Trading Time and Space in Catalytic Branching Programs

Ian Mertz

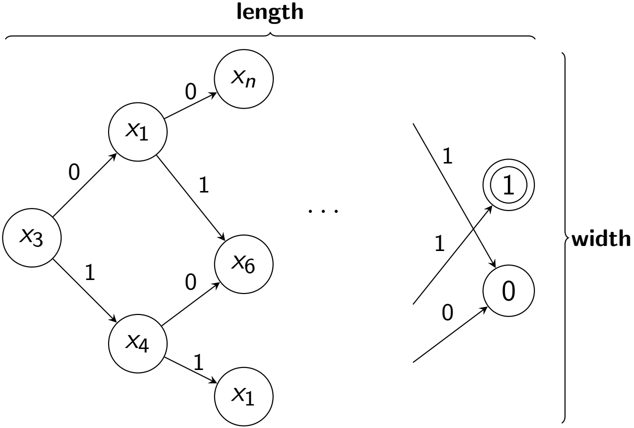
University of Toronto

July 23, 2022

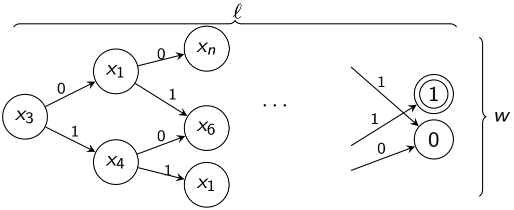
Joint work with James Cook

Space-bounded computation

$BP(w, \ell)$: layered branching programs of width w and length ℓ

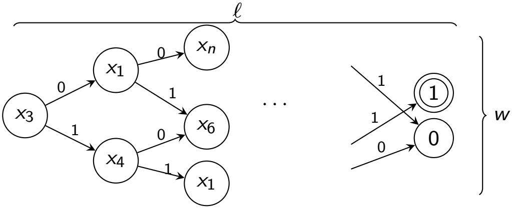


Space-bounded computation



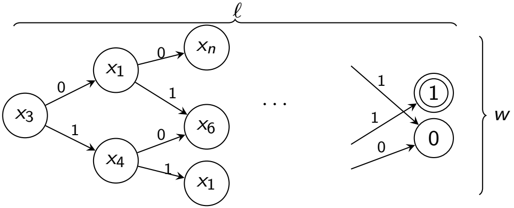
$BP(w, \ell)$ looks like $SPACETIME(\log w, \ell)$

Space-bounded computation



$BP(w, \ell)$ looks like $SPACETIME(\log w, \ell)$ (idea: for a fixed timestamp, w nodes in a layer $\leftrightarrow$ $\log w$ bits in memory)

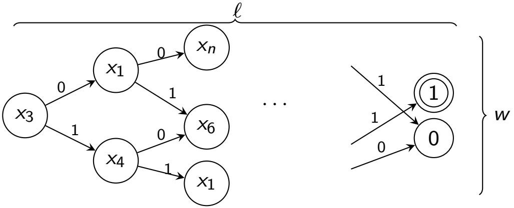
Space-bounded computation



$BP(w, \ell)$ looks like $SPACETIME(\log w, \ell)$ (idea: for a fixed timestamp, w nodes in a layer ↔ $\log w$ bits in memory), but...

SPACETIME is *uniform*: machine is “easy to describe” for every n

Space-bounded computation



$BP(w, \ell)$ looks like $SPACETIME(\log w, \ell)$ (idea: for a fixed timestamp, w nodes in a layer ↔ $\log w$ bits in memory), but...

SPACETIME is *uniform*: machine is "easy to describe" for every n

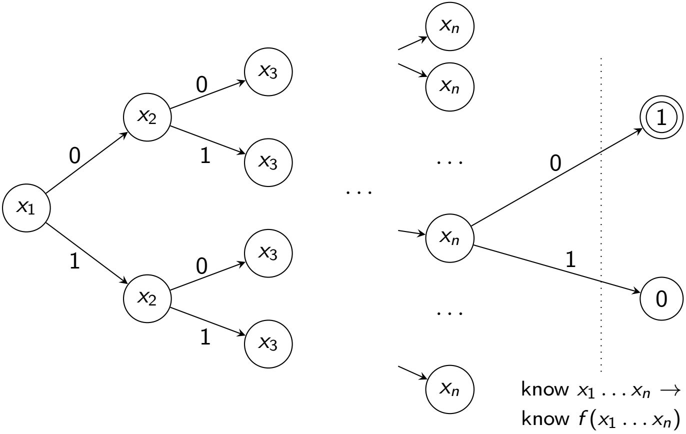
BP is *non-uniform*: no restrictions on the description

Space-bounded computation

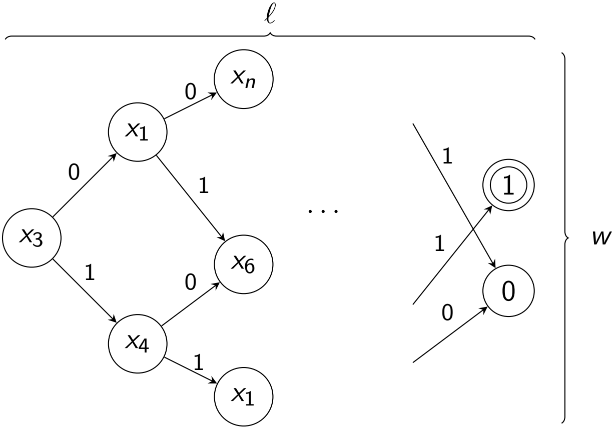
Every f can be computed by $BP(2^{n-1}, n)$

Space-bounded computation

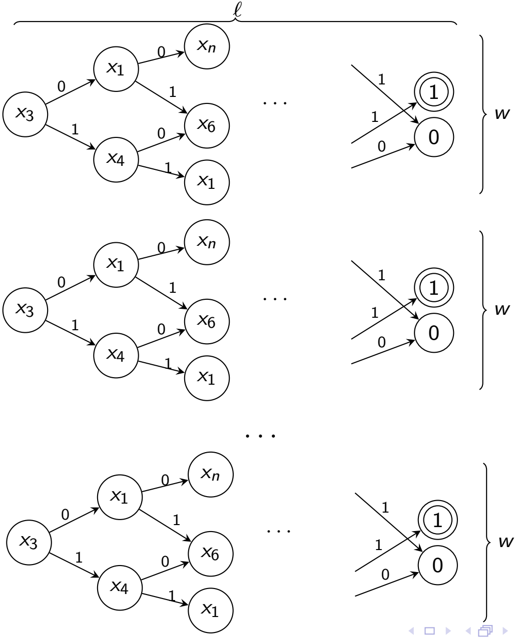
Every f can be computed by $BP(2^{n-1}, n)$



Amortized space-bounded computation

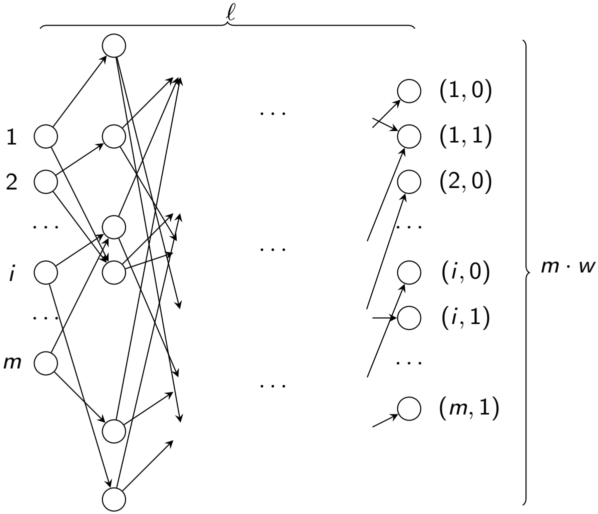


Amortized space-bounded computation



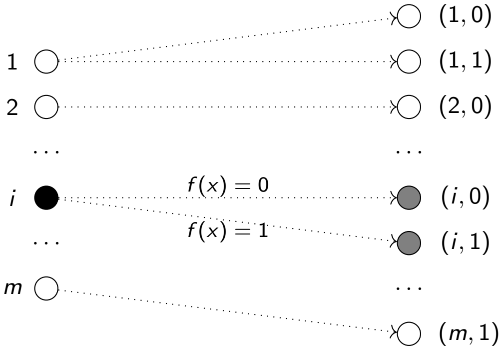
Amortized space-bounded computation

$mCBP(w, \ell, m)$: m different branching programs (one source $\rightarrow$ two sinks) which can share states



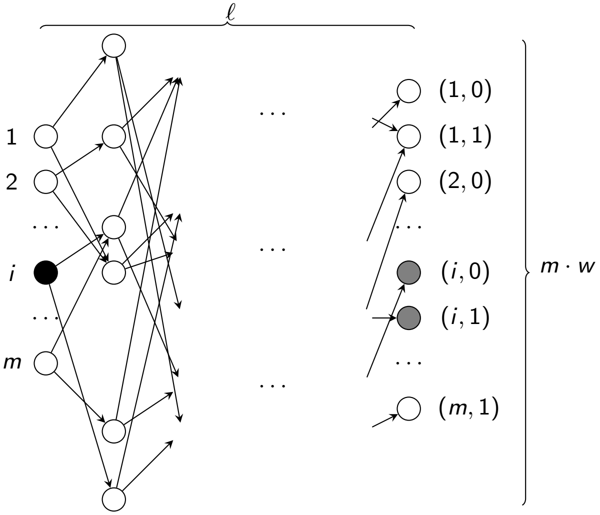
Amortized space-bounded computation

$mCBP(w, \ell, m)$: m different branching programs (one input node, two output nodes) which can share states



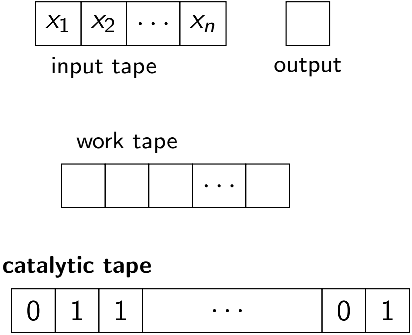
Amortized space-bounded computation

$mCBP(w, \ell, m)$: m different branching programs (one source → two sinks) which can share states



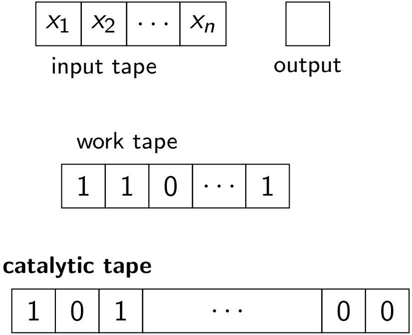
Catalytic computation

$CSPACETIME(s, t, c)$: space-bounded Turing Machines with an extra worktape (c bits) of full memory



Catalytic computation

CSPACETIME(s, t, c): space-bounded Turing Machines with an extra worktape (c bits) of full memory



x_1 | x_2 | $\cdots$ | x_n

input tape

output

work tape

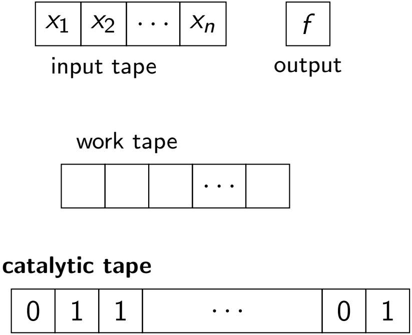
1 | 1 | 0 | $\cdots$ | 1

catalytic tape

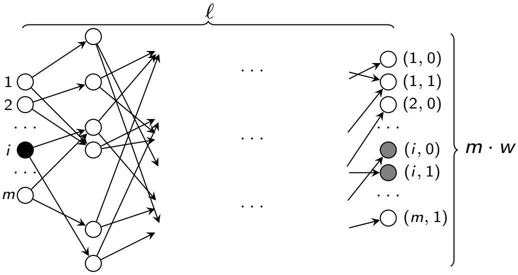
1 | 0 | 1 | $\cdots$ | 0 | 0

Catalytic computation

$CSPACETIME(s, t, c)$: space-bounded Turing Machines with an extra worktape (c bits) of full memory

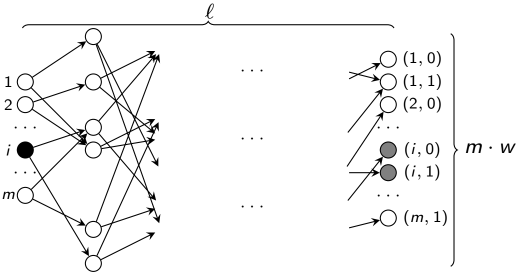


Catalytic computation



Again $mCBP(w, \ell, m)$ looks like non-uniform $CSPACETIME(\log w, \ell, \log m)$

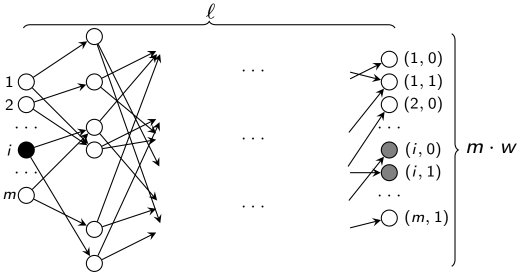
Catalytic computation



Again $mCBP(w, \ell, m)$ looks like non-uniform $CSPACETIME(\log w, \ell, \log m)$

- $m \cdot w$ nodes in a layer ↔ $\log m + \log w$ bits in memory

Catalytic computation



Again $mCBP(w, \ell, m)$ looks like non-uniform $CSPACETIME(\log w, \ell, \log m)$

- $m \cdot w$ nodes in a layer $\leftrightarrow$ $\log m + \log w$ bits in memory
- m sources plus source-sink pairing requirement $\leftrightarrow$ resetting $\log m$ catalytic memory)

Catalytic computation

Two interpretations of reducing w and m (non-uniform):

Catalytic computation

Two interpretations of reducing w and m (non-uniform):

1) *amortized space*: reducing the amortized space ($w = (w \cdot m)/m$) needed to compute f, or the number of copies (m) needed for amortization to help

Catalytic computation

Two interpretations of reducing w and m (non-uniform):

1) *amortized space*: reducing the amortized space ($w = (w \cdot m)/m$) needed to compute f, or the number of copies (m) needed for amortization to help

2) *catalytic space*: reducing the amount of space ($\log w$) and catalytic space ($\log m$) needed to compute f

Known results

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$.

Known results

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$.

Counting argument: almost every function f requires branching programs to have either non-amortized width or length $2^{\Omega(n)}$.

Known results

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$.

Counting argument: almost every function f requires branching programs to have either non-amortized width or length $2^{\Omega(n)}$.

In contrast, [Potechin'17] gives (asymptotically) optimal amortized width $w = O(1)$ and length $\ell = O(n)$ simultaneously

Known results

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$.

Counting argument: almost every function f requires branching programs to have either non-amortized width or length $2^{\Omega(n)}$.

In contrast, [Potechin'17] gives (asymptotically) optimal amortized width $w = O(1)$ and length $\ell = O(n)$ simultaneously

...but we need $\mathbf{m} = \mathbf{2^{2^n - 1}}$ to get it!

Our results

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{2^n-1}$.

Our results

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{2^n - 1}$.

Main result 1: for any $\epsilon > 0$, every function f can be computed by an m-catalytic branching program of width $2m$ and length $O_\epsilon(n)$, where $m = 2^{2^{\epsilon n}}$.

Our results (permutation branching programs)

[Potechin'17]': every function f can be computed by a read-4 permutation branching program of width 2^{2^n+1}.

Main result 1': for any $\epsilon > 0$, every function f can be computed by a read-$O_\epsilon(1)$ permutation branching program of width $2^{2^{\epsilon n}}$.

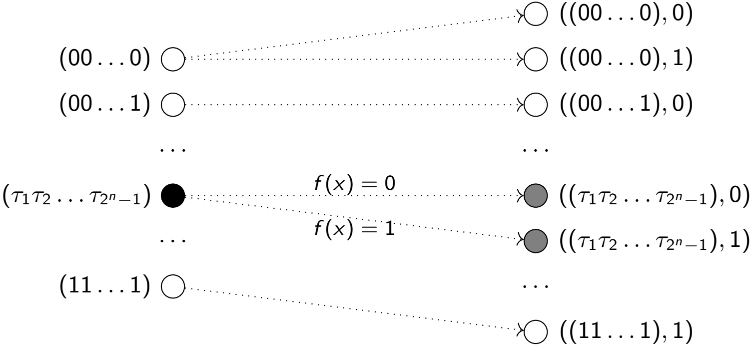
[Potechin'17] in one slide

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{2^n-1}$.

[Potechin'17] in one slide

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{2^n-1}$.

Setup: catalytic space $\log m = 2^n - 1$ in some initial state $\tau_1 \ldots \tau_{2^n-1}$, plus $\log 4 = 2$ bits of free work space



$(00\ldots0)$

$(00\ldots1)$

$\ldots$

$(\tau_1\tau_2\ldots\tau_{2^n-1})$

$\ldots$

$(11\ldots1)$

$f(x) = 0$

$f(x) = 1$

$((00\ldots0),0)$

$((00\ldots0),1)$

$((00\ldots1),0)$

$\ldots$

$((\tau_1\tau_2\ldots\tau_{2^n-1}),0)$

$((\tau_1\tau_2\ldots\tau_{2^n-1}),1)$

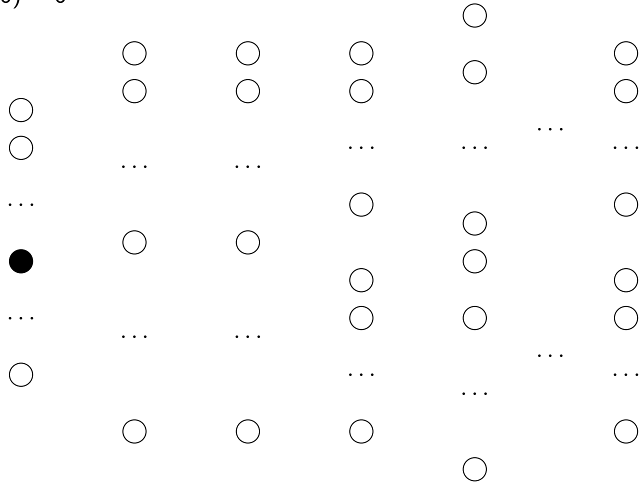
$\ldots$

$((11\ldots1),1)$

[Potechin'17] in two slides

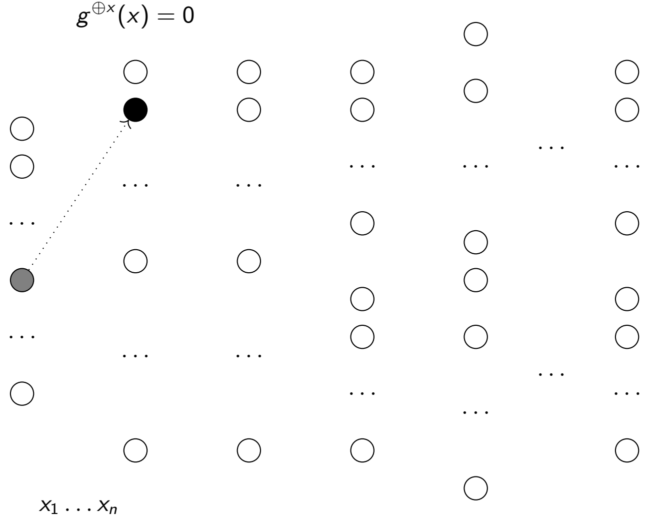
0) First free bit: $\vec{0}$ entry of g

$g(0) = 0$



[Potechin'17] in two slides

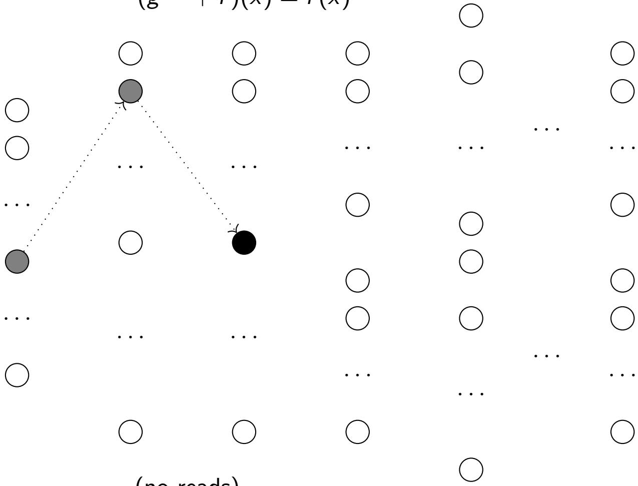
1) $g(\alpha_1 \dots \alpha_i \dots \alpha_n) \to g(\alpha_1 \dots \alpha_i^{x_i} \dots \alpha_n)$



[Potechin'17] in two slides

2) $g(y) \to g(y) + f(y)$

$$(g^{\oplus x} + f)(x) = f(x)$$

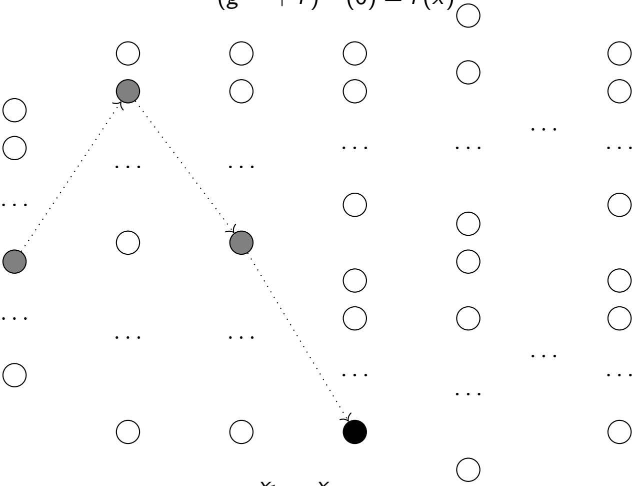


(no reads)

[Potechin'17] in two slides

3) $g(\alpha_1 \ldots \alpha_i^{x_i} \ldots \alpha_n) \to g(\alpha_1 \ldots \alpha_i \ldots \alpha_n)$

$(g^{\oplus x} + f)^{\oplus x}(0) = f(x)$

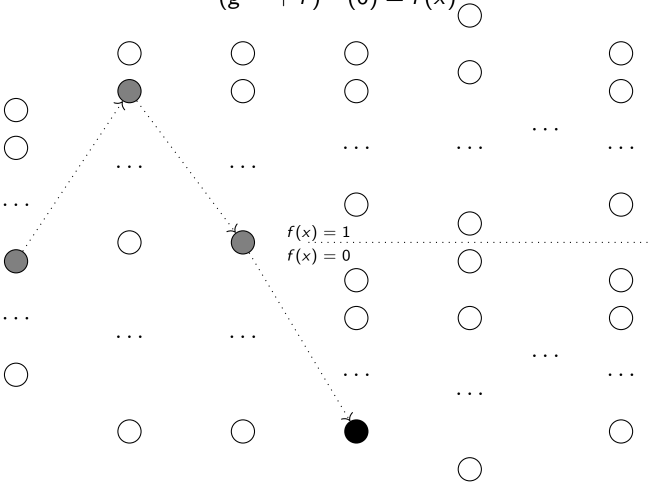


$x_1 \ldots x_n$

[Potechin'17] in two slides

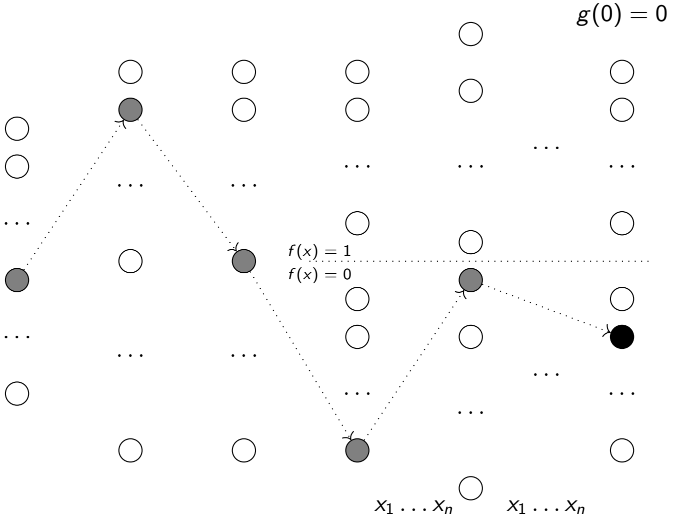
4) Second free bit (output): copy the answer from first free bit

$$(g^{\oplus x} + f)^{\oplus x}(0) = f(x)$$



[Potechin'17] in two slides

5) Undo steps 1-3 (do steps 3-1)



Trading time and space

Truth table representation [Potechin'17]:

$$f(x) = \sum_{\alpha \in \{0,1\}^n} f(\alpha) \cdot [x = \alpha]$$

Trading time and space

Truth table representation [Potechin'17]:

$$f(x) = \sum_{\alpha \in \{0,1\}^n} f(\alpha) \cdot [x = \alpha]$$

Monomial representation [Cook-Mertz'20,21]:

$$f(x) = \sum_{S \subseteq [n]} f_{mon}(S) \cdot \prod_{i \in S} x_i \mod 2$$

Trading time and space

Truth table representation [Potechin'17]:

$$f(x) = \sum_{\alpha \in \{0,1\}^n} f(\alpha) \cdot [x = \alpha]$$

Monomial representation [Cook-Mertz'20,21]:

$$f(x) = \sum_{S \subseteq [n]} f_{mon}(S) \cdot \prod_{i \in S} x_i \mod 2$$

Catalytic algorithms give us a way to compute f over the monomial basis only using catalytic memory.

Trading time and space

Monomial representation [Cook-Mertz'20,21]:

$$f(x) = \sum_{S \subseteq [n]} f_{mon}(S) \cdot \prod_{i \in S} x_i \mod 2$$

Two algorithms for monomial rep., different types of efficiency:

Trading time and space

Monomial representation [Cook-Mertz'20,21]:

$$f(x) = \sum_{S \subseteq [n]} f_{mon}(S) \cdot \prod_{i \in S} x_i \mod 2$$

Two algorithms for monomial rep., different types of efficiency:

1) *Potechin algorithm (monomial basis edition)*

- compute each monomial into separate memory in parallel
- linear time, exponential space

Trading time and space

Monomial representation [Cook-Mertz'20,21]:

$$f(x) = \sum_{S \subseteq [n]} f_{mon}(S) \cdot \prod_{i \in S} x_i \mod 2$$

Two algorithms for monomial rep., different types of efficiency:

1) *Potechin algorithm (monomial basis edition)*

- compute each monomial into separate memory in parallel
- linear time, exponential space

2) *Cook-Mertz algorithm (branching program edition)*

- compute each monomial directly into the output register in series
- exponential time, linear space

Trading time and space

Main result 1: for any $\epsilon > 2/n$, every function f can be computed by an m-catalytic branching program of width $2m$ and length $2^{1/\epsilon} \cdot 2\epsilon n$, where $m = 2^{n+\frac{1}{\epsilon}\cdot 2^{\epsilon n}}$.

Trading time and space

Main result 1: for any $\epsilon > 2/n$, every function f can be computed by an m-catalytic branching program of width $2m$ and length $2^{1/\epsilon} \cdot 2\epsilon n$, where $m = 2^{n+\frac{1}{\epsilon}\cdot 2^{\epsilon n}}$.

Proof idea: use time-efficient algorithm to compute monomials only up to degree ϵn, then use space-efficient algorithm to combine them to get the higher degree monomials.

Trading time and space

Main result 1: for any $\epsilon > 2/n$, every function f can be computed by an m-catalytic branching program of width $2m$ and length $2^{1/\epsilon} \cdot 2\epsilon n$, where $m = 2^{n+\frac{1}{\epsilon}\cdot 2^{\epsilon n}}$.

Proof idea: use time-efficient algorithm to compute monomials only up to degree ϵn, then use space-efficient algorithm to combine them to get the higher degree monomials.

- Small monomials: $\binom{n}{\leq \epsilon n}$ monomials → space $n^{\epsilon n}$

Trading time and space

Main result 1: for any $\epsilon > 2/n$, every function f can be computed by an m-catalytic branching program of width $2m$ and length $2^{1/\epsilon} \cdot 2\epsilon n$, where $m = 2^{n+\frac{1}{\epsilon}\cdot 2^{\epsilon n}}$.

Proof idea: use time-efficient algorithm to compute monomials only up to degree ϵn, then use space-efficient algorithm to combine them to get the higher degree monomials.

- Small monomials: $\binom{n}{\leq \epsilon n}$ monomials → space $n^{\epsilon n}$
  - better: split variables into $\frac{1}{\epsilon}$ groups → space $\frac{1}{\epsilon} \cdot 2^{\epsilon n}$ $(+n)$

Trading time and space

Main result 1: for any $\epsilon > 2/n$, every function f can be computed by an m-catalytic branching program of width $2m$ and length $2^{1/\epsilon} \cdot 2\epsilon n$, where $m = 2^{n+\frac{1}{\epsilon}\cdot 2^{\epsilon n}}$.

Proof idea: use time-efficient algorithm to compute monomials only up to degree ϵn, then use space-efficient algorithm to combine them to get the higher degree monomials.

- Small monomials: $\binom{n}{\leq \epsilon n}$ monomials → space $n^{\epsilon n}$
  - better: split variables into $\frac{1}{\epsilon}$ groups → space $\frac{1}{\epsilon} \cdot 2^{\epsilon n}$ $(+n)$
- Large monomials: degree $1/\epsilon$ → time $2^{1/\epsilon}$ $(\cdot 2\epsilon n)$

Trading time and space

Main result 1: for any $\epsilon > 2/n$, every function f can be computed by an m-catalytic branching program of width $2m$ and length $2^{1/\epsilon} \cdot 2\epsilon n$, where $m = 2^{n+\frac{1}{\epsilon}\cdot 2^{\epsilon n}}$.

Proof idea: use time-efficient algorithm to compute monomials only up to degree ϵn, then use space-efficient algorithm to combine them to get the higher degree monomials.

- Small monomials: $\binom{n}{\leq \epsilon n}$ monomials → space $n^{\epsilon n}$
  - better: split variables into $\frac{1}{\epsilon}$ groups → space $\frac{1}{\epsilon} \cdot 2^{\epsilon n}$ $(+n)$
- Large monomials: degree $1/\epsilon$ → time $2^{1/\epsilon}$ $(\cdot 2\epsilon n)$

☐

Extending to easier functions

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{2^n-1}$.

Extending to easier functions

[Potechin'17]: every function f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{2^n - 1}$.

[Robere-Zuiddam'22]: if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{\binom{n}{\leq d} - 1}$.

Extending to easier functions

[Robere-Zuiddam'22]: if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{\binom{n}{\leq d}-1}$.

Proof idea (original): for low degree f, the Potechin algorithm has many isomorphic disjoint components based on the symmetries of the polynomial associated with f.

Extending to easier functions

[Robere-Zuiddam'22]: if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{\binom{n}{\leq d}-1}$.

Proof idea (original): for low degree f, the Potechin algorithm has many isomorphic disjoint components based on the symmetries of the polynomial associated with f.

Proof idea (new): monomial version of Potechin algorithm again, but now only compute monomials which actually appear in f ($\binom{n}{\leq d}$ by assumption).

Extending to easier functions

[Robere-Zuiddam'22]: if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{\binom{n}{\leq d}-1}$.

Extending to easier functions

[Robere-Zuiddam'22]: if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{\binom{n}{\leq d} - 1}$.

Main result 2: for any $\epsilon > 2/d$, if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of of width $2m$ and length $2^{1/\epsilon} \cdot 2n$, where $m = 2^{n + \frac{1}{\epsilon} \cdot \binom{n}{\leq \epsilon d}}$.

Extending to easier functions

[Robere-Zuiddam'22]: if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of width $4m$ and length $4n$, where $m = 2^{\binom{n}{\leq d}-1}$.

Main result 2: for any $\epsilon > 2/d$, if f is a degree d polynomial over $\mathbb{F}_2$, then f can be computed by an m-catalytic branching program of of width $2m$ and length $2^{1/\epsilon} \cdot 2n$, where $m = 2^{n+\frac{1}{\epsilon}\cdot\binom{n}{\leq \epsilon d}}$.

Proof idea: same* time-space tradeoff as before, now with ϵd instead of ϵn. ☐

Saving time

All results are linear time, which is optimal up to a constant factor. But how small can we get the constant?

Saving time

All results are linear time, which is optimal up to a constant factor. But how small can we get the constant?

Main result 3: every f can be computed by an m-catalytic (or even permutation) branching program of length $4n - 4$ and width $4m$, where $m = 2^{2^n - 1}$.

Saving time

All results are linear time, which is optimal up to a constant factor. But how small can we get the constant?

Main result 3: every f can be computed by an m-catalytic (or even permutation) branching program of length $4n - 4$ and width $4m$, where $m = 2^{2^n - 1}$.

Main result 4: any permutation* branching program calculating the AND function which reads any variable less than three times requires length at least $4n - 4$.

Open problems

Save on *either* time or space (while keeping other optimal)

- would give better tradeoff algorithm

Open problems

Save on *either* time or space (while keeping other optimal)

- would give better tradeoff algorithm

Show that for some f, m must be at least 2^n to get linear amortized size

- counting only gives $m \geq 2^n/O(n)$

Open problems

Save on *either* time or space (while keeping other optimal)

- would give better tradeoff algorithm

Show that for some f, m must be at least 2^n to get linear amortized size

- counting only gives $m \geq 2^n / O(n)$

Optimal permutation branching program length for any function

- somewhere between $3n$* and $4n - 4$
- can get a read-3 program for $AND(x_1, x_2, x_3)$