Recitation session, March 7

Let $\mathcal{O}$ be a ring with no zero divisors. We consider power series with coefficients in the ring $\mathcal{O}$. At the first recitation session, we looked at the following problems:

Problem 1. Let $B(x) = \sum_{n\geq 0} b_n x^n$ be a composable power series (which, we recall, means that $b_0 = 0$). Our goal is to find power series $A(x) = \sum_{n\geq 0} a_n x^n$ such that $A(B(x)) = x$. Prove the following facts:

- If b_1 has no multiplicative inverse in $\mathcal{O}$, then no such $A(x)$ exists.
- If b_1 has a multiplicative inverse, then $A(x)$ exists and is unique. Moreover, $A(x)$ then has the following properties:
  a) $A(x)$ is composable,
  b) $a_1 = \frac{1}{b_1}$,
  c) $B(A(x)) = x$,
  d) a_n only depends on $b_1, \ldots, b_n$.

Definition 1. For a formal power series $A(x) = \sum_{n\geq 0} a_n x^n$ we define its *formal derivative*, denoted by $\frac{\mathrm{d}}{\mathrm{d}x}A(x)$, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}A(x) = \sum_{n\geq 1} n \cdot a_n x^{n-1},$$

where the expression $n \cdot a_n$ refers to the sum $a_n + a_n + \cdots + a_n$ with n summands (therefore $n \cdot a_n$ is well defined in any ring $\mathcal{O}$).

Problem 2. Show that the following holds for any power series $A(x)$ and $B(x)$:

- $\frac{\mathrm{d}}{\mathrm{d}x}(A(x) + B(x)) = \frac{\mathrm{d}}{\mathrm{d}x}A(x) + \frac{\mathrm{d}}{\mathrm{d}x}B(x)$,
- $\frac{\mathrm{d}}{\mathrm{d}x}(A(x)B(x)) = (\frac{\mathrm{d}}{\mathrm{d}x}A(x))B(x) + A(x)(\frac{\mathrm{d}}{\mathrm{d}x}B(x))$,
- $\frac{\mathrm{d}}{\mathrm{d}x}A(B(x)) = \big((\frac{\mathrm{d}}{\mathrm{d}x}A(x)) \circ B(x)\big)\,\frac{\mathrm{d}}{\mathrm{d}x}B(x)$, provided $A(B(x))$ is well defined.