Automating Cutting Planes Is NP-Hard

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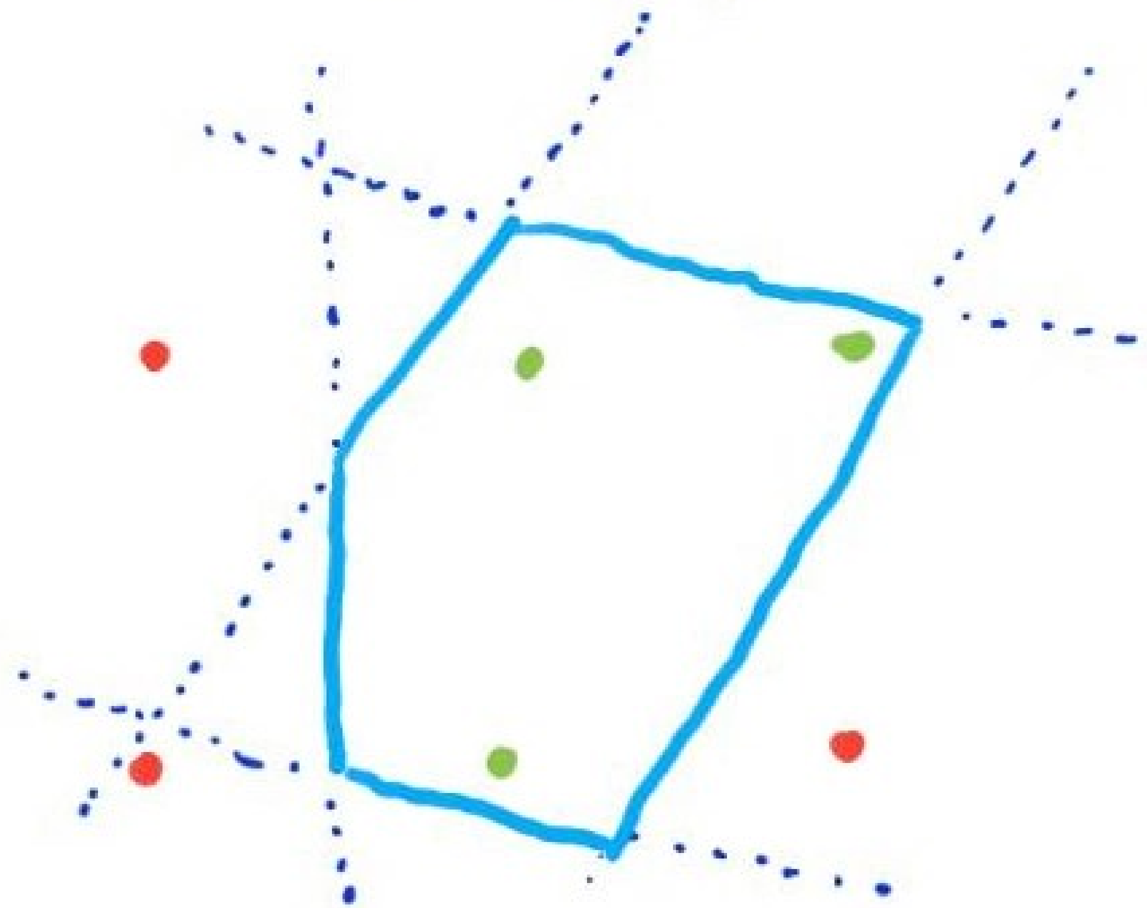
CUTTING PLANES

$$A = \{4x_1 - 2x_2 \geq 1,\ -x_1 + 5x_3 \geq 0, \ldots\}$$

[Gomory '50s]
[Chvátal '79]
[Schrijver '80]

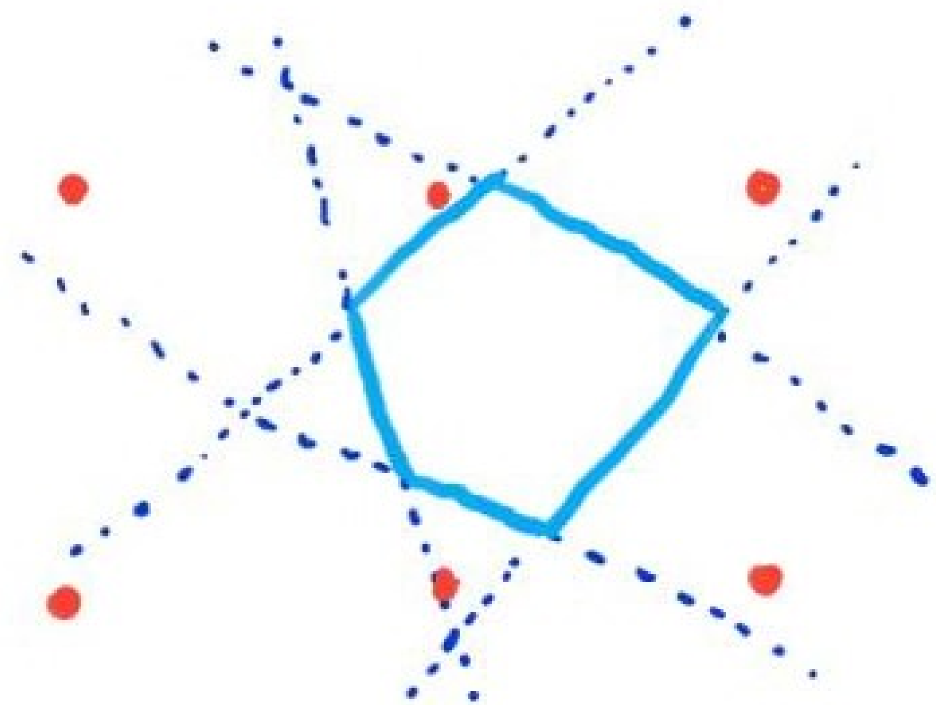
...



CUTTING PLANES

$$A = \{4x_1 - 2x_2 \geq 1, -x_1 + 5x_3 \geq 0, \ldots\}$$

[Cook et al. '87]



Cutting Planes

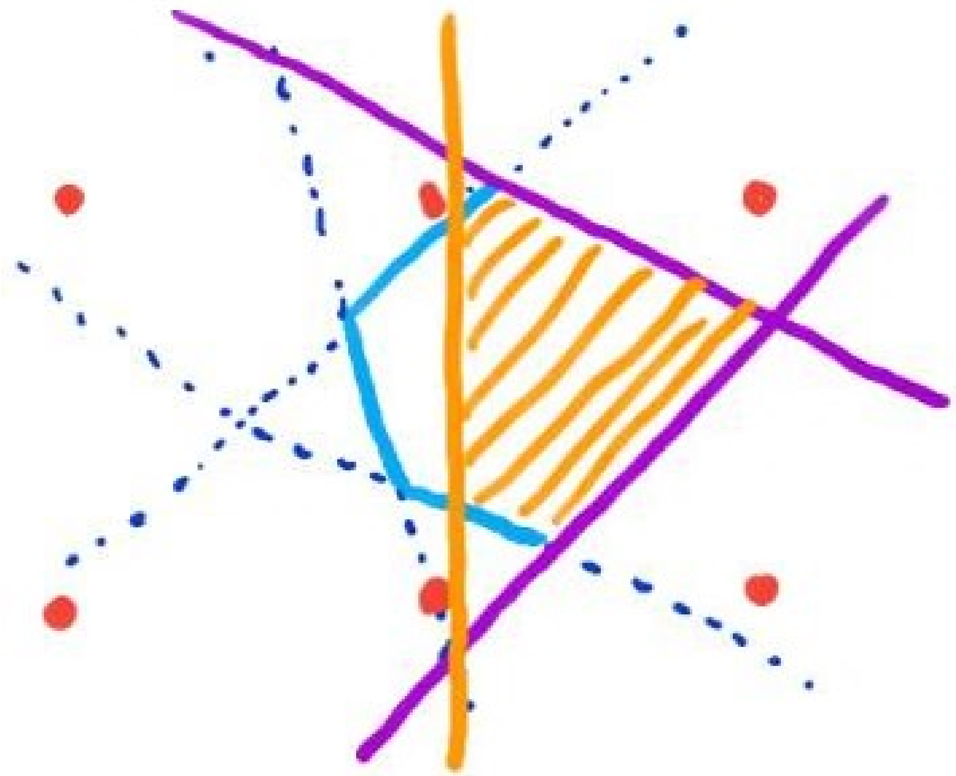
$$A = \left\{ \sum_{i=1}^{n} a_{ij} x_i \geq \alpha_j \right\}_j$$

$$L_i \vdash \{A, L_1 \ldots L_{i-1}\}$$

$$\pi = (L_1, L_2, \ldots, L_s := \{0 \geq 1\})$$

Cutting Planes

$$L_1 \vdash \{A_3, L_2\}$$



MAIN RESULTS

How efficiently can we find CP proofs?

$$\text{poly}(n, |A|) \longrightarrow \text{poly}(s_{CP})$$

s_{CP} = # CP inequalities needed

Main Results

Main Theorem: it is NP-hard to find a CP refutation of A in time poly(s_{CP}).

AUTOMATABILITY

proof system P: given UNSAT τ,
derive $\perp$ via sound deductions

can we find a P-refutation
of τ in time poly(s_P)?

AUTOMATABILITY

Resolution

$$\tau = \{(x_1 \vee \bar{x}_2 \vee x_3), (x_2), \ldots\}$$

goal: ϕ

$$A \vee B$$

$$A \vee x \qquad B \vee \bar{x}$$

AUTOMATABILITY

Resolution is width automatable

Thm: can find a Res ref π in time $n^{w(\tau)}$.

$$w_{Res}(\tau) = \min_{\pi} \max_{L \in \pi} \underbrace{w(L)}_{w=4}$$

$$L: (x_1 \vee \bar{x}_3 \vee x_5 \vee x_2)$$

AUTOMATABILITY

Resolution is width automatable

NS, PC, SA, SoS degree automatable

CP not degree automatable (deg 1)

AUTOMATABILITY

Theorem [Alekhnovich-Razborov'01]: it is ETH-hard to find a Res refutation of τ in time poly(s_{Res}).

first (and essentially only) result for Res automatability...

AUTOMATABILITY

Theorem [Atserias-Müller '19]: it is NP-hard to find a Res refutation of τ in time poly(s_{Res}).

AUTOMATABILITY

Key lemma: turning 3SAT instances into UNSAT instances.

$$\overset{\text{3SAT}}{F} \longrightarrow \overset{\text{unsat CNF}}{\tau_F}$$

$$\tau_F \xrightarrow{F \text{ SAT}} S(\tau) \le n^{O(1)}$$

$$\tau_F \xrightarrow{F \text{ unsat}} S(\tau) \ge 2^{n^{\Omega(1)}}$$

Automatability

Assume Aut takes poly(s) time;
run Aut(τ_F) for $n^{O(1)}$ steps

$$\underset{\text{3SAT}}{F} \longrightarrow \underset{\text{unsat CNF}}{\tau_F}$$

$$\tau_F \xrightarrow{F \text{ SAT}} \text{Aut}(\tau) \leq n^{O(1)}$$

$$\tau_F \xrightarrow{F \text{ unsat}} \text{Aut}(\tau) \gtrsim 2^{n^{\Omega(1)}}$$

AUTOMATABILITY

Main Theorem: it is NP-hard to find a CP refutation of A in time poly(s_{CP}).

3SAT

unsat system

$$\mathcal{F} \longrightarrow A_{\mathcal{F}}$$

$\xrightarrow{\mathcal{F}\ \text{SAT}}$ $S(A) \le n^{O(1)}$

$\xrightarrow{\mathcal{F}\ \text{unsat}}$ $S(A) \ge 2^{n^{\Omega(1)}}$

AUTOMATABILITY

Main Theorem: it is NP-hard to find a CP refutation of A in time poly(s_{CP}).

$$\overset{\text{3SAT}}{\mathcal{F}} \xrightarrow{[AM'19]} \overset{\text{unsat CNF}}{\tau_{\mathcal{F}}} \xrightarrow{\text{lifting}} \overset{\text{unsat lin ineq's}}{A_{\tau_{\mathcal{F}}}}$$

LIFTING

$$\tau(x_1 \ldots x_n) \longrightarrow A(x_1 \ldots x_n)$$

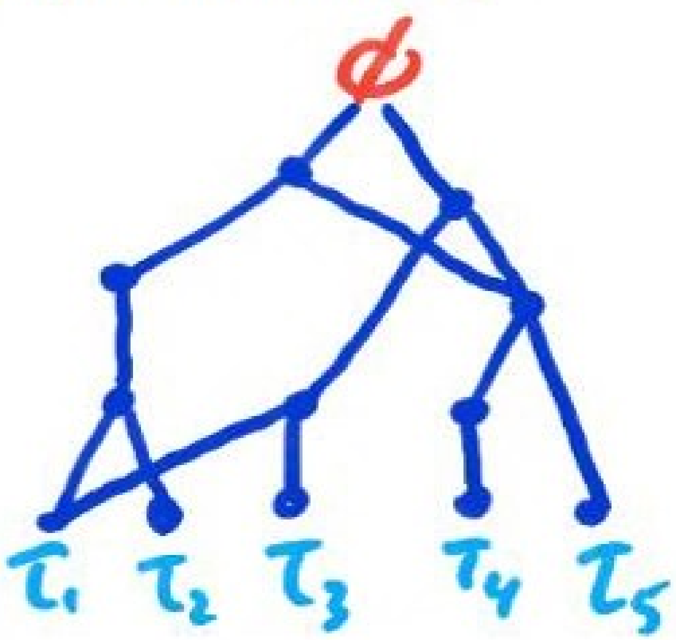
$$x_1 \vee \overline{x}_2 \vee x_3 \quad \longrightarrow \quad x_1 + (1 - x_2) + x_3 \geq 1$$

$$(0 \leq x_i \leq 1)$$

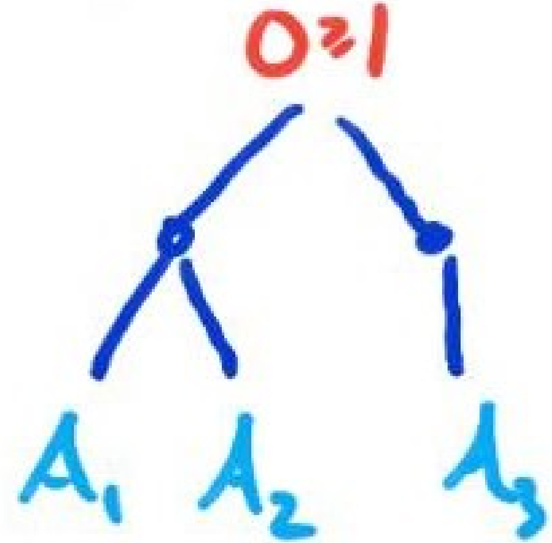
Lifting

$$\tau(x_1 \ldots x_n) \longrightarrow A(x_1 \ldots x_n)$$

Resolution



Cutting Planes



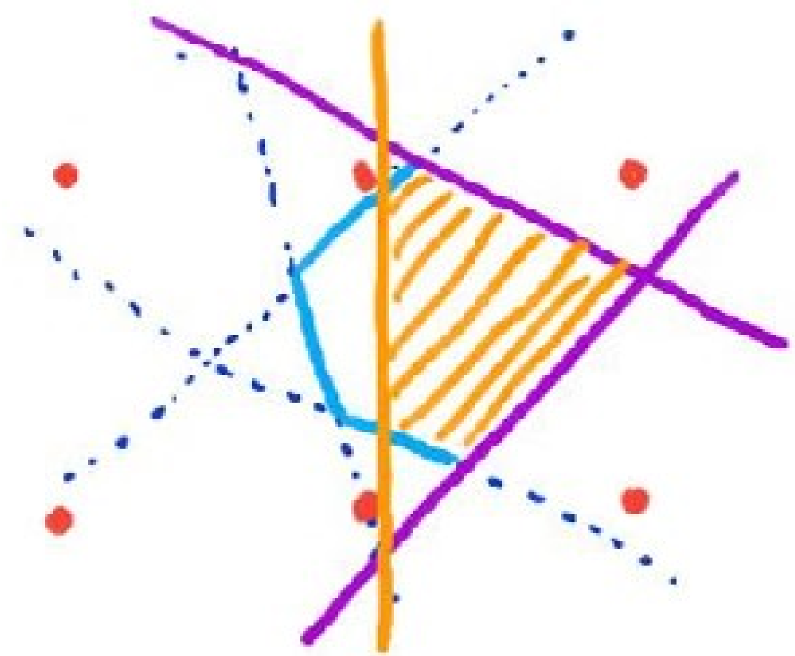
LIFTING

$$\tau(x_1 \cdots x_n) \longrightarrow A(x_1 \cdots x_n)$$

Resolution

$$\begin{tikzcd} & A \vee B & \\ A \vee x \arrow[ur] & & B \vee \bar{x} \arrow[ul] \end{tikzcd}$$

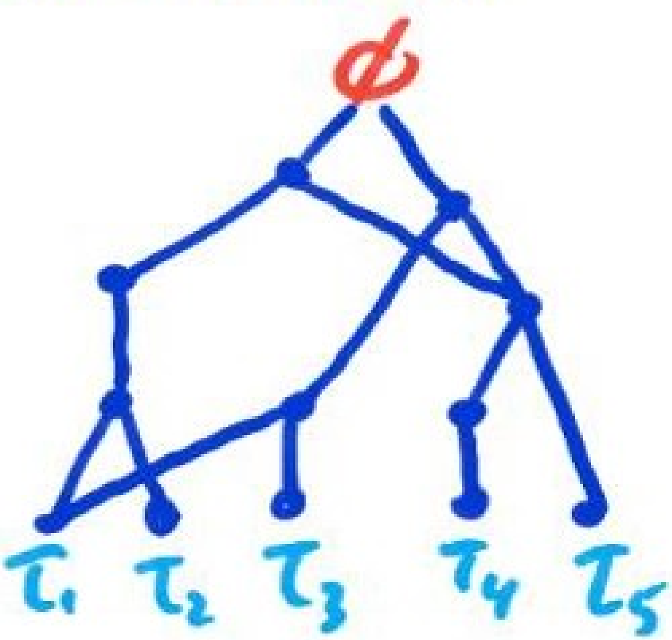
Cutting Planes



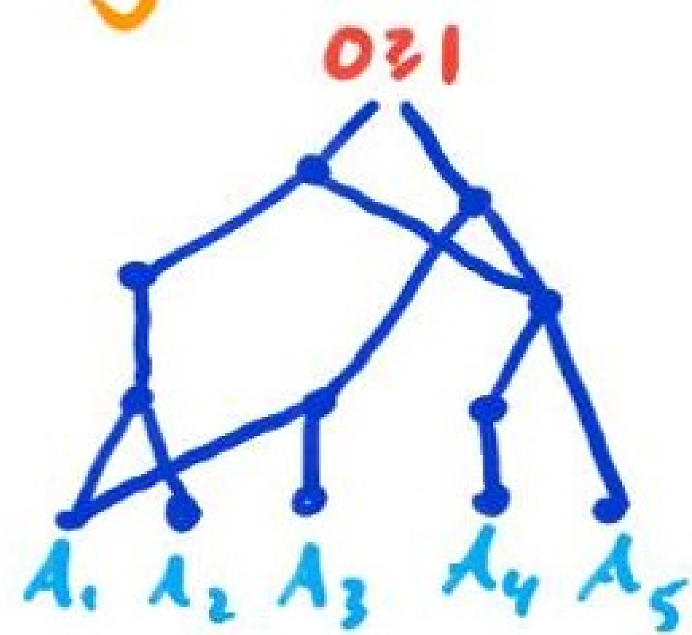
Lifting

$$\tau(x_1 \dots x_n) \longrightarrow A(g(\vec{y}_1) \dots g(\vec{y}_n))$$

Resolution



Cutting Planes



LIFTING

$$\text{Res ref} \longrightarrow \text{CP ref}$$

$$\tau(x) \longrightarrow \tau \circ g^n(y)$$

want:

$$S_{CP}(\tau \circ g) \approx S_{Res}(\tau) \left[\cdot \overbrace{\Theta(\log n)}^{\text{compute } g} \right]$$

LIFTING

Res ref $\longrightarrow$ CP ref

$$\tau(x) \longrightarrow \tau \circ g^n(y)$$

[GGKS'18] :

$$\log S_{CP}(\tau \circ g) = w_{Res}(\tau) \cdot \Theta(\log n)$$

LIFTING

$$\text{Res ref} \longrightarrow \text{CC ref}$$

$$\tau(x) \longrightarrow \tau \circ g^n(y)$$

[GGKS'18]:

$$\log S_{cc}(\tau \circ g) = w_{Res}(\tau) \cdot \Theta(\log n)$$

DONE?

LIFTING

$$S_{CP}(A) = n^{\Theta(w_{Res}(\tau))}$$

[AM'19] + [GGKS'18]

$$F \longrightarrow A \begin{array}{l} \xrightarrow{\text{SAT}} S(A) \le n^{O(w_{SAT})} \\ \\ \xrightarrow{\text{unsat}} S(A) \ge n^{\Omega(w_{uns})} \end{array}$$

LIFTING

$$S_{CP}(A) = n^{\Theta(w_{Res}(\tau))}$$

[AM'19]

$$\mathcal{F} \longrightarrow \tau \begin{cases} \xrightarrow{\text{SAT}} S(\tau) \le n^{O(1)} \\ \xrightarrow{\text{unsat}} S(\tau) \ge 2^{n^{\Omega(1)}} \end{cases}$$

LIFTING

$$S_{CP}(A) = n^{\Theta(w_{Res}(\tau))}$$

[AM'19]

$$\begin{tikzcd} & & w(\tau) \leq O(1)\,? \\ \mathcal{F} \arrow[r] & \tau \arrow[ur, "\text{SAT}"] \arrow[dr, "\text{unsat}"'] & \\ & & w(\tau) \geq n^{\Omega(1)}\,? \end{tikzcd}$$

LIFTING

$$S_{CP}(A) = n^{\Theta(w_{Res}(\tau))}$$

[AM'19]

$$\begin{tikzcd} & & w(\tau) = \Theta(n) \\ F \arrow[r] & \tau \arrow[ur, "SAT"] \arrow[dr, "unsat"'] & \\ & & w(\tau) = \Theta(n) \end{tikzcd}$$

LIFTING

$$S_{CP}(A) = n^{\Theta(w_{Res}(\tau))}$$

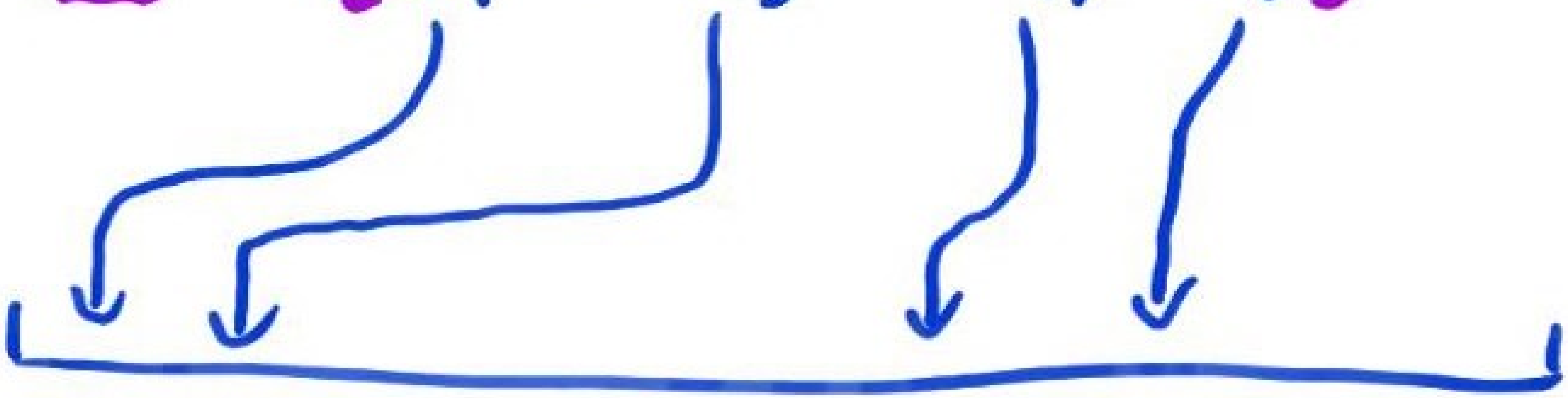
[AM'19]

$$\mathcal{F} \longrightarrow \tau \begin{cases} \xrightarrow{\text{SAT}} bw(\tau) \leq O(1) \\ \xrightarrow{\text{unsat}} bw(\tau) \geq n^{\Omega(1)} \end{cases}$$

Block-width

$$w = 4$$

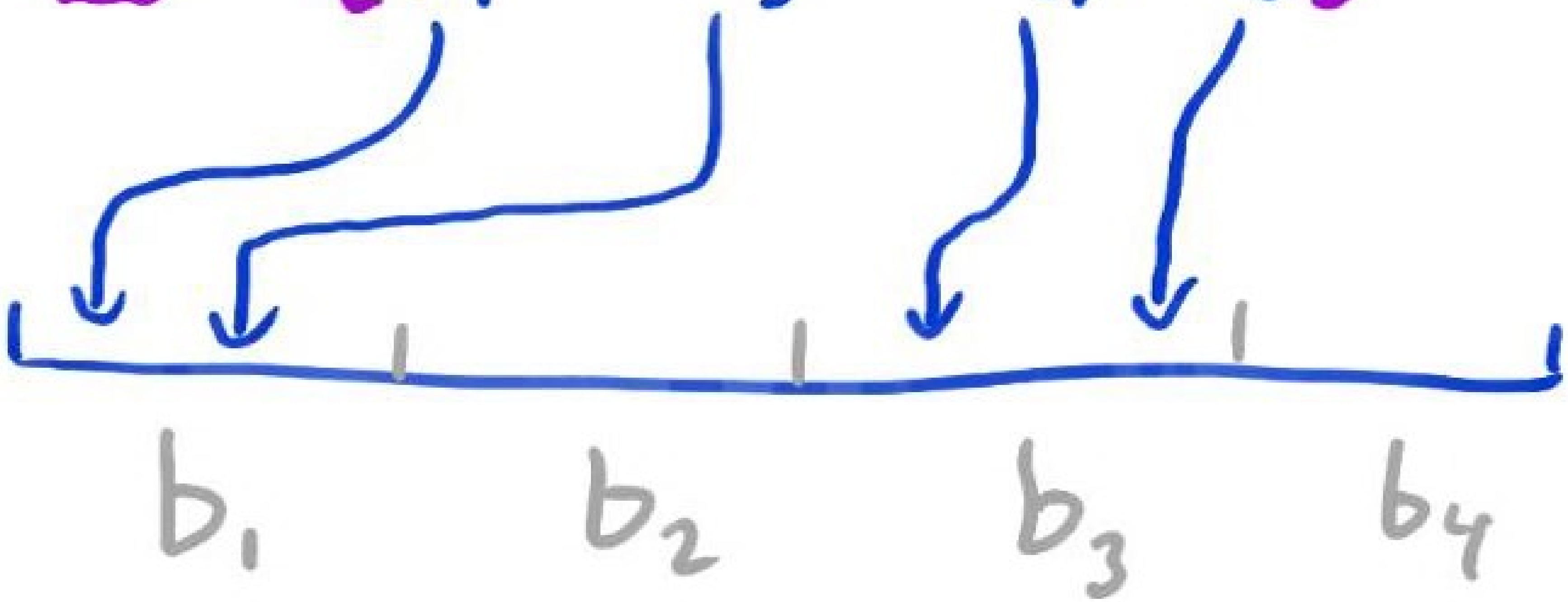
$$L = \{x_1 \vee x_3 \vee x_{17} \vee x_{21}\}$$



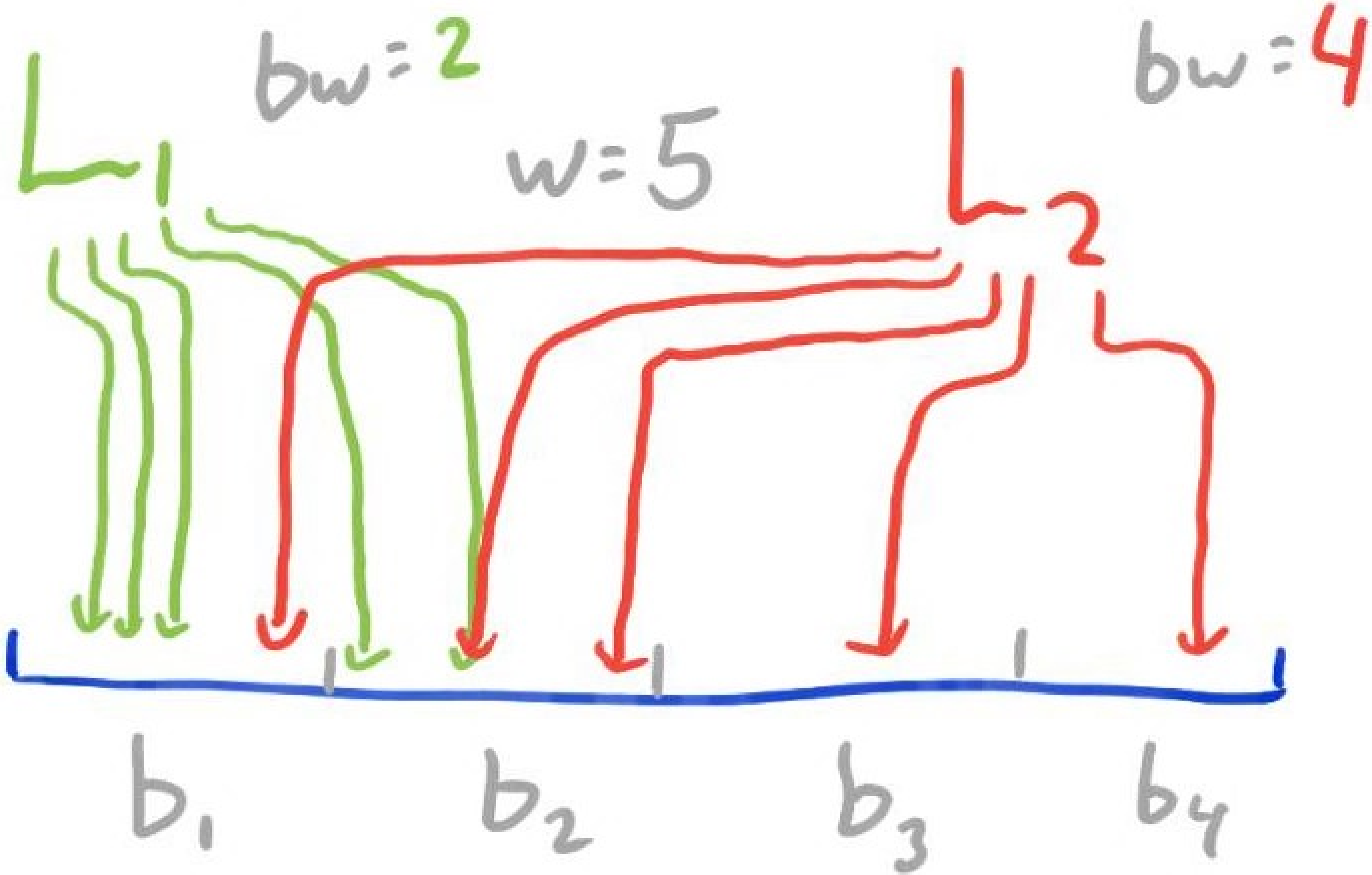
Block-width

$$bw = 2$$

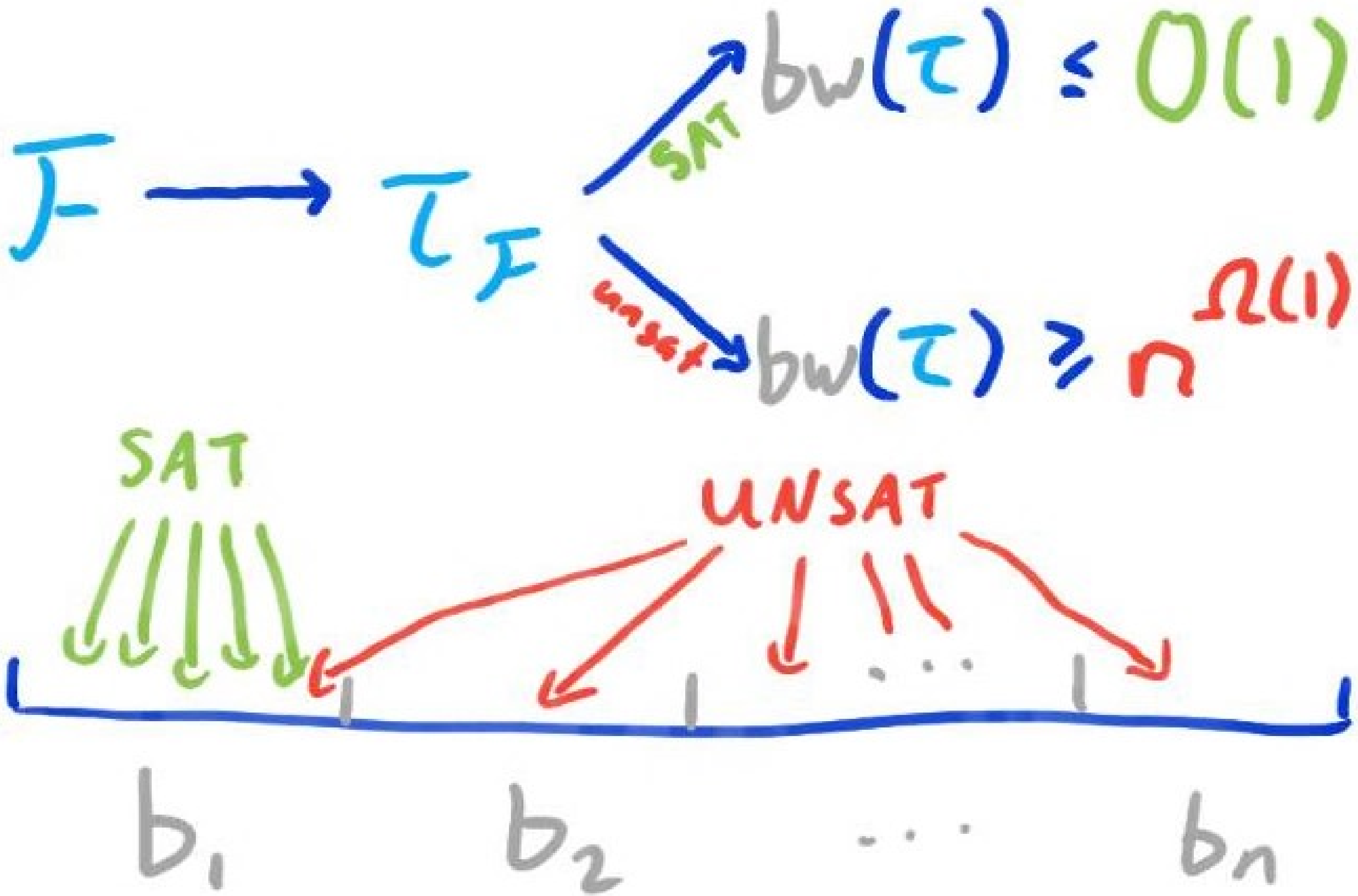
$$L = \{x_1 \vee x_3 \vee x_{17} \vee x_{21}\}$$



Block-width



BLOCK-WIDTH



$F \longrightarrow \tau_F$
SAT
$bw(\tau) \leq O(1)$
unsat
$bw(\tau) \geq n^{\Omega(1)}$
SAT
UNSAT
b_1
b_2
...
b_n

BLOCK-WIDTH

$$\begin{tikzcd} & bw(\tau) \le O(1) \\ \mathcal{F} \arrow[r] & \tau_{\mathcal{F}} \arrow[u, "\text{SAT}"] \arrow[d, "\text{unsat}"'] \\ & bw(\tau) \ge n^{\Omega(1)} \end{tikzcd}$$

[AM'19]: $S_{Res}(\tau_{\mathcal{F}}) = n^{\Theta(bw_{Res}(\tau_{\mathcal{F}}))}$

Lifting - New Theorem

$$S_{CP}(A) = n^{\Theta(bw_{Res}(\tau))}$$

[AM`19] + new lift

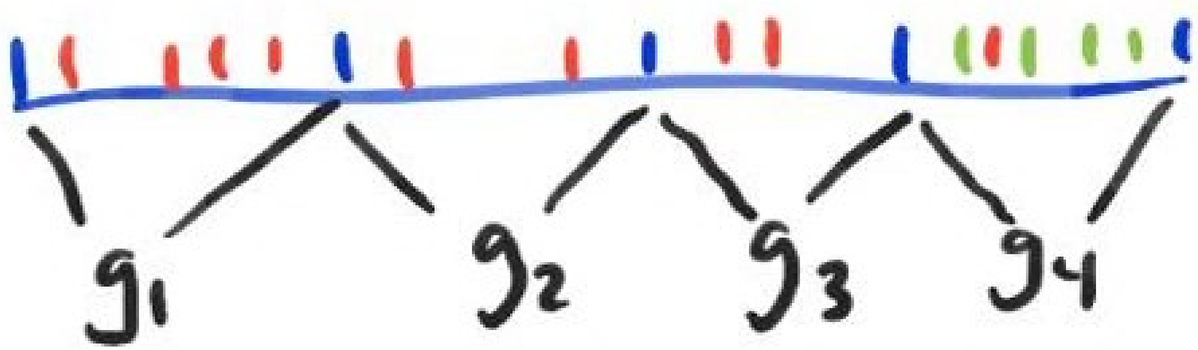
$$\mathcal{F} \longrightarrow A \xrightarrow{\text{SAT}} S(A) \leq n^{O(bw_{SAT})} = n^{O(1)}$$

$$A \xrightarrow{\text{unsat}} S(A) \geq n^{\Omega(bw_{uns})} = n^{n^{\Omega(1)}}$$

Lifting - New Theorem

$$S_{cp}(A) = n^{\Theta(bw_{Res}(\tau))}$$

Idea 1: $g : \{0,1\}^m \to \{0,1\}^b$



Lifting - New Theorem

$$s_{cp}(A) = n^{\Theta(bw_{Res}(\tau))}$$

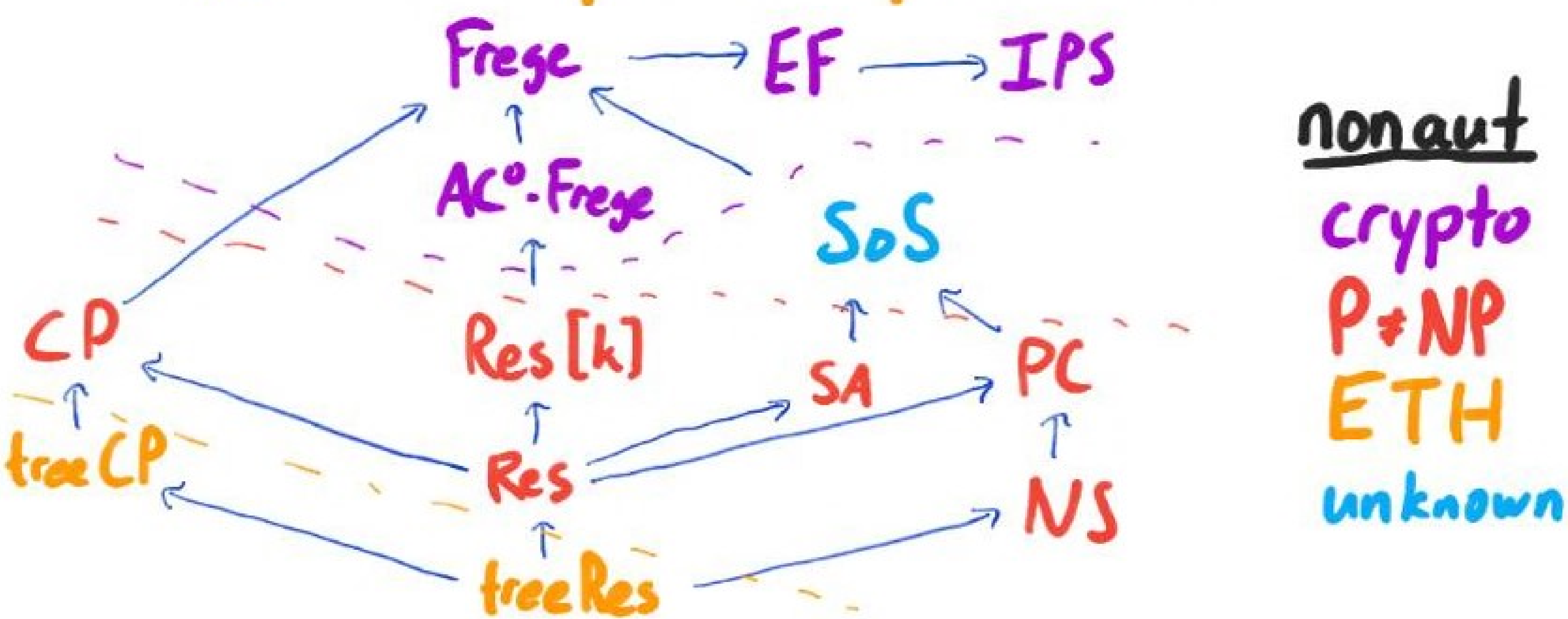
Idea 1: $g : \{0,1\}^m \to \{0,1\}^b$

Idea 2: multiparty communication

$$g(y_1, y_2, y_3, \cdots)$$

OPEN PROBLEMS

1) NP-hardness of automatability for more proof systems?



Open Problems

2) more applications of these new lifting theorems?

$$\log S_{rcc[poly]}(\tau \circ g) = bw(\tau) \cdot \Theta(\log n)$$

$$\log S_{rcc}(\tau \circ g) = d_{dt}(\tau) \cdot \Theta(\log d_{dt}(\tau))$$

Open Problems

1) NP-hardness of automatability for more proof systems?

2) more applications of these new lifting theorems?

Thank You!