, which are not bipartite.

Conclusion

Finally, we are ready to summarize and evaluate all our results.

<table>
<tr><th></th><th>infinite family exists</th><th>no infinite family exists</th></tr>
<tr><td rowspan="2">vertex-expanders</td><td>$0 < \alpha < 0$, appropriate values of d and ϵ, Theorem 7</td><td rowspan="2">$\epsilon > 0$, $\alpha > 1/(1+\epsilon)$, Lemma 9</td></tr>
<tr><td>$d \geq 100$, $\alpha \leq \frac{1}{100d}$, $\epsilon \leq \frac{d}{6}$, Theorem 8</td></tr>
<tr><td rowspan="2">edge-expanders</td><td>$0 < \alpha < 0$, appropriate values of d and ϵ, Theorem 7</td><td rowspan="2">$\epsilon > 0$, $\alpha > 1/(1+\epsilon)$, Lemma 9 and Lemma 2</td></tr>
<tr><td>$d \geq 100$, $\alpha \leq \frac{1}{100d}$, $\epsilon \leq \frac{1}{6}$, Theorem 8 and Lemma 1</td></tr>
<tr><td>odd-expanders</td><td>$d \geq 3$, $\epsilon \leq \frac{1}{40}$,
α is a positive constant,
Alon and Capalbo [2002]</td><td>$\epsilon > 0$, $\alpha \geq \frac{1}{2}$,
Conjecture 1 (proved for bipartite graphs)</td></tr>
<tr><td rowspan="2">unique-expanders</td><td rowspan="2">$d \geq 3$, $\epsilon \leq \frac{1}{40}$,
α is a positive constant,
Alon and Capalbo [2002]</td><td>$\epsilon > 0$, $\alpha \geq \frac{1}{2} - \frac{\epsilon}{4d}$,
Theorem 11</td></tr>
<tr><td>$\epsilon > 0$, $\alpha \geq \frac{16 \log d}{d}$,
simple graphs,
Theorem 12</td></tr>
</table>

Table 3.3: Summarizes under what conditions what expander classes exists and what configurations do not allow any family of (d, α, ϵ)-expanders.

While the bounds for vertex-expanders and edge-expanders are only trivial ones, restrictions stated for unique-expanders and odd-expanders are not obvious at all. Intuitively there is no reason, why low values of α should be used for unique-expanders, however, we have shown that it is pointless to search for some families of expanders, because they can never be found.

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$\{x_1, x_2, \ldots, x_n\}$ . . .	set containing n elements x_1, x_2, …, x_n
$\lvert S \rvert$	cardinality (the number of elements) of the finite set S
$(x_1, x_2, \ldots, x_n)$ . . .	ordered tuple containing n elements x_1, x_2, …, x_n
$\mathbb{N}$	set of natural numbers $\{1, 2, 3, 4, \ldots\}$
$\Pr[\ A\ ]$	probability that the event A occurs
$\Pr[\ A \mid B\ ]$	conditional probability that the event A occurs given that the event B has occurred
$A \cup B$	union of sets A and B
$A \cap B$	intersection of sets A and B
$\emptyset$	empty set
$\lfloor x \rfloor$	the largest integer less than or equal to x
$\lceil x \rceil$	the smallest integer greater than or equal to x