Second homework assignment

The numbers in boxes indicate the maximum number of points available for a given exercise.

[2] 1. Let G be a k-connected graph. Let $x, y_1, \dots, y_k$ be $k+1$ distinct vertices of G. Show that G contains k internally disjoint paths $P_1, \dots, P_k$, where P_i connects x to y_i.

[2+3] 2. Show that for every k there is a bipartite graph with choosability greater than k (2 points). Find such a bipartite graph with at most 4^k vertices (3 additional points).

[4+2] 3. Show that in each k-connected graph, each set of k vertices belongs to a common cycle (4 points). For every k, find a k-connected graph with a set of $k+1$ vertices that do not belong to a common cycle (2 points).

[3] 4. Show that every k-tree has a tree-decomposition of width at most k.

[2] 5. Show that if G is a minor of H, then $\mathrm{tw}(G) \le \mathrm{tw}(H)$, where $\mathrm{tw}(\cdot)$ denotes the treewidth.

[3] 6. Show that for every graph G with n vertices, $\mathrm{ch}(G) + \mathrm{ch}(\overline{G}) \le n+1$. Here $\overline{G}$ denotes the complement of G and $\mathrm{ch}(\cdot)$ denotes the choosability. Hint: use induction over n.

[3] 7. Let G_n be the graph obtained from the complete graph on $2n$ vertices by removing a set of n disjoint edges (e.g., G_1 is just two isolated vertices, G_2 is the 4-cycle). Show that G_n has choosability n.

[1] 8. Find a directed graph with no kernel.

[3] 9. Show that every planar bipartite graph is 3-choosable.

10. A family $\mathcal{F}$ of sets is called *pairwise intersecting* if each two sets in $\mathcal{F}$ intersect. Let $[n]$ denote the set $\{1, 2, \dots, n\}$.

[3] (a) Let $\mathcal{F}(n,k)$ denote the family of all the k-element subsets of the set $[n]$. Find a pairwise intersecting family $\mathcal{F} \subset \mathcal{F}(2n, n)$, with $|\mathcal{F}| = \binom{2n-1}{n}$, and with the property that the common intersection of all the sets in $\mathcal{F}$ is empty.

[4+3] (b) Let $\mathcal{F}$ be a family of finitely many k-element sets. Show that if each $k+1$ sets from $\mathcal{F}$ have nonempty intersection, then all the sets in $\mathcal{F}$ have nonempty intersection (4 points). For every k, find a family $\mathcal{F}'$ of k-element sets, such that each k sets from $\mathcal{F}'$ have nonempty intersection, but the intersection of all the sets in $\mathcal{F}'$ is empty (3 points).

[3] (c) Let $\mathcal{F}$ be a pairwise intersecting family whose members are subsets of $[n]$. Show that if $\mathcal{F}$ contains fewer than 2^{n-1} sets, then there is a set $X \subseteq [n]$ not belonging to $\mathcal{F}$, such that $\mathcal{F} \cup \{X\}$ is pairwise intersecting.

[4] 11. Show that a plane triangulation has a proper vertex coloring with three colors if and only if each of its vertices has an even degree.

[2] 12. Let $T_k(n)$ denote the k-partite Turán graph on n vertices, let $t_k(n)$ be the number of edges of $T_k(n)$. Show that

$$\lim_{n\to\infty} \frac{t_k(n)}{\binom{n}{2}} = \frac{k-1}{k}.$$

13. The aim of the following sequence of questions is to show that the regularity lemma is (almost) trivial for sparse graphs.

[3] (a) Show that for every $\varepsilon > 0$ there is a $d > 0$ such that every bipartite graph of density at most d is ε-regular.

[2] (b) Show that for every $\varepsilon > 0$ there is a $d > 0$ such that a graph G with n vertices and less than dn^2 edges has at most εn vertices of degree greater than εn.

5 (c) Let $\mathcal{G}$ be an infinite family of graphs, with the property that each graph $G \in \mathcal{G}$ with n vertices has at most $\gamma(n)n^2$ edges, for some function γ satisfying $\lim_{n\to\infty} \gamma(n) = 0$. Prove that the graphs from $\mathcal{G}$ satisfy the regularity lemma; in other words, prove that for every $\varepsilon > 0$ and m there is an M and an n_0 such that every graph $G \in \mathcal{G}$ with at least n_0 vertices has an ε-regular partition with at least m and at most M parts.

2 14. Let $G = (X \cup Y, E)$ be an ε-regular bipartite graph with parts X and Y. Let $\overline{G}$ be the *bipartite complement* of G, i.e., $\overline{G}$ is a bipartite graph with the same parts as G, and for every $x \in X$ and $y \in Y$, the pair $\{x, y\}$ is an edge of $\overline{G}$ if and only if it is not an edge of G. Show that $\overline{G}$ is ε-regular.

5 15. (a) Let H be a bipartite graph with v vertices. Show that for every $\varepsilon > 0$ there is a constant $K > 0$ and $n_0 \in \mathbb{N}$ such that every graph with $n \geq n_0$ vertices and at least εn^2 edges has at least Kn^v subgraphs isomorphic to H.

3 (b) Show that the statement above would be false if we omitted the assumption that H is bipartite.