General

Collapse all

Announcements

Course outline

Ideas and techniques from algebra (including linear algebra) have been used with striking success (and sometimes in surprising ways) in combinatorics, discrete geometry, and theoretical computer science, leading to breakthroughs and solution's to various long-standing open problems.The aim of this course is to explain some of these results and methods; the precise choice of topics will depend on the audience's background; possible examples of topics include:- Dimension arguments and set systems with restricted intersections: counterexamples to Borsuk's conjecture, explicit constructions of Ramsey graphs, and the chromatic number of the plane- Polynomial methods and application: the finite field Kakeya problem, the joints problem, the cap-set problem, and Erdős distance problems - the Combinatorial Nullstellensatz and applications in additive number theory and graph coloring- Eigenvalues of graphs, random walks, quasi-random graphs, and constructions of expander graphs- Shannon capacity, the Lovász θ-funcDon, and semidefinite programming relaxaDons- Stanley-Reisner rings and face numbers of polytopes and triangulated spheres

Target group: Students with a solid background in mathematics and/or theoretical computer science

Prerequisites: We will assume that the students are familiar with linear algebra; beyond that, we will keep the specific prerequisites minimal and aim to adapt to the background of the audience, introducing more advanced algebraic notions and results as we go along; the main prerequisite is mathematical maturity.

Evaluation: graded homework + class participation + exam

Teaching format: lecture + recitation

ECTS: 3 **Year:** 2020

Track segment(s):
CS-ALG Computer Science - Algorithms and Complexity
MAT-ALG Mathematics - Algebra
MAT-DISC Mathematics - Discrete Mathematics
MAT-GEO Mathematics - Geometry and Topology

Teacher(s):
Vojtech Kaluza Uli Wagner

Teaching assistant(s):

Pascal Wild

Schedule

Date	Time	Location	Type
2021-03-02	14:45-16:00	online	Course
2021-03-02	16:15-17:00	online	Recitation
2021-03-04	14:45-16:00	online	Course
2021-03-09	14:45-16:00	online	Course
2021-03-09	16:15-17:00	online	Recitation
2021-03-11	14:45-16:00	online	Course
2021-03-16	14:45-16:00	online	Course
2021-03-16	16:15-17:00	online	Recitation
2021-03-18	14:45-16:00	online	Course
2021-03-23	14:45-16:00	online	Course
2021-03-23	16:15-17:00	online	Recitation

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2021-03-25	14:45-16:00	online	Course
Easter break	Easter break	Easter break	Easter break
2021-04-13	14:45-16:00	online	Course
2021-04-13	16:15-17:00	online	Recitation
2021-04-15	14:45-16:00	online	Course
2021-04-20	14:45-16:00	online	Course
2021-04-20	16:15-17:00	online	Recitation
2021-04-22	14:45-16:00	online	Course

The course will be held online via Zoom. We will use the following Zoom link for the course (for all classes and recitations):

https://istaustria.zoom.us/j/98904949837?pwd=c3BiQkpKYTBMN3Z1aFZHZVVmWDJKZz09
Meeting ID: 989 0494 9837
Passcode: 586847

Exercise sheets

Exercise sheets will be posted here:

- Exercise Sheet 1 (due to March 9, 2021, 16:00)
- Exercise Sheet 2 (due to March 16, 2021, 16:00)
- Exercise Sheet 3 (due to March 23, 2021, 16:00)
- Exercise Sheet 4 (due to April 13, 2021, 16:00)
- Exercise Sheet 5 (due to April 20, 2021, 16:00)

Buzzword summary of lectures

We will provide buzzword summaries of the lectures with some pointers to the references here.

Lecture on March 2, 2021:

- Warm-up examples for set systems with restricted intersections
- Statement and proof of *Oddtown theorem*
- Statement and proof of *Non-uniform Fisher inequality*

Some relevant sections in the references listed below:

- Miniatures 3 and 4 in Matoušek's Thirty-three Miniatures
- Section 1.1 and 4.1 in Babai and Frankl's Linear Algebra Methods in Combinatorics

Lecture on March 4, 2021:

- Statement and proof of a version of *few intersections modulo a prime*
- Some reminder of some properties of polynomials
- Geometric application: Discussion around Borsuk's conjecture including the statement of a Theorem of Kahn-Kalai disproving this conjecture (lower bound on number of sets needed to partition a bounded subset of Euclidean space into sets of strictly smaller diameter)

Some relevant sections in the references listed below:

- Miniatures 17 and 18 in Matoušek's Thirty-three Miniatures
- Sections 5.4, 5.5 and 5.6 in Babai and Frankl's Linear Algebra Methods in Combinatorics

Lecture on March 9, 2021:

- Proof of Theorem of Kahn-Kalai disproving Borsuk's conjecture
- discussion around the Kakeya set problem, definition of Kakeya set, some historical context, statement of Dvir's theorem on the size of Kakeya sets over finite fields

Some relevant sections in the references listed below:

- Miniature 18 and 24 in Matoušek's Thirty-three Miniatures

Lecture on March 11, 2021:

- Proof of Dvir's theorem on the size of Kakeya sets over finite fields
- discussion around the joints problem
- statement of $O(L^{2/3})$ upper bound on the number of joints of L lines in $\mathbb{R}^3$ due to Guth and Katz with some preparation for the proof

Some relevant sections in the references listed below:

- Miniature 24 and 25 in Matoušek's Thirty-three Miniatures
- Section 2.5 in Guth's book

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Lecture on March 16, 2021:

- Proof of Guth and Katz's theorem on the number of joints of L lines in $\mathbb{R}^3$
- Introduction to Combinatorial Nullstellensatz
- Proof and statement of one version of the Combinatorial Nullstellensatz
- Proof of Cauchy--Davenport Theorem as application of Combinatorial Nullstellensatz

Some relevant sections in the references listed below:

- Section 2.5 in Guth's book
- Section 1 and 2, Theorems 3.2 in Noga Alon's paper

Lecture on March 18, 2021:

- Several applications of Combinatorial Nullstellensatz: a result due to Alon-Füredi showing that a loopless multigraph with average vertex degree > 2p-2 and maximal degree at most 2p-1 contain a p-regular subgraph, a proof of the Erdős-Ginzburg-Ziv using the permanent lemma
- Second form of the Combinatorial Nullstellensatz with proof

Some relevant sections in the references listed below:

- Theorem 1.1, Theorem 6.1 and Proposition 8.2 in Noga Alon's paper

Lecture on March 23, 2021:

- Introduction to *cap set problem*
- statement and proof of a bound from 2016 on the maximal size of a cap set in $\mathbb{F}_3^n$ of the form c^n for some explicit constant $c < 3$ due to Ellenberg-Gijswijt building on work of Croot-Lev-Pach

Some relevant sections in the references listed below:

- Ellenberg and Gijswijt's paper

Lecture on March 25, 2021:

- statement and proof of an upper bound bound from 2016 on the *size of 3-sunflower set system* in $[n]$ of the form c^n for some explicit $c < 2$ due to Naslund and Sawin

Some relevant sections in the references listed below:

- Naslund and Sawin's paper

Lecture on April 13, 2021:

- Introduction to Shannon capacity $\Theta(G)$ of a graph G including an argument that $\Theta(C_5) = \sqrt{5}$
- Definition of Shannon capacity and Lovász's theta function
- Statement of various equivalent expressions for Lovász's theta function
- first part of proof of their equivalence

Some relevant sections in the references listed below:

- Section 11.1., 11.2 and 11.5 in Lovász book "Graphs an Geometry"

Lecture on April 15, 2021:

- Proof of equivalence of various expressions for Lovász's theta function
- Discussion of some consequences of these equivalences including that $\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$ and $\vartheta(G)\vartheta(\bar{G}) \geq |V(G)|$
- automorphism invariant orthonormal representations

Some relevant sections in the references listed below:

- Section 11.1., 11.2 and 11.5 in Lovász book "Graphs an Geometry"

Lecture on April 20, 2021:

- $\vartheta(G)\vartheta(\bar{G}) = |V(G)|$ for graphs with vertex-transitive automorphism group
- various results relating $\vartheta(G)$ to eigenvalues of matrices including a generalized Hoffman's bound due to Lovász

Some relevant sections in the references listed below:

- Section 11.2 in Lovász book "Graphs an Geometry"

Lecture on April 22, 2021:

- glimpse into theory of semidefinite programming

References and reading material

Here are some references for the course. More will be posted along the way.

- Unpublished notes "Linear Algebra Methods in Combinatorics" by László Babai and Péter Frankl. A pdf is here.
- Jiří Matoušek's "Thirty-three miniatures". A preliminary version can be downloaded under this link.
- Larry Guth "Polynomial Methods in Combinatorics".
- Noga Alon's paper "Combinatorial Nullstellensatz" which can be found under this link.
- Jordan Ellenberg and Jion Gijswijt paper "On large subsets of F_q^n with no three-term arithmetic progression" which can be found under ? .ink.
- Eric Naslund and William Sawin's paper "Upper bounds for sunflower-free sets" whose arxiv version can be found under this link.

- László Lovász's book "Graphs and Geometry". A pdf-version of it can be download from Lovász's website under this link.