anyt
[entry]anyt/global/

Chapter 15

Circle packing for planar graphs

By Sariel Har-Peled, April 8, 2018[①] Version: 0.1

> Out of the trunk, the branches grow; out of them, the twigs. So, in productive subjects, grow the chapters.
>
> Moby Dick, Herman Melville

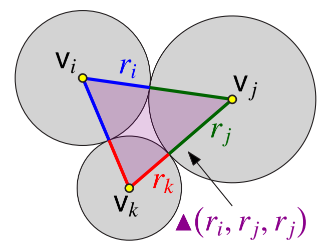
This is an early draft of a new chapter. Read at your own peril.

Here, prove one of the most surprising results about planar graphs: A planar graph can be realized as a set of disks, where each disk represents a vertex, and two disks touch if and only if the two vertices are connected by an edge in the original graph. See Figure 15.1 for some examples.

15.1. Introduction, and the game of whac-an-angle

15.1.1. The basic idea

Consider a triangulation (i.e., a maximal planar graph with no parallel edges or self loops) $G = (V, E)$ with n vertices, where $V = \{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$. Such a triangulation has $m = 3n - 6$ edges and $2n - 4$ faces, see Lemma 15.5.1. Furthermore, assume we are given a planar embedding of this triangulation. Let $F = \left\{f_1, \ldots, f_{2n-4}\right\}$ be the set of faces of this graph (they are all triangles), in the embedding of G under consideration. Imagine that we assigned radius r_i to v_i, for $i = 1, \ldots, n$. For a triangular face $\triangle v_i v_j v_k$ of this graph, if we place three disks with radii r_i, r_j, r_k, and force them to touch each other, they uniquely define their ***induced triangle*** (having edges of length $r_i + r_j, r_j + r_k, r_i + r_k$), and let $\blacktriangle(r_i, r_j, r_k)$ denote this triangle, see figure on the right. In particular, one can compute the angles of this triangle explicitly as a function of these radii[②].



Small and big angles. So, imagine such a vertex v, and let $\psi(\mathsf{v})$ be the ***total angle***[③] at v. That is, it is the sum of the angles at v in the induced triangles adjacent to v. Think about these triangles as being made of cardboard, and imagine gluing them together along their matching edges around v in the plane. If the total angle of v (i.e., $\psi(\mathsf{v})$) is smaller than 2π then we get a gap. If the total angle is larger than 2π, then things wrap around. See Figure 15.3.

Let us be lucky. If we are lucky and the angle around v is exactly 2π then all its induced triangles can be drawn in the plane around it. Furthermore, if the total angles of *all* the internal vertices each add exactly to 2π, then its not hard, although it requires a proof, to show that we found the desired embedding (indeed, start with such a triangle as above, and start tiling the plane gluing adjacent triangles together, and argue that this tilling is consistent).

[①]This work is licensed under the Creative Commons Attribution-Noncommercial 3.0 License. To view a copy of this license, visit http://creativecommons.org/licenses/by-nc/3.0/ or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA.

[②]What is really meant is that somebody can compute these angles, but the author is planning to keep his principle of avoiding such competent people at all costs. Nobody should compromise on competence when laziness suffices.

[③]Total angle is like total war, but for angles.

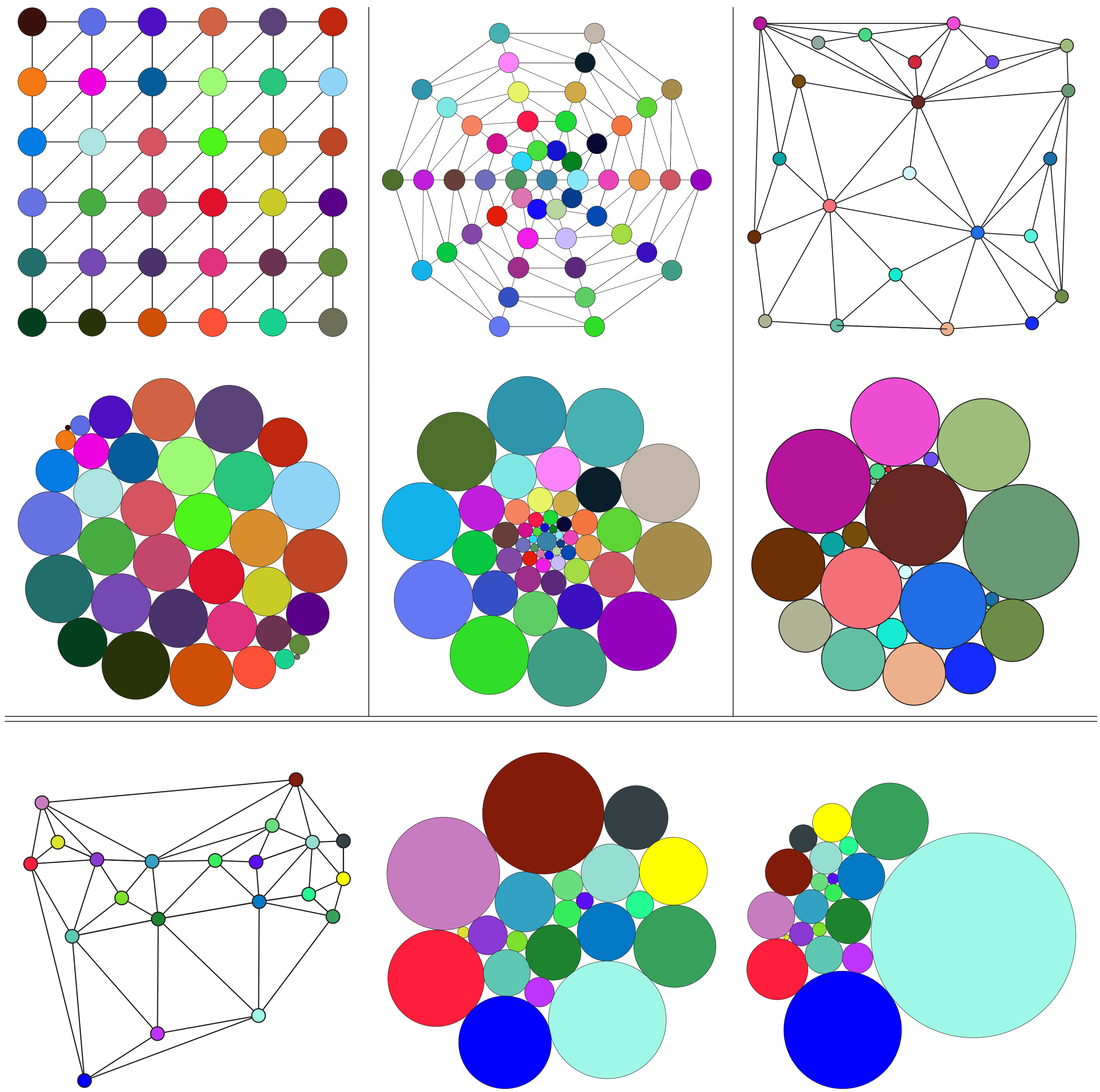


Figure 15.1: Examples of circle packing realization of planar graphs. As the bottom example shows, such realizations are not necessarily unique. Figures generated using an open source program https://github.com/seub/CirclePackings, written by Benjamin Beeker and Brice Loustau.

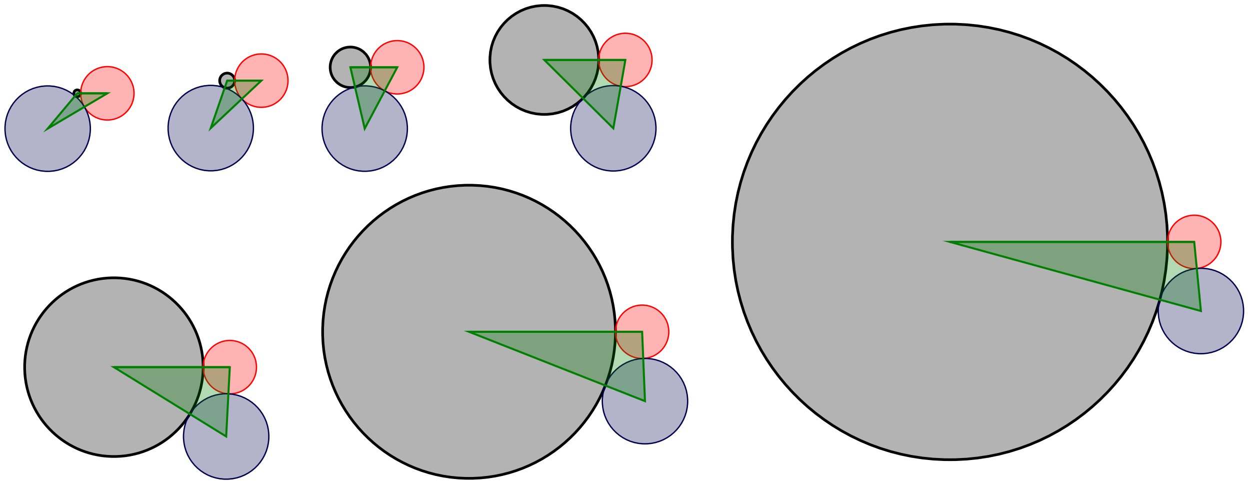


Figure 15.2: The effect of changing the radius of one disk in a triangle on the angle at the corresponding vertex.

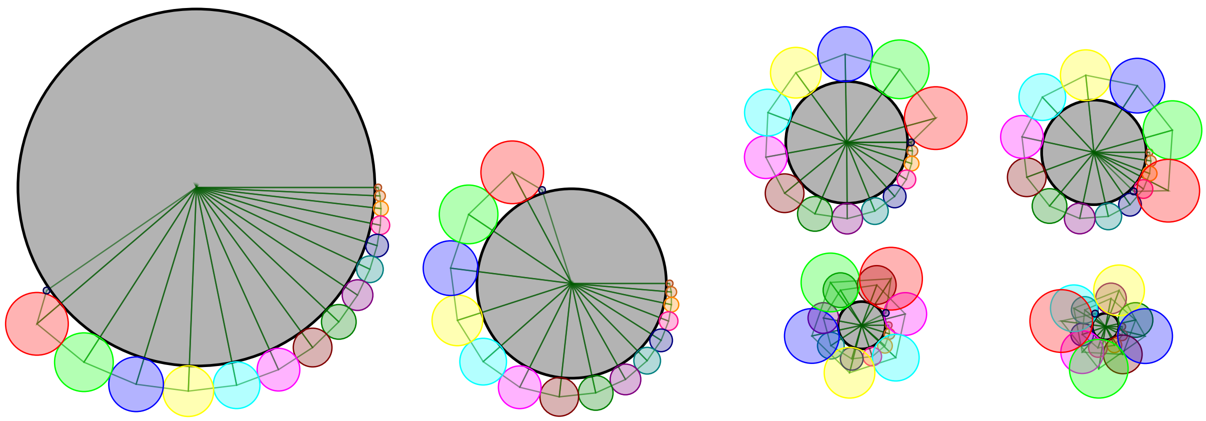


Figure 15.3: The change in the angle around a vertex as a function of its radius (while all other radii are fixed). There is always a choice of a radius such that the total angle around a vertex is exactly 2π. If the total angle is too small, then the realized triangulation has a gap around the vertex. Similarly, if the total angle is too large, the triangulation wraps around itself at the vertex.

Fixing the angles. Consider a vector $\mathsf{r} = (r_1, \ldots, r_n)$ of radii. For $i = 1, \ldots, n$, let ψ_i be the total angle at v_i As stated above, if $\psi_i < 2\pi$, then v_i has an angle deficit. In particular, shrinking r_i (while leaving the radii of all the other vertices fixed) increases ψ_i, see Figure 15.2. As such, we can increase it till $\psi_i = 2\pi$, see Figure 15.3. Similarly, if $\psi_i > 2\pi$ then enlarging r_i decreases ψ_i, and we can do this till $\psi_i = 2\pi$.

It is now natural to try and play a WHAC-AN-ANGLE game, where we repeatedly fix each angle by increasing or decreasing its associated radius as described above. The problem of course is that as we change the radius of a vertex v, the other angles in the induced triangle containing v might get ruined. To this end, we will use a more careful strategy to play with radii.

The other challenge is to argue that such a WHAC-AN-ANGLE game converges. While this works (see bibliographical notes), we will take a more existential approach – we prove that the mapping between vector of radii to angles, map some radii vector to the desired circle embedding, thus implying the result.

15.1.2. Mapping from radii vector to angles vector

Since the planar graph under consideration is a triangulation (i.e., all faces are triangles), it follows that it has $2n-4$ faces, see Lemma 15.5.1. Consider the sum of the induced angles, and observe that

$$\sum_{i=1}^{n} \psi_i = \sum_{j=1}^{2n-4} \pi = (2n-4)\pi,$$

as every induced triangle contributes π to this sum (note, that somewhat bizarrely, for the outer face we take the inner angles of this triangle as its angles – this technicality would cause us a minor headache shortly). For the sake of concreteness, let us consider only radii vectors $\mathsf{r} = (r_1, \ldots, r_n)$ that have the property that $\|\mathsf{r}\|_1 = \sum_i r_i = 1$ and furthermore, we want the outer triangle of this triangulation to be a regular triangle; that is, we require that the angles of the outer triangle are $60° = \pi/30$. The former constraint induces the *open* simplex

$$\Delta = \left\{ \mathsf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n \;\middle|\; r_1 > 0, r_2 > 0, \ldots, r_n > 0, \text{ and } \sum\nolimits_i r_i = 1 \right\}. \tag{15.1}$$

Given a vector r, it induces for each vertex v_i an angle ψ_i. In particular, let

$$\widehat{\sigma}(\mathsf{r}) = (\psi_1, \ldots, \psi_n) : \Delta \longrightarrow \mathbb{R}^n \tag{15.2}$$

denote this mapping. Our purpose is to show that there exists a vector $\mathsf{r} \in \Delta$, such that $\widehat{\sigma}(\mathsf{r})$ maps to a point where all its coordinates are 2π. This is not quite correct, as the three vertices of the outer face are going to have different angles around them (i.e., $\neq 2\pi$) in the final realization. In particular, assume that these three vertices are $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$, and we require that $\psi_1 = \psi_2 = \psi_3 = 2\pi/3$, see Remark 15.1.1 below. That is, our purpose is to prove that there exists a vector $\mathsf{r}^* \in \Delta$, such that

$$\widehat{\sigma}(\mathsf{r}^*) = \eta = \left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, 2\pi, \ldots, 2\pi \right). \tag{15.3}$$

The range of the mapping $\widehat{\sigma}$ is the open simplex

$$\Psi = \left\{ (\psi_1, \ldots, \psi_n) \in \mathbb{R}^n \;\middle|\; \psi_1 > 0, \psi_2 > 0, \ldots, \psi_n > 0, \text{ and } \sum\nolimits_i \psi_i = (2n-4)\pi \right\}. \tag{15.4}$$

Remark 15.1.1. The reader is probably confused by the angle $2\pi/3$ required of the outer vertices. What we want for the outer vertices is that their angle is $\pi/3$. However, there is also the outer face of G, which in the final realization is going to be a regular triangle which is (conceptually) the union of all the other realized triangles (i.e., it is the back face of the realization) and has angle $\pi/3$. As such, the total angle in these outer vertices is twice bigger.

Lemma 15.1.2. *The mapping $\widehat{\sigma} : \Delta \to \Psi$ is one-to-one.*

Proof: Consider a triangle $\mathsf{v}_i\mathsf{v}_j\mathsf{v}_k$ and its contribution to the angles ψ_i, ψ_j, ψ_k. If we increase r_i (and leave r_j and r_k the same or decrease them), then by the argument above ψ_i decreases, and $\psi_j + \psi_k$ increases. Similarly, if we increase r_i and r_j, but keep r_k the same or decrease it, then, again, $\psi_i + \psi_j$ would decrease but ψ_k would increase.

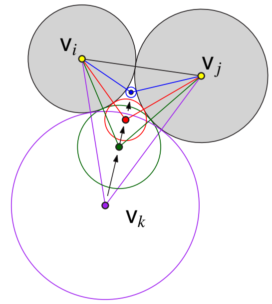


Figure 15.4: The incredibly shrinking triangle.

In particular, consider two different vectors $\mathsf{r}, \mathsf{r}' \in \Delta$. Let I be the set of vertices v_i, such that $\mathsf{r}'_i > \mathsf{r}_i$. Clearly, a triangle with all vertices in I contributes the same quantity to the sum

$$S(\mathsf{r}) = \sum_{i \in I} \psi_i(\mathsf{r}) = \sum_{\triangle \in \mathrm{faces}(G)} \sum_{\substack{\mathsf{v}_i \in V(\triangle) \\ i \in I}} \measuredangle \mathsf{v}_i(\triangle, \mathsf{r})$$

and corresponding sum $S(\mathsf{r}')$, where $\measuredangle v_i(\triangle, \mathsf{r})$ denotes the angle of v_i in the triangle $\triangle$ (when realizing $\triangle$ according to the radii of r).

So, consider a mixed triangle $\triangle$ that has exactly one vertex v in I. Then, the radius at v increases (as we moved from r to r') and the other two radii are no bigger. But then, the angle of v in this triangle has decreased, and the contribution of this triangle to $S(\mathsf{r}')$ had decreases compared to $S(\mathsf{r})$.

Similarly, if a mixed triangle $\triangle$ contains exactly two vertices $\mathsf{v}_i, \mathsf{v}_j$ with indices in I, then the angle at the third vertex of $\triangle$ must have increased, implying that the contribution of angles of $\triangle$ to $S(\mathsf{r}')$ is smaller than their contribution to $S(\mathsf{r})$.

The given graph is connected, $\|\mathsf{r}'\|_1 = \|\mathsf{r}\|_1 = 1$, and thus there must be such a mixed triangle, which implies that $S(\mathsf{r}') < S(\mathsf{r})$. Since $S(\mathsf{r})$ is a sum of some fixed coordinates of $\widehat{\sigma}(\cdot)$, we conclude that $\widehat{\sigma}(\mathsf{r}) \neq \widehat{\sigma}(\mathsf{r}')$, as claimed. ■

Observation 15.1.3. *The proof of the above lemma implies the following. Consider any two vectors* $\mathsf{r} = (r_1, \ldots, r_n), \mathsf{s} = (s_1, \ldots, s_n) \in \Delta$, *and let* $J = \left\{ i \;\middle|\; s_i < r_i \right\}$ *be the set of indices of the vertices that their radius shrinks as we move from* r *to* s. *Then, we have*

$$\sum_{i \in J} \psi_i(\mathsf{r}) < \sum_{i \in J} \psi_i(\mathsf{s}). \tag{15.5}$$

Definition 15.1.4. For an embedded planar graph G, and a set of vertices $X \subseteq V(G)$, let $\mathcal{F}(X)$ denote the set of ***incident faces***; that is, the set of all faces of G that are adjacent to vertices of X.

Lemma 15.1.5. *Let* $\mathsf{s} = (s_1, \ldots, s_n) \in \partial\Delta$ *(see Eq. (15.1)), and let* $I = \{i \mid s_i = 0\}$. *Then, we have*

$$\lim_{\substack{\mathsf{r} \to \mathsf{s} \\ \mathsf{r} \in \Delta}} \sum_{i \in I} \psi_i(\mathsf{r}) = |\mathcal{F}(I)|\, \pi, \tag{15.6}$$

where $\mathcal{F}(I)$ *is the set of faces of* G *that are incident to the vertices of* I.

Proof: Consider a triangle $\triangle = \triangle \mathsf{v}_i \mathsf{v}_j \mathsf{v}_k$. If all the vertices of $\triangle$ have indices in I, then the triangle always contribute π to the summation in the limit. As such, we need to concern ourselves only with mixed triangles.

So imagine a mixed triangle with one angle in I. That is r_j and r_i remain the same, but r_k tends to zero. Clearly, the angle of v_k in $\blacktriangle(r_i, r_j, r_k)$ tends to π, see Figure 15.4. Similarly, if both r_j and r_k tends to zero (but r_i remains the same), then the total angles adjacent to these two vertices in $\blacktriangle(r_i, r_j, r_k)$ tends to π. That is, vertices with radii tending to zero "suck" all the total angle in the triangle out of the other vertices in the triangle.

Thus, the total angle of these incredibly shrinking vertices, is just the total number of triangles they participate it (multiplied by π, naturally), as the sum of angles in these triangle move to the vertices in I, thus implying the claim. ■

Consider a configuration $\mathsf{r} = (r_1, \ldots, r_n) \in \Delta$, and an arbitrary set of indices $I \subseteq \{1, \ldots, n\}$, such that $\alpha = \sum_{i \notin I} r_i > 0$. If we decrease all the radii of the vertices in I to zero, and scale the remaining coordinate, then we reach the configuration $\mathsf{s} = (s_1, \ldots, s_n) \in \partial\Delta$, where

$$s_i = \begin{cases} 0 & i \in I, \\ r_i/(1-\alpha) & \text{otherwise.} \end{cases}$$

(As such, $\|\mathsf{s}\|_1 = \|\mathsf{r}\|_1 = 1$.) As a concrete way to do this, consider the "moving" point $p(t) = (1-t)\mathsf{r} + t\mathsf{s}$, as t moves from 0 to 1. Clearly, during this motion, all the radii in I decreases, and all the other ones increases. Combining Eq. (15.5) and Eq. (15.6), we have that

$$\sum_{i \in I} \psi_i(\mathsf{r}) < \sum_{i \in I} \psi_i(\mathsf{s}) \leq |\mathcal{F}(I)|\,\pi.$$

Since this holds for any I, and any angle vectors that might be generated, this naturally gives rise to the following open polytope:

$$\mathcal{P} = \bigcap_{I \subset \{1,\ldots n\}} \left\{ (\psi_1, \ldots, \psi_n) \in \Psi \;\middle|\; \sum_{i \in I} \psi_i < |\mathcal{F}(I)|\,\pi \right\},$$

see Eq. (15.4).

Let us summarize:

(A) $\widehat{\sigma} : \Delta \to \mathcal{P}$ is one-to-one (Lemma 15.1.2).
(B) $\widehat{\sigma}$ is continuous (by construction).
(C) $\widehat{\sigma}$ maps (in the limit) the boundary of Δ into the boundary of $\mathcal{P}$ (by Lemma 15.1.5).
(D) $\widehat{\sigma}$ is onto – this is implies by the above properties and Lemma 15.2.2 below. The statement of the lemma is about mapping between open balls, but it applies to the open sets at hand, since they are convex and open.

We are almost done – we need to just verify that our favorite point is indeed in $\mathcal{P}$.

Lemma 15.1.6. *The point* $\eta = \left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, 2\pi, \ldots, 2\pi\right) \in \mathcal{P}$.

Proof: Well, $\eta \in \Psi$, as the sum of its angles is $2\pi(n-3) + 2\pi = (2n-4)\pi$, as desired. So, consider a proper subset of I of $\{1, \ldots, n\}$. If $\mathcal{F}(I)$ includes all the vertices of G, then the associated inequality of $\mathcal{P}$ holds trivially, as $|\mathcal{F}(I)| = 2n-4$ in this case.

For I to miss some face of G it must be that $|I| \leq n-3$. It is not hard to show that in this case $|\mathcal{F}(I)| > 2\,|I|$, see Lemma 15.2.1 below. We conclude that $\sum_{i \in I} \eta_i \leq 2\pi\,|I| < |\mathcal{F}(I)|\,\pi$, thus implying that $\eta \in \mathcal{P}$. ■

15.1.2.1. The result

Theorem 15.1.7 (Circle packing theorem). *Let $H = (V, E)$ be a finite simple planar graph. Then H can be realized by a set of interior-disjoint disks, where every disk corresponds to a vertex, and two disks touch, if and only if the corresponding vertices has an edge between them in the original graph.*

Proof: We embed H in the plane in linear time Theorem 15.5.2. We add edges to H till it becomes maximal planar graph, see Lemma 15.5.3. We introduce a new vertex in the middle of every added edge, and again add edges till this graph becomes a maximal planar graph. Let G be the resulting graph, and let n be the number of its vertices (note, that we have at hand a planar embedding of G).

We now apply the approach described above to G. We assign radii to its vertices, and define a mapping $\widehat{\sigma}$, see Eq. (15.2), from the radii configuration to the associated angle configuration. By Lemma 15.1.6, the desired realization $\eta = \left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, 2\pi, \ldots, 2\pi\right) \in \mathcal{P}$. The mapping $\widehat{\sigma} : \Delta \rightarrow \mathcal{P}$ is one to one (Lemma 15.1.2) and onto (by Lemma 15.2.2). As such, $\widehat{\sigma}$ has an inverse function, and the point $\widehat{\sigma}^{-1}(\eta)$ is the desired configuration of radii that generates a valid embedding of G as a circle packing. Finally, we delete all the circles corresponding to vertices added above. Clearly, what remains is a circle packing of the original graph H. ■

15.2. Some helper lemmas

15.2.1. Number of triangles induced

Lemma 15.2.1. *Let G be an embedded triangulation with n vertices, and let $X \subseteq V(G)$ be a set of vertices such that $|X| \leq n - 3$. Then the set of incident faces $\mathcal{F}(X)$ is of size $> 2|X|$ (see Definition 15.1.4).*

Proof: If there is an edge $e = uv \in E(G)$ with both endpoints $u, v \in X$, then contract this edge – this deletes two triangles in $\mathcal{F}(X)$, and decreases the size of X by one (as X replaces the two vertices u, v by a new merged vertex). In the end of this process, the set X has resulted in a set of vertices X' that is independent in the remaining triangulation (which has at least 4 vertices). Each vertex in this triangulation is adjacent to at least three triangles, and no triangle is adjacent to more than one vertex of X'. We conclude that $|\mathcal{F}(X)| = 2\big(|X| - |X'|\big) + 3|X'| = 2|X| + |X'|$, which implies the claim as $|X'| \geq 1$. ■

15.2.2. A helper lemma about mappings

For a set $X \subseteq \mathbb{R}^d$, its ***closure***, denoted by $\mathrm{cl}(X)$, is the set X together with all its limit points (as such, $\mathrm{cl}(X)$ is a closed set).

Lemma 15.2.2. *Let f be a one-to-one and continuous mapping from b to itself, where b is the open unit ball in $\mathbb{R}^n$. Furthermore, assume that for any $p \in \partial\mathsf{b}$, we have that $\lim_{q \to p, q \in \mathsf{b}} f(q) \in \partial\mathsf{b}$ (i.e., f "maps" the boundary of b to the boundary of b), then f is a surjective mapping (i.e., it is onto) b.*

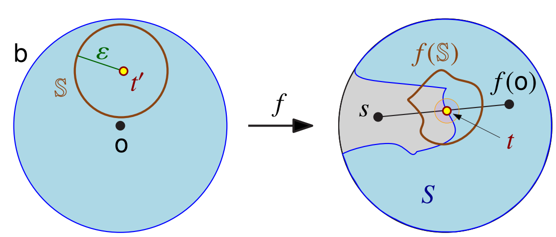


Figure 15.5

Proof: Let $S = \mathrm{cl}\big(f(\mathsf{b})\big)$, and assume for the sake of contradiction that there exists a point $s \in \mathsf{b} \setminus S$. Let t be the first intersection of the segment $sf(\mathsf{o})$ with S as one moves from s towards $f(\mathsf{o})$, where o is the origin. Observe that t is a boundary point of S. Now, the points $f(\mathsf{o})$ and s are in b, and as such $t \in f(\mathsf{o})s \subseteq \mathsf{b}$. Furthermore, since $t \in S$, there exits $t' = f^{-1}(t) \in \mathrm{cl}(\mathsf{b})$ such that $f(t') = t$. If $t' \in \partial\mathsf{b}$, then by assumption $f(t') \in \partial\mathsf{b}$, which is a contradiction[4]. As such, $t' \in \mathsf{b}$.

We now prove that t can not be a boundary point of S, thus getting a contradiction. Indeed, consider an $\varepsilon > 0$ sufficiently small such that the sphere $\mathbb{S}$ of radius ε centered at t' is contained in b. Since $\mathbb{S}$ is a closed set, it must be that $t \notin \mathrm{cl}(f(\mathbb{S}))$, otherwise an easy limit argument shows that f is not one-to-one[5]. In particular, let δ be the minimum distance between t and $\mathrm{cl}\bigl(f(\mathbb{S})\bigr)$, and observe that it must be that the ball of radius $\delta/2$ centered at t is fully contained in the image of $f(\mathsf{b})$, but this contradicts t being on the boundary of S. ∎

15.3. Bibliographical notes

Our presentation follows Pach and Agarwal [**pa-cg-95**]. A nice survey of the circle packing theorem is provided by the wikipedia page on the circle packing theorem. An easy consequence of the circle packing theorem is the planar separator theorem – see next chapter for details.

Some history. The circle packing theorem, also known as the Koebe-Andreev-Thurston theorem, has a curious history considering that William Thurston was born ten years after Koebe proved the theorem. In 1985 Thurston, who was brilliant mathematician and a Fields medalist, gave a talk outlining a proof of the circle packing theorem and pointing out that Andreev's work [**a-ocpls-70**] implies it. A few years later it came to light that Koebe already proved it in 1936 [**k-kdka-36**].

Thurston was interested in the circle packing theorem as a way to construct (i.e., approximate) a conformal mapping f between two simply connected open sets X, Y in the plane. Such a conformal mapping preserve angles – that is, two curve that intersect in a certain angle in X, get mapped by f into two curves that intersect in Y with the same angle. This is known as the *Riemann mapping theorem* which was proved in 1912 by Carathéodory, However, there are other techniques that works for achieving such a conformal mapping. The Riemann mapping theorem is quite surprising as it implies that one can deform an image into any reasonable shape while preserving angles.

State of the art in practice. According to Kenneth Stephenson (personal communication, May 2017), the current state of the art as far as implementation is GOPack [**osc-lcpa-17**]. For smaller configurations the algorithm of Collins and Stephenson [**cs-cpa-03**] seems to work well. For truly small configurations, one can adjust the radii directly (the WHAC-AN-ANGLE game), but one still has to do the embedding itself, which might not be easy. In particular, the GOPack package mentioned above, maintains not only the radii of the circles, but also their centers, and alternates between updating the location of the centers and radii. While in practice this works quite well, there is currently no proof of convergence for this method. We refer the reader to the aforementioned papers and references there in for more details about algorithms and results about circle packing. The book by Stephenson [**s-icptd-05**] is a good introduction to this material.

[4] Formally, we are dealing here with an extension of f that is defined on the boundary of b using the limit. We omit the tedious and easy details to keep things clean/simple.

[5] Indeed, if $t \in \mathrm{cl}(f(\mathbb{S}))$, then there exists a sequence of points $p_1, p_2, \ldots \in f(\mathbb{S})$ that converges to t. Then, the sequence $f^{-1}(p_1), f^{-1}(p_2), \ldots \in \mathbb{S}$. The set $\mathbb{S}$ is bounded and closed (i.e., its compact), and as such, this sequence have a subsequence $f^{-1}(p_{i_1}), f^{-1}(p_{i_2}), \ldots \in \mathbb{S}$ that converges to a limit point $p \in \mathbb{S}$. We then have that $f(p) = \lim_j f\bigl(f^{-1}(p_{i_j})\bigr) = t$. Namely, $f(t') = f(p)$ and $t' \neq p$. Thus contradicting f being one-to-one.

An open problem. It would be nice to have a combinatorial algorithm that yields similar embedding to the circle packing theorem and has a finite (guaranteed) running time (which is polynomial, and hopefully near linear). Here, we are willing to relax some conditions – the shapes can be any convex shapes that are fat, and we require that regions intersect (i.e., we do not require tangency) if they share an edge, and furthermore, we might even allow some additional edges that do not exist in the original graph. In particular, a natural condition would be that no point in the plane is covered by more than some constant number of regions of the realizations. Such realizations are *low density graphs*, see [**hq-aapel-15**, **hq-aaldg-15-arxiv**]

15.4. Exercises

15.5. From previous lectures

Lemma 15.5.1. *A simple planar graph G with n vertices has at most $3n-6$ edges and at most $2n-4$ faces. A triangulation has exactly $3n-6$ edges and $2n-4$ faces.*

Theorem 15.5.2. *Given a graph G with n vertices, an algorithm can check if it is a planar graph in $O(n)$ time, and if so it outputs a planar embedding of G.*

Lemma 15.5.3. *Given a (simple) planar graph $G = (V, E)$, one can add edges to it so that it becomes a triangulation (i.e., all its faces are triangles).*