Second recitation session, March 21

Problem 1. Let $\mathcal{A}$ and $\mathcal{B}$ be labelled combinatorial classes, with weights w_A and w_B, respectively. Let $\mathcal{A} \otimes \mathcal{B}$ be the labelled product of $\mathcal{A}$ and $\mathcal{B}$, with weigth function w defined by $w((\alpha, \beta)) = w_A(\alpha)w_B(\beta)$ for every $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$. Show that the weighted exponential generating functions of these classes satisfy the identity

$$EVF(\mathcal{A} \otimes \mathcal{B}, w) = EVF(\mathcal{A}, w_A)EVF(\mathcal{B}, w_B).$$

In the rest of the recitation, we considered the following scenario:

Suppose that we toss a fair coin, and we record whether the outcome was heads ('H') or tails ('T'), where both outcomes have the same probability $\frac{1}{2}$. Tossing the coin repeatedly independently and recording the sequence of outcomes, we obtain a random potentially infinite sequence $C_1, C_2, C_3, \dots$, where each C_i is a random element of the set $\{H, T\}$.

Define *the waiting time of* HHH, denoted W_{HHH}, to be the number of coin tosses we have to make before we first obtain three consecutive outcomes of 'H', or formally, $W_{HHH} = \min\{n;\ C_{n-2} = C_{n-1} = C_n = H\}$.

Our aim is to answer the following questions:

- What is the expected value of W_{HHH}?
- Does this expected value change when HHH is replaced by a different word of length 3 over the alphabet $\{H, T\}$?

This leads to the first problem (not really related to the topic of the course).

Problem 2. Show that the following two expressions for the expectation $E[X]$ of a non-negative integer-valued random variable X are equivalent:

$$E[X] = \sum_{n \geq 0} n\mathbb{P}[X = n]$$
$$E[X] = \sum_{n \geq 0} \mathbb{P}[X > n],$$

where $\mathbb{P}[\cdot]$ denotes the probability of the given event.

We say that a (finite) word over the alphabet $\{H, T\}$ is *HHH-free* if it has no three consecutive letters equal to H. We say that a word is *HHH-terminating* if it ends with three consecutive letters H, and does not contain any other occurrence of three consecutive H. Let f_n an t_n denote, respectively, the number of HHH-free and HHH-terminating words of length n.

Observe that $\mathbb{P}[W_{HHH} > n] = \frac{f_n}{2^n}$ and $\mathbb{P}[W_{HHH} = n] = \frac{t_n}{2^n}$.

Let $F(x) = \sum_{n \geq 0} f_n x^n = 1 + 2x + 4x^2 + 7x^3 + \cdots$ be the OGF of HHH-free words, and let $T(x) = \sum_{n \geq 0} t_n x^n = x^3 + x^4 + \cdots$ be the OGF of HHH-terminating words. We will obtain a system of two equations of the two unknown power series $F(x)$ and $T(x)$.

Problem 3. Prove that the following holds:

$$2xF(x) = F(x) - 1 + T(x).$$

Hint: what kind of words can we obtain by appending a letter 'H' or 'T' to a HHH-free word?

Problem 4. Prove that the following holds:

$$x^3F(x) = T(x) + xT(x) + x^2T(x).$$

Hint: what kind of words can we obtain by appending 'HHH' to a HHH-free word?

Problem 5. Use the equations from the previous two problems to obtain formulas for $F(x)$ and $T(x)$. How to obtain the expression for $E[W_{HHH}]$ from the formulas of $F(x)$ or $T(x)$? (You may here assume without proof that the series involved in the definition of $E[W_{HHH}]$ actually converge.) Would the results obtained so far change if HHH is replaced with THH or HTH or any other word?

Problem 6 (This was not solved at the recitation but rather left as a homework)**.** Suppose two players A and B are playing the following game: the players repeatedly toss a coin until the appearance of either three consecutive tosses with outcomes HTH or three consecutive tosses with outcomes THH. If HTH appears, player A wins, if THH appears, player B wins. What is the probability that A wins?