Short Proofs are Hard to Find

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University of Toronto

Joint work w/ Toniann Pitassi, Hao Wei

ICALP, July 10, 2019

Proof propositional complexity

$*54{\cdot}43$. $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

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From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

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How long is the shortest $\mathcal{P}$-proof of τ?

Can we find short $\mathcal{P}$-proofs of τ?

Resolution

One of the simplest and most important proof systems

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- SAT solvers ([Davis-Putnam-Logemann-Loveland], [Pipatsrisawat-Darwiche])
- automated theorem proving
- model checking
- planning/inference

Resolution

Axioms:

$\bar{a} \vee d$

$b \vee d$

$\bar{b}$

$a \vee b \vee c$

$a \vee \bar{c}$

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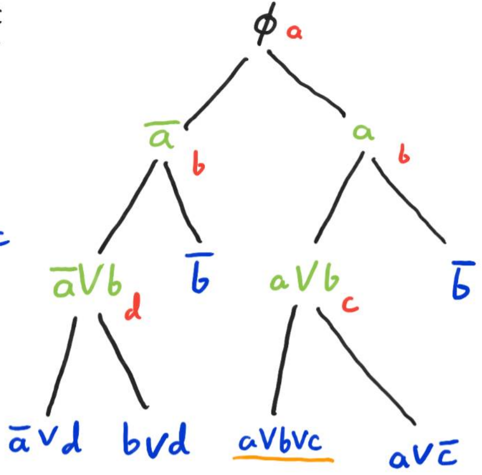
width = 3



Automatizability

Automatizability [Bonet-Pitassi-Raz]

A proof system $\mathcal{P}$ is f-automatizable if there exists an algorithm A : UNSAT $\rightarrow \mathcal{P}$ that takes as input τ and returns a $\mathcal{P}$-refutation of τ in time $f(n, S_{\mathcal{P}}(\tau))$, where $S_{\mathcal{P}}(\tau)$ is the size of the shortest $\mathcal{P}$-refutation of τ.

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Automatizability is connnected to many problems in computer science...

- theorem proving and SAT solvers
- algorithms for PAC learning ([Kothari-Livni], [Alekhnovich-Braverman-Feldman-Klivans-Pitassi])
- algorithms for unsupervised learning ([Bhattiprolu-Guruswami-Lee])
- approximation algorithms (many works...)

Known automatizability lower bounds

General results and results for strong systems

- approximating $S_{\mathcal{P}}(\tau)$ to within $2^{\log^{1-o(1)} n}$ is NP-hard for all "reasonable" $\mathcal{P}$ ([Alekhnovich-Buss-Moran-Pitassi])

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- approximating $S_{\mathcal{P}}(\tau)$ to within $2^{\log^{1-o(1)} n}$ is NP-hard for all "reasonable" $\mathcal{P}$ ([Alekhnovich-Buss-Moran-Pitassi])
- lower bounds against different Frege systems under cryptographic assumptions ([Bonet-Domingo-Gavaldà-Maciel-Pitassi],[BPR],[Krajíček-Pudlák])

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Results for weak systems

- first lower bounds against automatizability for Res, TreeRes by [Alekhnovich-Razborov]

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Rest of this talk: a new version of [AR] + [GL]

- simplified construction and proofs
- stronger lower bounds via ETH assumption
- results also hold for Res(r)

Our results

Theorem (Main Theorem)

Assuming ETH, $\mathcal{P}$ *is not* $n^{\tilde{o}(\log\log S_{\mathcal{P}}(\tau))}$*-automatizable for* $\mathcal{P} =$ Res, TreeRes, Nullsatz, PC.

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Theorem (Main Theorem for Res(r))

Assuming ETH, Res(r) *is not* $n^{\tilde{o}(\log\log S_{\mathcal{P}}(\tau)/exp(r^2))}$*-automatizable for* $r \leq \tilde{O}(\log\log\log n)$.

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Theorem (Atserias-Muller'19)

Assuming $\mathsf{P} \neq \mathsf{NP}$, Res *is not automatizable.*
Assuming ETH, Res *is not automatizable in subexponential time.*

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Theorem (Atserias-Muller'19)

Assuming P $\neq$ NP, Res *is not automatizable.*
Assuming ETH, Res *is not automatizable in subexponential time.*

Theorem (Bonet-Pitassi; Ben-Sasson-Wigderson)

TreeRes *is* $n^{O(\log S_{\mathcal{P}}(\tau))}$*-automatizable.*
Res *is* $n^{O(\sqrt{n \log S_{\mathcal{P}}(\tau)})}$*-automatizable.*

Getting an automatizability lower bound

Recipe:

(1) Hard gap problem G

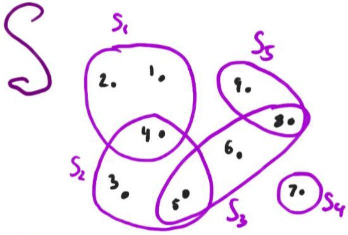
(2) Turn an instance of G into a tautology τ such that

- "yes" instances have small proofs
- "no" instances have no small proofs

(3) Run automatizing algorithm Aut on τ and see how long the output is

Gap hitting set

- $\mathcal{S} = \{S_1 \ldots S_n\}$ over $[n]$



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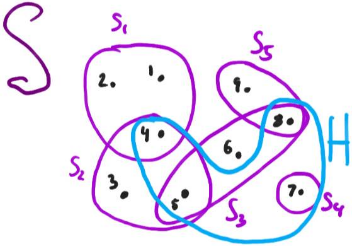

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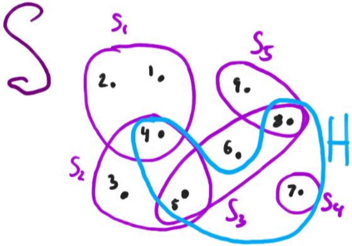
- **Gap hitting set**: given $\mathcal{S}$, distinguish whether $\gamma(\mathcal{S}) \leq k$ or $\gamma(\mathcal{S}) > k^2$



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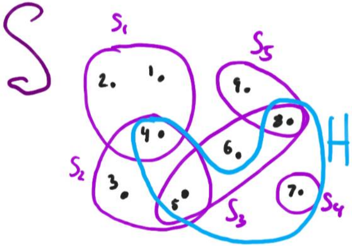

Theorem (Chen-Lin)

Assuming ETH *the gap hitting set problem cannot be solved in time* $n^{o(k)}$ *for* $k = \tilde{O}(\log \log n)$

From gap hitting set to automatizability

Theorem (Main Technical Lemma)

For $k = \tilde{O}(\log \log n)$, there exists a polytime algorithm mapping $\mathcal{S}$ to $\tau_{\mathcal{S}}$ s.t.

- *if $\gamma(\mathcal{S}) \leq k$ then $S_{\mathcal{P}}(\tau_{\mathcal{S}}) \leq n^{O(1)}$*
- *if $\gamma(\mathcal{S}) > k^2$ then $S_{\mathcal{P}}(\tau_{\mathcal{S}}) \geq n^{\Omega(k)}$*

where $\mathcal{P} \in \{\mathrm{TreeRes}, \mathrm{Res}, \mathrm{Nullsatz}, \mathrm{PC}\}$.

Proof sketch of main theorem

Theorem (Main Theorem)

Assuming ETH, *$\mathcal{P}$ is not $n^{\tilde{o}(\log\log S_{\mathcal{P}}(\tau))}$-automatizable.*

Proof: Let *Aut* be the automatizing algorithm for $\mathcal{P}$ running in time $f(n, S) = n^{\tilde{o}(\log\log S)}$, and let $k = \tilde{\Theta}(\log\log n)$.

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Detour: universal sets

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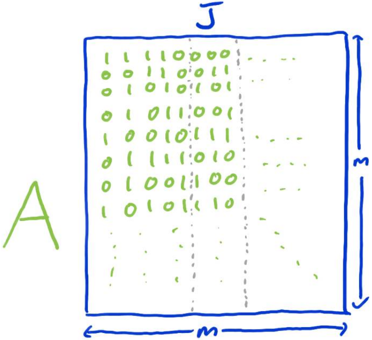

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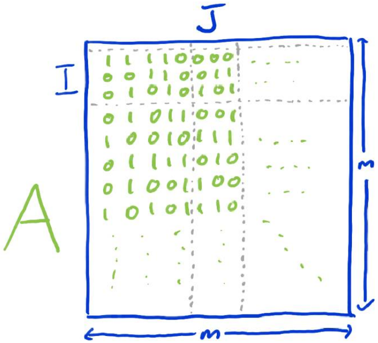
- constructions like the *Paley graph* work for $q = \frac{\log m}{4}$



Defining $\tau_{\mathcal{S}}$

Variables of $\tau_{\mathcal{S}}$ will implicitly define two matrices using A and $\mathcal{S}$

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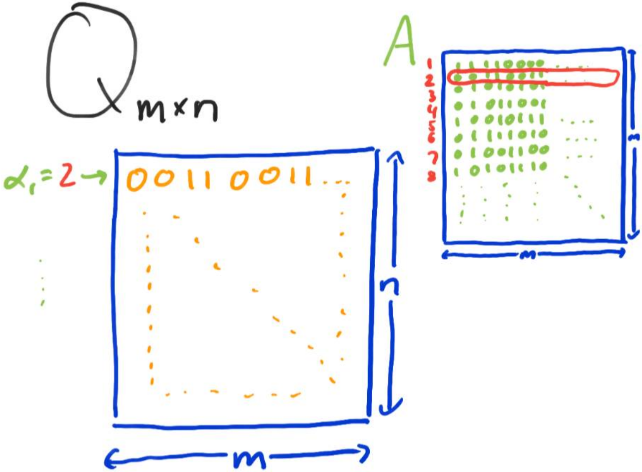

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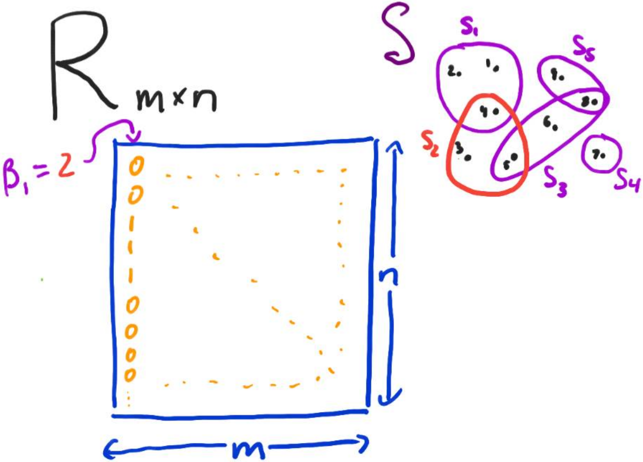

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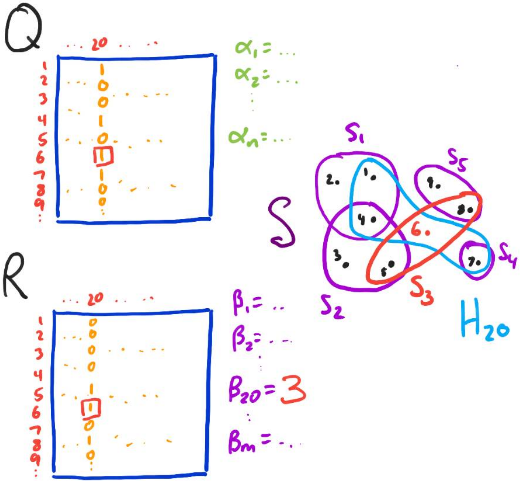

Upper bound on $S_{\mathcal{P}}(\tau_{\mathcal{S}})$

Lemma (Upper bound on $S_{\mathcal{P}}(\tau_{\mathcal{S}})$)

If $\gamma(\mathcal{S}) \leq k \leq \frac{\log m}{4}$, then $\tau_{\mathcal{S}}$ is unsatisfiable and $S(\tau_{\mathcal{S}}) \leq m^k n$ for TreeRes.

High-level idea: the universal property of A guarantees some column of Q will be a hitting set.

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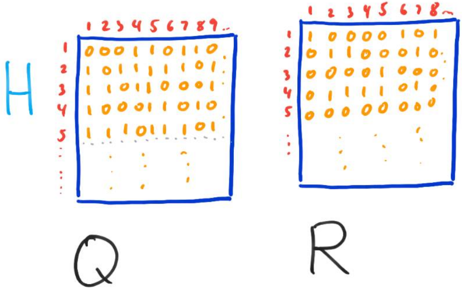

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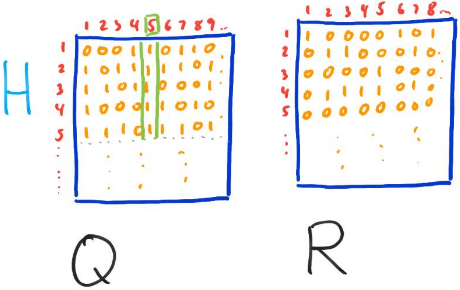

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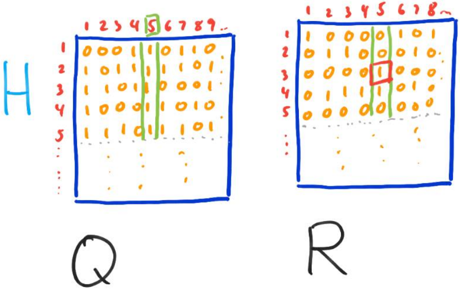

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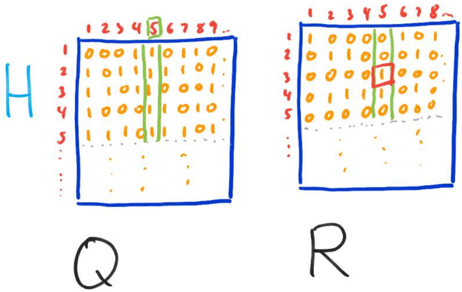

Size of the proof: $m^k n$

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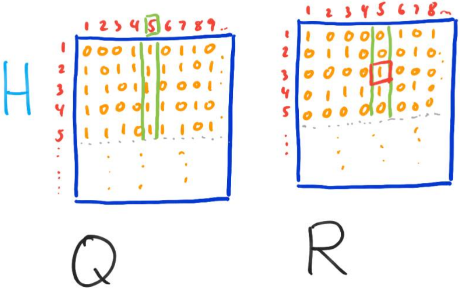

Size of the proof: $m^k n = n^2$ for $m = n^{1/k}$

Lower bound on $S_{\mathcal{P}}(\tau_{\mathcal{S}})$

High-level idea 1: any proof π must query all rows in some hitting set

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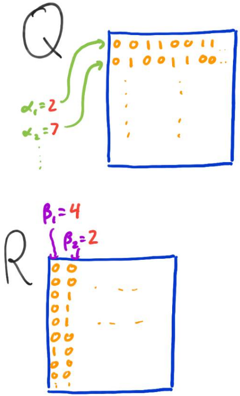
- Res(k) - switching lemma [Buss-Impagliazzo-Segerlend]
- TreeCP - lifting [upcoming work]

Lower bound on $S_{\mathcal{P}}(\tau_{\mathcal{S}})$

High-level idea 2: π knows nothing about a row or column without setting lots of variables



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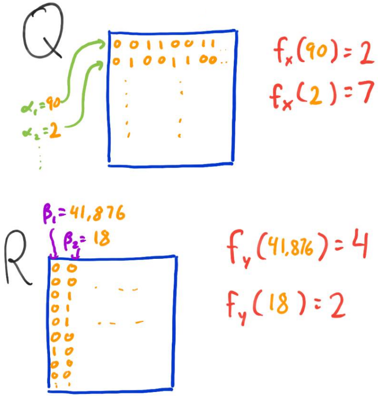
Error-correcting codes

- $x_i \in \{0,1\}^{6 \log m}$, $y_j \in \{0,1\}^{6 \log n}$
- $f_x : \{0,1\}^{6 \log m} \to [m]$, $f_y : \{0,1\}^{6 \log n} \to [n]$



Open problems

Better hard k in gap hitting set $\rightarrow$ better non-automatizability result

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Theorem (Main Technical Lemma)

For $k = O(\sqrt{\log n})$*, there exists a polytime algorithm mapping* $\mathcal{S}$ *to* $\tau_{\mathcal{S}}$ *...*

Thank you!

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