Complexity of Regular Functions

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March 2, 2015

Overview

Next up...

In the beginning...

Problem: find the minimum cost path in a graph

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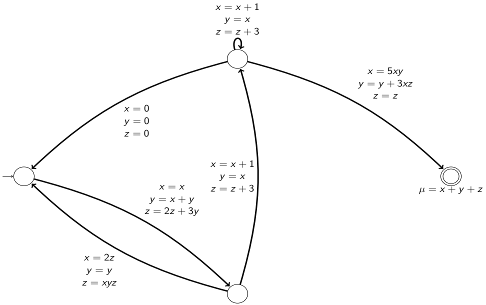
How do we make this model work?

Cost Register Automata (CRA)

Definition[Alur, D'Antoni, Deshmukh, Raghothaman, Yuan]

A *cost-register automaton* is a deterministic finite-state automaton augmented with a finite set of registers that store elements of an algebraic domain. A computation step consists of consuming the next input symbol, transitioning to a new state based on that input symbol, and updating each register based on a function over the algebra.



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Our results

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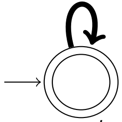
$$NC^1 \subseteq \#NC^1 \subseteq GapNC^1 \subseteq L \subseteq AC^1 \subseteq P$$

A natural barrier

$$r_i = \prod_{j=1}^{k} r_j \ \forall i$$



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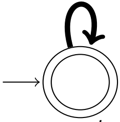

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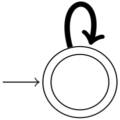

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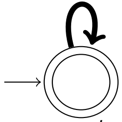

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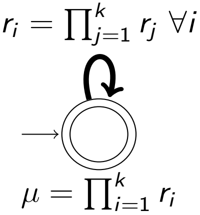

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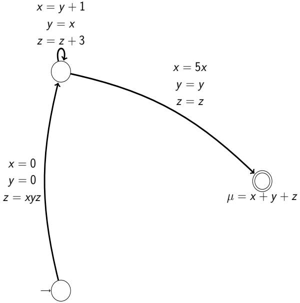
Wait, this is c^{k^n}! That takes $O(k^n)$ bits just to write down; no way we're getting that in P.

Copyless CRAs (CCRAs)

Definition[Alur, Freilich, Raghothaman]

A *copyless CRA* is a CRA where for any transition, no register can be used more than once to update the registers on that transition.



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If we consider the algebraic degree of the functions we are representing, then CCRAs have $n^{O(1)}$-bounded algebraic degree, unlike the problematic exponential degree functions from before.

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In NC^1 we can build a constant-width graph with n layers, with an arrow from $v_{i,j}$ to $v_{i+1,l}$ if on the ith input variable there is a transition of the form $r_l \leftarrow r_j + c$.

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There is a unique path from the first layer to any chosen vertex v_i on the last layer (given that they all have indegree 1), and we can find it in NC^1. Tracing back along this path gives us all the constants that sum up to give the final value of r_i.

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The boolean circuitry can all be replaced since $\mathrm{NC}^1 \subseteq \#\mathrm{NC}^1$, and so we get that $f \in \mathrm{GapNC}^1$ (we are working over $\mathbb{Z}$, so negative results are possible, and so we do not have $\#\mathrm{NC}^1$)

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All functions computable by poly size, poly degree circuits over $(\Gamma^*, \max, \circ)$ lie in AC^1 [AJMV].

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Moral of the story: replacing copyless with $\otimes c$ often gives the same result.

Next up...

A mandatory table and open problems

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If you are in computer-aided verification, there's a whole lot of literature on regular functions, so maybe these new bounds will make CRAs more attractive for using in algorithms?

That's all Folks!