Minimal Counterexamples to Flow Conjectures

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Overview

1. Introduction to the Problem
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2. Restrictions on Counterexamples
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3. Kochol’s Approach


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2. Restrictions on Counterexamples
3. Kochol’s Approach
4. Computations and Modifications

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**Martin Kochol.**

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Smallest counterexample to the 5-flow conjecture has girth at least eleven. *Journal of Combinatorial Theory, Series B, 100(4):381–389, 2010.*
Introduction: \( k \)-Flow

![Graph Diagram]
**Introduction: $k$-Flow**

**Definition**

A $\mathbb{Z}_k$-flow (a $k$-flow) on a graph $G$ is a mapping $f : E_G \rightarrow \mathbb{Z}_k$ s.t.

\[
\sum_{\bar{e} = (v,u)} f(\bar{e}) = \sum_{\bar{e} = (u,v)} f(\bar{e})
\]

holds for every vertex $v$. 

---

![Diagram of $k$-flow on a graph](image-url)

*Minimal Counterexamples to Flow Conjectures*
Introduction: $k$-Flow

Definition

A $\mathbb{Z}_k$-flow (a $k$-flow) on a graph $G$ is a mapping $f: E_{\vec{G}} \rightarrow \mathbb{Z}_k$ s.t.

$$\sum_{\vec{e}=(v,u)} f(\vec{e}) = \sum_{\vec{e}=(u,v)} f(\vec{e})$$

holds for every vertex $v$.

- Flow $f$ is nowhere-zero if $f(\vec{e}) \neq 0$ holds for each arc $\vec{e}$.
For a planar graph $G$, let $G^*$ be its dual graph:
- vertex $v_F^*$ for each face $F$ of $G$,
- edge $e_e^*$ for each edge $e \in E_G$.
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- edge $e_e^*$ for each edge $e \in E_G$.

**Observation**

Planar graph $G$ admits a nowhere-zero $k$-flow if and only if its dual graph $G^*$ has a $k$-coloring.
Conjecture (Tutte)

There exists $k \in \mathbb{N}$ s.t. each bridgeless graph admits a NZ $k$-flow.

Paul D. Seymour.
Nowhere-zero 6-flows.

William Thomas Tutte.
A contribution to the theory of chromatic polynomials.
Theorem (Seymour)
Each bridgeless graph admits a NZ 6-flow.

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Minimal Counterexample

- Seymour: minimal counterexample is cubic.

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**Theorem**

Neither $C_3$ nor $C_4$ can be subgraph of a minimal counterexample to the 5-Flow Conjecture.

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Theorem (Kochol)

Any minimal counterexample to the 5-Flow Conjecture does not contain any circuit of length less than 9.

Martin Kochol.

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**Theorem (Kochol)**

Any minimal counterexample to the 5-Flow Conjecture does not contain any circuit of length less than 11.

---

**Martin Kochol.**


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Minimal Counterexample

**Theorem (PK)**

Any minimal counterexample to the 5-Flow Conjecture does not contain any circuit of length less than 12.
Lemma

For a network \((G, U)\), \(|U| = n\), there exist integers \(x_1, \ldots, x_{pn}\) s.t.

\[
F_{G, U}(s) = \sum_{i=1}^{pn} x_i \chi(s, P_i)
\]

holds for each \(s \in S_n\).
Introduction to Networks

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For a network \((G, U)\), \(|U| = n\), there exist integers \(x_1, \ldots, x_n\) s.t.

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\(p_4 = 4, \mathcal{P}_4 = \{ P_1 \}\)
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F_{G,U}(s) = \sum_{i=1}^{p_n} x_i \chi(s, P_i)
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\[
p_4 = 4, \quad \mathcal{P}_4 = \{P_1, P_2, P_3, P_4\}
\]

\[
F_{G,U}(s) = 1 \chi(s, P_1) + 1 \chi(s, P_2) + 0 \chi(s, P_3) + 1 \chi(s, P_4)
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Lemma

For a network \((G, U)\), \(|U| = n\), there exist integers \(x_1, \ldots, x_{pn}\) s.t.

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- \(s = (1, 1, 4, 4) \in S_4\)

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\[
2 = 1\chi(s, P_1) + 1\chi(s, P_2) + 0\chi(s, P_3) + 1\chi(s, P_4)
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Lemma

For a network \((G, U)\), \(|U| = n\), there exist integers \(x_1, \ldots, x_p\) s.t.

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\begin{itemize}
  \item \(p_4 = 4\), \(P_4 = \{P_1, P_2, P_3, P_4\}\)
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\end{itemize}

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2 = 1 \cdot 1 + 1 \chi(s, P_2) + 0 \chi(s, P_3) + 1 \chi(s, P_4)
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\[
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\]
Forbidden Networks

Graph $H \rightarrow$ network $(\tilde{H}, U)$, $|U| = n$. 

\begin{center}
\begin{tikzpicture}
\node at (-2,0) {$H$};
\node at (2,0) {$\tilde{H}$};
\end{tikzpicture}
\end{center}
Forbidden Networks

- Graph $H \rightarrow$ network $(\tilde{H}, U)$, $|U| = n$.

- $S_H = \left\{ s \in S_n : F_{\tilde{H}, U}(s) > 0 \right\}$. 
Forbidden Networks

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- $V_n, V_H$: linear hulls of $\left\{ \chi_n(s) : s \in S_n \right\}$, $\left\{ \chi_n(s) : s \in S_H \right\}$ (in $\mathbb{Q}^{p_n}$).
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**Theorem (Kochol)**

If $V_H = V_n$ then $H$ cannot be a subgraph of any minimal counterexample to the 5-Flow Conjecture.
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- $G$ minimal counterexample, $H \leq G \implies I = G - H$. 

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![Graph Diagram](image-url)
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- $\forall s \in S_n: F_{\tilde{H},U}(s) \cdot F_{\tilde{I},W}(s) = 0 \implies \forall s \in S_H: F_{\tilde{I},W}(s) = 0.$
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- $\forall s \in S_n: F_{\bar{H},U}(s) \cdot F_{\bar{I},W}(s) = 0 \implies \forall s \in S_H: F_{\bar{I},W}(s) = 0$.
- $V_H = V_n \implies \forall s \in S_n: F_{\bar{I},W}(s) = 0$.
- $G/H$ admits a NZ 5-flow $\implies$ contradiction.
Computations

- $M_n, M_H$: matrices with rows $\chi_n(s)$ for $s \in S_n$ and $s \in S_H$. 
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\[ V_n = V_H \iff \text{rank } M_n = \text{rank } M_H \]
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\[
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\]

<table>
<thead>
<tr>
<th>$H$</th>
<th>size of $M_n$</th>
<th>size of $M_H$</th>
<th>rank $M_n$</th>
<th>rank $M_H$</th>
<th>note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_7$</td>
<td>$819 \times 162$</td>
<td>$483 \times 162$</td>
<td>$147$</td>
<td>$147$</td>
<td>MK</td>
</tr>
<tr>
<td>$C_8$</td>
<td>$3277 \times 715$</td>
<td>$1513 \times 715$</td>
<td>$568$</td>
<td>$568$</td>
<td>MK</td>
</tr>
<tr>
<td>$C_9$</td>
<td>$13107 \times 3425$</td>
<td>$4665 \times 3425$</td>
<td>$2227$</td>
<td>$2227$</td>
<td>PK</td>
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Computations

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<tr>
<td>$C_8$</td>
<td>$3 277 \times 715$</td>
<td>$1 513 \times 715$</td>
<td>568</td>
<td>568</td>
<td>MK</td>
</tr>
<tr>
<td>$C_9$</td>
<td>$13 107 \times 3 425$</td>
<td>$4 665 \times 3 425$</td>
<td>2 227</td>
<td>2 227</td>
<td>PK</td>
</tr>
</tbody>
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Minimal Counterexamples to Flow Conjectures
Computations

- \( M_n, M_H \): matrices with rows \( \chi_n(s) \) for \( s \in S_n \) and \( s \in S_H \).

\[ V_n = V_H \iff \text{rank } M_n = \text{rank } M_H \]

<table>
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<tr>
<th>( H )</th>
<th>size of ( M_H )</th>
<th>size of ( M_{H'} )</th>
<th>rank ( M_H )</th>
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</thead>
<tbody>
<tr>
<td>( C_9 )</td>
<td>262 \times 238</td>
<td>430 \times 238</td>
<td>151</td>
<td>151</td>
<td>MK</td>
</tr>
<tr>
<td>( C_{10} )</td>
<td>792 \times 1 079</td>
<td>1 415 \times 1 079</td>
<td>539</td>
<td>539</td>
<td>MK</td>
</tr>
<tr>
<td>( C_{11} )</td>
<td>1 972 \times 4 752</td>
<td>3 937 \times 4 752</td>
<td>1 699</td>
<td>1 699</td>
<td>PK</td>
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Modification

- Even circuit $H \rightarrow$ perfect matching $H'$.

$H = C_{10}$

$H' = 5 \times e$
Modification

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$H = C_{10}$

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**Theorem**

Let $G$ be a 3-edge-connected cubic graph containing $H = C_{2k}$ ($k > 1$) as a subgraph. Then there exists some non-crossing perfect matching $H'$ s.t. $G_{H \rightarrow H'}$ is bridgeless.
Modification

- Even circuit $H \rightarrow$ perfect matching $H'$.

\[
H = C_{10} \quad H' = 5 \times e
\]

Theorem

Let $G$ be a 3-edge-connected cubic graph containing $H = C_{2k}$ ($k > 1$) as a subgraph. Then there exists some non-crossing perfect matching $H'$ s.t. $G_{H \rightarrow H'}$ is bridgeless.

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<tr>
<th>$H$</th>
<th>size of $M_H$</th>
<th>size of $M_{H_e}$</th>
<th>size of $M_{H_m'}$</th>
<th>rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_8$</td>
<td>$122 \times 81$</td>
<td>$176 \times 81$</td>
<td>$149 \times 81$</td>
<td>62</td>
</tr>
<tr>
<td>$C_{10}$</td>
<td>$792 \times 1,079$</td>
<td>$1,415 \times 1,079$</td>
<td>$1,129 \times 1,079$</td>
<td>539</td>
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Closing Remarks

Implementation issues:
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- written in Sage – Kochol used Maple
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Future work:
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Future work:

- optimization of the program, use of different language or structures
- use of different graphs in the role of $H$ and $H'$
- study of other reductions of the size of matrices
- use of the method on other open problems
Thank you!

I will gladly answer your questions and ideas.