Coloring rings

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Abstract

A *ring* is a graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets, $X_1, \ldots, X_k$, such that for all $i \in \{1, \ldots, k\}$ the set X_i can be ordered as $X_i = \{u_i^1, \ldots, u_i^{|X_i|}\}$ so that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \cdots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. A *hyperhole* is a ring R such that for all $i \in \{1, \ldots, k\}$, X_i is complete to $X_{i-1} \cup X_{i+1}$. In this paper, we prove that the chromatic number of a ring R is equal to the chromatic number of a maximum hyperhole in R. Using this result, we give a polynomial-time coloring algorithm for rings.

Rings appeared as one of the basic classes in a decomposition theorem for a class of graphs studied by Boncompagni, Penev, and Vušković in [*Journal of Graph Theory* 91 (2019), 192–246]. Using our coloring algorithm for rings, we show that graphs in this larger class can also be colored in polynomial time. Furthermore, we obtain an optimal χ-bounding function for this larger class of graphs, and we also verify Hadwiger's conjecture for it.

Keywords: chromatic number, vertex coloring, algorithms, optimal χ-bounding function, Hadwiger's conjecture.

1 Introduction

All graphs in this paper are finite, simple, and nonnull. As usual, for a graph G and a vertex v of G, $N_G(v)$ is the set of neighbors of v in G, and $N_G[v] = N_G(v) \cup \{v\}$.

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A *ring* is a graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets $X_1, \dots, X_k$ (whenever convenient, we consider subscripts of the X_i's to be modulo k), such that for all $i \in \{1, \dots, k\}$ the set X_i can be ordered as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that

$$X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}.$$

(Note that this implies that $X_1, \dots, X_k$ are all cliques.) Under such circumstances, we also say that the ring R is of *length* k, or that R is a *k-ring*; furthermore, $(X_1, \dots, X_k)$ is called a *ring partition* of R. A ring is *even* or *odd* depending on the parity of its length. Rings played an important role in [1]: they formed a "basic class" in the decomposition theorems for a couple of graph classes defined by excluding certain "Truemper configurations" (more on this in subsection 1.1). In that paper, the complexity of the optimal vertex coloring problem for rings was left as an open problem.[1] In the present paper, we give a polynomial-time coloring algorithm for rings (see Theorems 4.2 and 5.2).

It can easily be shown that every ring is a circular-arc graph. Furthermore, rings have unbounded clique width. To see this, let $k \geq 3$ be an integer, and let R be a $(k+1)$-ring with ring partition $(X_1, \dots, X_k, X_{k+1})$ such that the cliques X_i are all of size $k+1$, with vertices labeled $0, 1, \dots, k$, and furthermore, assume that vertices labeled p and q from consecutive cliques of the ring partition are adjacent if and only if $p + q \leq k$. Now, the graph obtained from R by first deleting X_{k+1}, and then deleting all the vertices labeled 0, is precisely the permutation graph H_k defined in [7], and the clique-width of H_k is at least k (see Lemma 5.4 from [7]).

Given graphs H and G, we say that G *contains* H if G contains an induced subgraph isomorphic to H; if G does not contain H, then G is *H-free*. For a family $\mathcal{H}$ of graphs, we say that a graph G is *$\mathcal{H}$-free* if G is H-free for all $H \in \mathcal{H}$.

Given a graph G, a *clique* of G is a (possibly empty) set of pairwise adjacent vertices in G, and a *stable set* of G is a (possibly empty) set of pairwise nonadjacent vertices in G. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G, and the *stability number* of G, denoted by $\alpha(G)$, is the maximum size of a stable set of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to "properly color" G, i.e. to color the vertices of G in such a way that no two adjacent vertices receive the same color.

Given a graph G, a vertex $v \in V(G)$, and a set $S \subseteq V(G) \setminus \{v\}$, we say that v is *complete* (resp. *anticomplete*) to S in G provided that v is adjacent (resp. nonadjacent) to every vertex of S; given disjoint sets $X, Y \subseteq V(G)$,

[1]In fact, only odd rings are difficult in this regard; even rings are readily colorable in polynomial time (see Lemma 3.2).

we say that X is *complete* (resp. *anticomplete*) to Y in G provided that every vertex in X is complete (resp. anticomplete) to Y in G.

A *hole* is a chordless cycle on at least four vertices; the *length* of a hole is the number of its vertices, and a hole is *even* or *odd* according to the parity of its length. When we say "H is a hole in G," we mean that H is a hole that is an induced subgraph of G.

A *hyperhole* is any graph H whose vertex set can be partitioned into $k \geq 4$ nonempty cliques $X_1, \dots, X_k$ (whenever convenient, we consider subscripts of the X_i's to be modulo k) such that for all $i \in \{1, \dots, k\}$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$; under such circumstances, we also say that H is a hyperhole of *length* k, or that H is a *k-hyperhole*. A hyperhole is *even* or *odd* according to the parity of its length. Note that every hole is a hyperhole, and every hyperhole is a ring. When we say "H is a hyperhole in G," we mean that H is a hyperhole that is an induced subgraph of G.

Hyperholes can be colored in linear time [10]. Furthermore, the following theorem gives a formula for the chromatic number of a hyperhole.

Lemma 1.1. *[10] Let H be a hyperhole. Then $\chi(H) = \max\{\omega(H), \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil\}$.*

The main result of the present paper is the following theorem.

Theorem 1.2. *Let $k \geq 4$ be an integer, and let R be a k-ring. Then $\chi(R) = \max\{\chi(H) \mid H$ is a k-hyperhole in $R\}$.*

It was shown in [1] that all holes of a k-ring ($k \geq 4$) are of length k; consequently, all hyperholes in a k-ring are of length k. Thus, Theorem 1.2 in fact establishes that the chromatic number of a ring is equal to the maximum chromatic number of a hyperhole in the ring.

It is easy to see that the stability number of any k-hyperhole ($k \geq 4$) is $\lfloor k/2 \rfloor$. Thus, the following is an immediate corollary of Lemma 1.1 and Theorem 1.2.

Corollary 1.3. *Let $k \geq 4$ be an integer, and let R be a k-ring. Then $\chi(R) = \max\Big(\{\omega(R)\} \cup \{\left\lceil \frac{|V(H)|}{\lfloor k/2 \rfloor} \right\rceil \mid H$ is a k-hyperhole in $R\}\Big)$.*

We use Corollary 1.3 to give an $O(n^3)$ time algorithm that computes the chromatic number of a ring (see Theorem 4.2), and using that algorithm as a subroutine, we construct an $O(n^6)$ time algorithm that produces an optimal coloring of a ring (see Theorem 5.2).

1.1 Background and paper outline

A class of graphs is *hereditary* if it is closed under isomorphism and induced subgraphs.

A *theta* is any subdivision of the complete bipartite graph $K_{2,3}$; in particular, $K_{2,3}$ is a theta. A *pyramid* is any subdivision of the complete graph K_4 in which one triangle remains unsubdivided, and of the remaining three edges, at least two edges are subdivided at least once. A *prism* is any subdivision of $\overline{C_6}$ (where $\overline{C_6}$ is the complement of C_6) in which the two triangles remain unsubdivided; in particular, $\overline{C_6}$ is a prism. A *three-path-configuration* (or *3PC* for short) is any theta, pyramid, or prism.

A *wheel* is a graph that consists of a hole and an additional vertex that has at least three neighbors in the hole. If this additional vertex is adjacent to all vertices of the hole, then the wheel is said to be a *universal wheel*; if the additional vertex is adjacent to three consecutive vertices of the hole, and to no other vertex of the hole, then the wheel is said to be a *twin wheel*. A *proper wheel* is a wheel that is neither a universal wheel nor a twin wheel.

A *Truemper configuration* is any 3PC or wheel (for a survey, see [14]). Note that every Truemper configuration contains a hole. Note, furthermore, that every prism or theta contains an even hole, and every pyramid contains an odd hole. Thus, even-hole-free graphs contain no prisms and no thetas, and odd-hole-free graphs contain no pyramids.

$\mathcal{G}_{\mathrm{T}}$ is the class of all (3PC, proper wheel, universal wheel)-free graphs; thus, the only Truemper configurations that a graph in $\mathcal{G}_{\mathrm{T}}$ can contain are the twin wheels. Clearly, the class $\mathcal{G}_{\mathrm{T}}$ is hereditary. A decomposition theorem for $\mathcal{G}_{\mathrm{T}}$ (where rings form one of the "basic classes") was obtained in [1],[2] as were polynomial-time algorithms that solve the recognition, maximum weight clique, and maximum weight stable set problems for the class $\mathcal{G}_{\mathrm{T}}$. The complexity of the optimal coloring problem for $\mathcal{G}_{\mathrm{T}}$ was left open in [1], and the main obstacle in this context were rings. In the present paper, we show that graphs in $\mathcal{G}_{\mathrm{T}}$ can be colored in polynomial time (see Theorems 4.3 and 5.3).

A *simplicial vertex* is a vertex whose neighborhood is a (possibly empty) clique. For an integer $k \geq 4$, let $\mathcal{R}_k$ be the class of all graphs G that have the property that every induced subgraph of G either is a k-ring or has a simplicial vertex; clearly, $\mathcal{R}_k$ is hereditary, and furthermore (by Lemma 2.8) it contains all k-rings. We remark that graphs in $\mathcal{R}_k$ are precisely the chordal graphs,[3] and the graphs that can be obtained from a k-ring by repeatedly adding simplicial vertices (see Lemma 2.9). Further, for all integers $k \geq 4$, we set $\mathcal{R}_{\geq k} = \bigcup_{i=k}^{\infty} \mathcal{R}_i$; clearly, $\mathcal{R}_{\geq k}$ is hereditary, and furthermore (by Lemma 2.8) it contains all rings of length at least k. In particular, the class $\mathcal{R}_{\geq 4}$ is hereditary and contains all rings. We show that graphs in $\mathcal{R}_{\geq 4}$ can be colored in polynomial time (see Theorems 4.2 and 5.2).

A *clique-cutset* of a graph G is (possibly empty) clique C such that $G \setminus C$ is disconnected. A *clique-cut-partition* of a graph G is a partition (A, B, C)

[2]In the present paper, this decomposition theorem is stated as Theorem 2.11.

[3]A graph is *chordal* if it contains no holes.

of $V(G)$ such that A and B are nonempty and anticomplete to each other, and C is a (possibly empty) clique. Clearly, a graph admits a clique-cutset if and only if it admits a clique-cut-partition.

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$. The Strong Perfect Graph Theorem [2] states that a graph G is perfect if and only if neither G nor $\overline{G}$ contains an odd hole.

$\mathbb{N}$ is the set of all positive integers. A hereditary class $\mathcal{G}$ is *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ (called a *χ-bounding function* for $\mathcal{G}$) such that all graphs $G \in \mathcal{G}$ satisfy $\chi(G) \leq f(\omega(G))$. For a hereditary χ-bounded class $\mathcal{G}$ that contains all complete graphs (equivalently: that contains graphs of arbitrarily large clique number), we say that a χ-bounding function $f : \mathbb{N} \to \mathbb{N}$ for $\mathcal{G}$ is *optimal* if for all $n \in \mathbb{N}$, there exists a graph $G \in \mathcal{G}$ such that $\omega(G) = n$ and $\chi(G) = f(n)$. It was shown in [1] that $\mathcal{G}_{\mathrm{T}}$ is χ-bounded by a linear function; more precisely, it was shown that every graph $G \in \mathcal{G}_{\mathrm{T}}$ satisfies $\chi(G) \leq \left\lfloor \frac{3}{2}\omega(G) \right\rfloor$. In the present paper, we improve this χ-bounding function, and in fact, we find the optimal χ-bounding function for the class $\mathcal{G}_{\mathrm{T}}$ (see Theorem 6.15).

Finally, we consider Hadwiger's conjecture. Let H be a graph with vertex set $V(H) = \{v_1, \dots, v_n\}$. We say that a graph G *contains H as a minor* if there exist pairwise disjoint, nonempty subsets $S_1, \dots, S_n \subseteq V(G)$ (called *branch sets*) such that $G[S_1], \dots, G[S_n]$ are all connected, and for all distinct $i, j \in \{1, \dots, n\}$ such that $v_i v_j \in E(H)$, there is at least one edge between S_i and S_j in G. As usual, the complete graph on k vertices is denoted by K_k. Hadwiger's conjecture states that every graph G contains $K_{\chi(G)}$ as a minor. Using Theorem 1.2, we prove that rings satisfy Hadwiger's conjecture, and as a corollary, we obtain that graphs in $\mathcal{G}_{\mathrm{T}}$ also satisfy Hadwiger's conjecture (see Theorem 7.4).

A *hyperantihole* is a graph A whose vertex set can be partitioned into nonempty cliques $X_1, \dots, X_k$ ($k \geq 4$) such that for all $i \in \{1, \dots, k\}$, X_i is complete to $V(A) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ and anticomplete to $X_{i-1} \cup X_{i+1}$.[4] Under these circumstances, we also say that the hyperantihole A is of *length* k, and that A is a k-hyperantihole. A hyperhole is *odd* or *even* depending on the parity of its length.

The remainder of this paper is organized as follows. In section 2, we state a few results from the literature that we need in the remainder of the paper, and we also prove a few easy lemmas about rings and their induced subgraphs. In section 3, we prove Theorem 1.2, and we also give a polynomial-time algorithm for coloring even rings (see Lemma 3.2). In section 4, we construct an $O(n^3)$ time algorithm that computes the chromatic number of a ring (see Theorem 4.2),[5] and more generally, we construct an

[4] Note that the complement of a hyperantihole need not be a hyperhole.

[5] In fact, our algorithm computes the chromatic number of graphs in $\mathcal{R}_{\geq 4}$. By Lemma 2.8, $\mathcal{R}_{\geq 4}$ contains all rings.

$O(n^5)$ time algorithm that computes the chromatic number of graphs in $\mathcal{G}_{\mathrm{T}}$ (see Theorem 4.3). In section 5, we give an $O(n^6)$ time algorithm for coloring rings (see Theorem 5.2).[6] Even rings are easy to color (see Lemma 3.2); our coloring algorithm for odd rings relies on the ideas from the proof of Theorem 1.2, and it also uses the algorithm from Theorem 4.2 as a subroutine. Using our coloring algorithm for rings, as well as various results from the literature, we also construct an $O(n^7)$ time coloring algorithm for graphs in $\mathcal{G}_{\mathrm{T}}$ (see Theorem 5.3). In section 6, we obtain the optimal χ-bounding function for the class $\mathcal{G}_{\mathrm{T}}$ (see Theorem 6.15). Furthermore, in section 6, for each odd integer $k \geq 5$, we obtain the optimal bound for the chromatic number in terms of the clique number for k-hyperholes and k-hyperantiholes.[7] Finally, in section 7, we prove Hadwiger's conjecture for the class $\mathcal{G}_{\mathrm{T}}$ (see Theorem 7.4).

2 A few preliminary lemmas

In this section, we state a few results from the literature, which we use later in the paper. We also prove a few easy results about rings and their induced subgraphs.

Given a graph G and distinct vertices $u, v \in V(G)$, we say that u *dominates* v in G whenever $N_G[v] \subseteq N_G[u]$. The following lemma was stated without proof in [1] (see Lemma 1.4 from [1]); it readily follows from the definition of a ring, as the reader can check.

Lemma 2.1. *[1] Let G be a graph, and let $(X_1, \dots, X_k)$, with $k \geq 4$, be a partition of $V(G)$. Then G is a k-ring with good partition $(X_1, \dots, X_k)$ if and only if all the following hold:*

(a) $X_1, \dots, X_k$ are cliques;

(b) for all $i \in \mathbb{Z}_k$, X_i is anticomplete to $V(G) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$;

(c) for all $i \in \mathbb{Z}_k$, some vertex of X_i is complete to $X_{i-1} \cup X_{i+1}$;

(d) for all $i \in \mathbb{Z}_k$, and all distinct $y_i, y'_i \in X_i$, one of y_i, y'_i dominates the other.

Rings can be recognized in polynomial time. More precisely, the following is Lemma 8.14 from [1]. (In all our algorithms, n denotes the number of vertices and m the number of edges of the input graph.)

Lemma 2.2. *[1] There exists an algorithm with the following specifications:*

[6] In fact, this is a coloring algorithm for graphs in $\mathcal{R}_{\geq 4}$.

[7] We only defined χ-boundedness for hereditary classes, and so, technically, these are not "χ-bounding functions" for the classes k-hyperholes and k-hyperantiholes. They are, however, optimal χ-bounding functions for the closures of these classes under induced subgraphs. See section 6 for the details.

- *Input: A graph G;*
- *Output: Either the true statement that G is a ring, together with the length and ring partition of the ring, or the true statement that G is not a ring;*
- *Running time: $O(n^2)$.*

As an easy corollary of the lemma above, we can obtain Lemma 2.3 (below). We remark that the proof of (but not the statement of) Lemma 8.14 from [1] in fact gives precisely Lemma 2.3. For the sake of completeness, we give a full proof.

Lemma 2.3. *There exists an algorithm with the following specifications:*

- *Input: A graph G;*
- *Output: Exactly one of the following:*
  - *the true statement that G is a ring, together with the length k and a ring partition $(X_1, \dots, X_k)$ of the ring G, and for each $i \in \{1, \dots, k\}$, an ordering $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ of X_i such that $X_i \subseteq N_G[u_i^{|X_i|}] \subseteq \dots \subseteq N_G[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$,*
  - *the true statement that G is not a ring;*
- *Running time: $O(n^2)$.*

Proof. We first run the algorithm from Lemma 2.2 with input G; this takes $O(n^2)$ time. If the algorithm returns the statement that G is not a ring, then we return this statement as well, and we stop. So assume that the algorithm returned the statement that G is a ring, together with the length k and ring partition $(X_1, \dots, X_k)$ of the ring. We then find the degrees of all vertices of G, and for each $i \in \{1, \dots, k\}$, we order X_i as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $\deg_G(u_i^1) \geq \dots \geq \deg_G(u_i^{|X_i|})$; this takes $O(n^2)$ time. Since we already know that $(X_1, \dots, X_k)$ is a ring partition of G, it is easy to see that for all $i \in \{1, \dots, k\}$, we have that $X_i \subseteq N_G[u_i^{|X_i|}] \subseteq \dots \subseteq N_G[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. We now return the statement that G is a ring of length k, the ring partition $(X_1, \dots, X_k)$ of G, and for each $i \in \{1, \dots, k\}$, the ordering $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ of X_i, and we stop. Clearly, the algorithm is correct, and its running time is $O(n^2)$. □

Next, Lemma 1.5 and Theorem 8.25 from [1] readily imply the following.

Lemma 2.4. *[1] There exists an algorithm with the following specifications:*

- *Input: A graph G;*

- *Output: Either a maximum clique C of G, or the true statement that G is not an induced subgraph of a ring;*

- *Running time: $O(n^3)$.*

We remind the reader that a *simplicial vertex* is a vertex whose neighborhood is a (possibly empty) clique. A *simplicial elimination ordering* of a graph G is an ordering $v_1, \dots, v_n$ of the vertices of G such that for all $i \in \{1, \dots, n\}$, v_i is simplicial in the graph $G[v_i, v_{i+1}, \dots, v_n]$. A *chordal graph* is a graph that contains no holes. (Equivalently, a *chordal graph* is a graph all of whose induced cycles are triangles.) It is well known (and easy to show) that a graph is chordal if and only if it has a simplicial elimination ordering (see [6]). Furthermore, there is an $O(n + m)$ time algorithm that either produces a simplicial elimination ordering of the input graph, or determines that the graph is not chordal [11]. Recall that a graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$; it is well known (and easy to show) that chordal graphs are perfect [3].

The following algorithm is a minor modification of the algorithm described in the introduction of [8].[8]

Lemma 2.5. *There exists an algorithm with the following specifications:*

- *Input: A graph G;*

- *Output: A maximal sequence $v_1, \dots, v_t$ ($t \geq 0$) of vertices of G such that for all $i \in \{1, \dots, t\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$;*

- *Running time: $O(n^3)$.*

Proof. **Step 0.** First, for all distinct $x, y \in V(G)$, we set

$$\text{diff}(x, y) \quad = \quad \begin{cases} |N_G[x] \setminus N_G[y]| & \text{if} \quad xy \in E(G) \\ \\ 0 & \text{if} \quad xy \notin E(G) \end{cases}$$

Computing $\text{diff}(x, y)$ for all possible choices of distinct $x, y \in V(G)$ can be done in $O(n^3)$ time. We will update $\text{diff}(x, y)$ as the algorithm proceeds. Note that a vertex $x \in V(G)$ is simplicial in G if and only if for all $y \in V(G) \setminus \{x\}$, we have that $\text{diff}(x, y) = 0$. We also let L be the empty list. We now go to Step 1.

Step 1. We first check if there is a vertex $x \in V(G)$ such that for all $y \in V(G) \setminus \{x\}$, we have that $\text{diff}(x, y) = 0$; this can be done in $O(n^2)$ time.

[8] The algorithm from [8] produces a maximal sequence $v_1, \dots, v_t$ ($t \geq 0$) of vertices of the input graph G such that for all $i \in \{1, \dots, t\}$, v_i is simplicial in either $G \setminus \{v_1, \dots, v_{i-1}\}$ or $\overline{G} \setminus \{v_1, \dots, v_{i-1}\}$. Thus, the algorithm from Lemma 2.5 is in fact obtained from the algorithm from [8] by omitting some steps. The running time of the two algorithms is the same. For the sake of completeness, we give all the details for the algorithm that we need (i.e. the algorithm from Lemma 2.5).

If we found no such vertex, then G has no simplicial vertices; in this case, we return the list L and stop. Suppose now that we found such a vertex x. First, we set $L := L, x$ (i.e. we update L by adding x to the end of L). Then, for all distinct $x', y \in V(G) \setminus \{x\}$, we update $\text{diff}(x', y)$ as follows: if $x \in N_G[x'] \setminus N_G[y']$, then we set $\text{diff}(x', y) := \text{diff}(x', y) - 1$, and otherwise, we do not change $\text{diff}(x', y)$; this update takes $O(n^2)$ time. We now go to Step 1 with input $G \setminus x$, L, and $\text{diff}(x', y)$ for all distinct $x', y \in V(G) \setminus \{x\}$.

Clearly, the algorithm terminates and is correct. Step 0 takes $O(n^3)$ time. We make $O(n)$ calls to Step 1, and otherwise, the slowest step of Step 1 takes $O(n^2)$ time. Thus, the total running time of the algorithm is $O(n^3)$. □

The following is Lemma 2.4(a)-(c) from [1].

Lemma 2.6. *[1] Let R be a k-ring with ring partition $(X_1, \dots, X_k)$. Then all the following hold:*

(a) every hole in R intersects each of $X_1, \dots, X_k$ in exactly one vertex;

(b) every hole in R is of length k;

(c) for all $i \in \{1, \dots, k\}$, $R \setminus X_i$ is chordal.

Note that Lemma 2.6(b) implies that, for an integer $k \geq 4$, every hyperhole in a k-ring is of length k.

Lemma 2.7. *Let $k \geq 4$ be an integer. Then every induced subgraph of a k-ring either contains a simplicial vertex or is a k-ring. More precisely, let R be a k-ring with ring partition $(X_1, \dots, X_k)$, and let $Y \subseteq V(R)$ be a nonempty set. Then either $R[Y]$ contains a simplicial vertex, or $R[Y]$ is a k-ring with ring partition $(X_1 \cap Y, \dots, X_k \cap Y)$.*

Proof. For all $i \in \{1, \dots, k\}$, we set $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a ring. For all $i \in \{1, \dots, k\}$, set $Y_i = X_i \cap Y$. If at least one of $Y_1, \dots, Y_k$ is empty, then Lemma 2.6(c) implies that $R[Y]$ is chordal, and consequently (by [6]), $R[Y]$ contains a simplicial vertex. So from now on, we assume that $Y_1, \dots, Y_k$ are all nonempty.

For all $i \in \{1, \dots, k\}$, let $j_i \in \{1, \dots, |X_i|\}$ be maximal with the property that $u_i^{j_i} \in Y_i$; then $u_i^{j_i}$ is dominated in $R[Y]$ by all other vertices in Y_i. If for some $i \in \{1, \dots, k\}$, $u_i^{j_i}$ is anticomplete to Y_{i-1} or Y_{i+1}, then Lemma 2.1 readily implies that $u_i^{j_i}$ is a simplicial vertex of $R[Y]$, and we are done; otherwise, Lemma 2.1 implies that $R[Y]$ is a ring with ring partition $(Y_1, \dots, Y_k)$. □

Lemma 2.8. *For all integers $k \geq 4$, both the following hold:*

- *the class $\mathcal{R}_k$ is hereditary and contains all k-rings;*

- *the class $\mathcal{R}_{\geq k}$ is hereditary and contains all rings of length at least k;*

In particular, the class $\mathcal{R}_{\geq 4}$ is hereditary and contains all rings.

Proof. This follows immediately from Lemma 2.7 and from the relevant definitions. $\square$

The following lemma (Lemma 2.9) will not be used in the remainder of the paper, but the reader may find it informative.

Lemma 2.9. *Let $k \geq 4$ be an integer, and let G be a graph. Then the following are equivalent:*

(a) $G \in \mathcal{R}_k$;

(b) either G is chordal, or G can be obtained from a k-ring by repeatedly adding simplicial vertices.

Proof. The fact that (a) implies (b) follows from the definition of a ring. The reverse implication follows from the definition of $\mathcal{R}_k$, from Lemma 2.8, and from the fact that chordal graphs are precisely those graphs that have a simplicial elimination ordering [6]. $\square$

Lemma 2.10. *Let G be a graph on at least two vertices, and let $v \in V(G)$ be a simplicial vertex. Then $\omega(G) = \max\{|N_G[v]|, \omega(G \setminus v)\}$ and $\chi(G) = \max\{\omega(G), \chi(G \setminus v)\}$.*

Proof. We first show that $\omega(G) = \max\{|N_G[v]|, \omega(G \setminus v)\}$. Since v is simplicial, $N_G[v]$ is a clique, and we readily deduce that $\max\{|N_G[v]|, \omega(G \setminus v)\} \leq \omega(G)$. To prove the reverse inequality, let K be a clique of size $\omega(G)$ in G. If $v \notin K$, then K is a clique in $G \setminus v$, and so $\omega(G) = |K| \leq \omega(G \setminus v) \leq \max\{|N_G[v]|, \omega(G \setminus v)\}$. So suppose that $v \in K$. Since K is a clique, it follows that $K \subseteq N_G[v]$, and so $\omega(G) = |K| \leq |N_G[v]| \leq \max\{|N_G[v]|, \omega(G \setminus v)\}$. This proves that $\omega(G) = \max\{|N_G[v]|, \omega(G \setminus v)\}$.

It remains to show that $\chi(G) = \max\{\omega(G), \chi(G \setminus v)\}$. It is clear that $\max\{\omega(G), \chi(G \setminus v)\} \leq \chi(G)$. For the reverse inequality, we set $\ell = \max\{\omega(G), \chi(G \setminus v)\}$, and we construct a proper coloring of G that uses at most ℓ colors. First, we properly color $G \setminus v$ with colors $1, \dots, \ell$. Next, since $N_G[v]$ is a clique, we see that $|N_G(v)| = |N_G[v]| - 1 \leq \omega(G) - 1 \leq \ell - 1$; thus, at least one of our ℓ colors was not used on $N_G(v)$, and we can assign this "unused" color to v. This produces a proper coloring of G that uses at most ℓ colors, and we are done. $\square$

We complete this section by stating the decomposition theorem for the class $\mathcal{G}_{\mathrm{T}}$ proven in [1] (this is Theorem 1.7 from [1]).

Theorem 2.11. *[1] Let $G \in \mathcal{G}_T$. Then one of the following holds:*

- *G is a complete graph, a ring, or a 7-hyperantihole;*
- *G admits a clique-cutset.*

Finally, we remark that graphs in $\mathcal{G}_{\rm T}$ can be recognized in polynomial time [1], but we do not need this result in the remainder of the paper.

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We begin with an easy lemma.

Lemma 3.1. *Let R be a k-ring (with $k \geq 4$) such that $\chi(R) = \omega(R)$. Then R contains a k-hyperhole H such that $\chi(H) = \chi(R)$.*

Proof. Let $(X_1, \dots, X_k)$ be a ring partition of R, and for all $i \in \{1, \dots, k\}$, let $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ be an ordering of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a ring. Let Q be a clique of size $\omega(R)$ in R. By the definition of a ring, and by symmetry, we may assume that $Q \subseteq X_1 \cup X_2$. Since u_1^1 is complete to X_2, and since u_2^1 is complete to X_1, the maximality of Q guarantees that $u_1^1, u_2^1 \in Q$. Set $H = R[Q \cup \{u_3^1, u_4^1, \dots, u_k^1\}]$. Clearly, H is a k-ring, and $\chi(H) \leq \chi(R)$. On the other hand, since H contains a clique (namely Q) of size $\omega(R)$, we see that $\chi(H) \geq \omega(R)$. Since $\chi(R) = \omega(R)$, we deduce that $\chi(H) = \chi(R)$. $\square$

In view of Lemma 3.1, our next lemma (Lemma 3.2) shows that Theorem 1.2 holds for even rings. We will also rely on Lemma 3.2 in our coloring algorithm for rings in section 5.

Lemma 3.2. *Even rings are perfect.*[9] *Furthermore, there exists an algorithm with the following specifications:*

- *Input: A graph G;*
- *Output: Either an optimal coloring of G, or the true statement that G is not an even ring;*
- *Running time: $O(n^3)$.*

Proof. We begin by constructing the algorithm. We first call the algorithm from Lemma 2.3 with input G; this takes $O(n^2)$ time. If the algorithm returns the answer that G is not a ring, then we return the answer that G is not an even ring, and we stop. So from now on, we assume that the algorithm returned all the following:

[9] We remind the reader that a graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$. In particular, every perfect graph G satisfies $\chi(G) = \omega(G)$. The fact that even rings are perfect easily follows from the Strong Perfect Graph Theorem [2]. However, here we give an elementary proof of this fact.

- the true statement that G is a ring;
- the length k and a ring partition $(X_1, \dots, X_k)$ of the ring G;
- for each $i \in \{1, \dots, k\}$, an ordering $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ of X_i such that $X_i \subseteq N_G[u_i^{|X_i|}] \subseteq \cdots \subseteq N_G[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$.

If k is odd, then we return the answer that G is not an even ring, and we stop. So assume that k is even. We then compute $\omega(G)$ by running the algorithm from Lemma 2.4 with input G; this takes $O(n^3)$ time. We now color G as follows. For all odd $i \in \{1, \dots, k\}$ and all $j \in \{1, \dots, |X_i|\}$, we assign color j to the vertex u_i^j; and for all even $i \in \{1, \dots, k\}$ and all $j \in \{1, \dots, |X_i|\}$, we assign color $\omega(G) - j + 1$ to the vertex u_i^j. Since $|X_i| \leq \omega(R)$ for all $i \in \{1, \dots, k\}$, we see that our coloring uses only colors $1, \dots, \omega(R)$. Let us show that the coloring is proper. Suppose otherwise. By Lemma 2.1(b) and symmetry, we may assume that there exist some $i \in \{1, \dots, k\}$, $j \in \{1, \dots, |X_i|\}$, and $\ell \in \{1, \dots, |X_j|\}$ such that u_i^j and u_{i+1}^ℓ are adjacent in R and were assigned the same color. Since u_i^j and u_{i+1}^ℓ are adjacent, the definition of a ring implies that $\{u_i^1, \dots, u_i^j\}$ and $\{u_{i+1}^1, \dots, u_{i+1}^\ell\}$ are cliques, complete to each other; thus, $\{u_i^1, \dots, u_i^j\} \cup \{u_{i+1}^1, \dots, u_{i+1}^\ell\}$ is a clique, and consequently, $j + \ell \leq \omega(R)$. On the other hand, by construction, we have that:

- if i is odd, then u_i^j received color j, and u_{i+1}^ℓ received color $\omega(R) - \ell + 1$;
- if i is even, then u_i^j received color $\omega(R) - j + 1$, and u_{i+1}^ℓ received color ℓ.

Since u_i^j and u_{i+1}^ℓ received the same color, it follows that either $j = \omega(R) - \ell + 1$ or $\omega(R) - j + 1 = \ell$; in either case, we get that $j + \ell = \omega(R) + 1$, contrary to the fact that $j + \ell \leq \omega(R)$. This proves that our coloring of G is indeed proper. Furthermore, as pointed out above, this coloring uses at most $\omega(G)$ colors. Since $\omega(G) \leq \chi(G)$, we deduce that our coloring is optimal, and that $\chi(G) = \omega(G)$. We now return this coloring of G, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. Note, furthermore, that we have established that all even rings R satisfy $\chi(R) = \omega(R)$. The fact that even rings are perfect now follows from Lemmas 2.7 and 2.10 by an easy induction. □

As we pointed out above, Lemmas 3.1 and 3.2 together imply that even rings satisfy Theorem 1.2. We devote the remainder of the section to proving Theorem 1.2 for odd rings.

Given a graph G, a coloring c of G, and distinct colors a, b used by c, we set $R_{G,c}^{a,b} = G[\{x \in V(G) \mid c(x) = a \text{ or } c(x) = b\}]$;[10] note that if c is a

[10]Thus, $R_{G,c}^{a,b}$ is the subgraph of G induced by the vertices colored a or b.

proper coloring of G, then $R^{a,b}_{G,c}$ is a bipartite graph, and if G contains no even holes, then $R^{a,b}_{G,c}$ is a forest. Furthermore, we have the following lemma.

Lemma 3.3. *Let $k \geq 5$ be an odd integer, let R be a k-ring with ring partition $(X_1, \dots, X_k)$, let G be an induced subgraph of R, let c be a proper coloring of G, let a, b be distinct colors used by c, and let Q be any component of $R^{a,b}_{G,c}$. Then there are integers $i, j \in \{1, \dots, k\}$ such that $V(Q) \subseteq X_i \cup X_{i+1} \cup \dots \cup X_{j-1} \cup X_j$,[11] and Q consists of a path $p_i, \dots, p_j$, where $p_\ell \in X_\ell$ for all $\ell \in \{i, \dots, j\}$, plus, optionally for each ℓ, a vertex $p'_\ell \in X_\ell$ with $N_R[p'_\ell] \subseteq N_R[p_\ell]$ and $N_Q(p'_\ell) = \{p_\ell\}$.*

Proof. By Lemma 2.6, all holes in R are of length k, and in particular, R contains no even holes. The result now readily follows from the relevant definitions. $\square$

Here, we need a few more definitions. Let R be a ring with ring partition $(X_1, \dots, X_k)$, and for each $i \in \{1, \dots, k\}$, let $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ be an ordering of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a ring. For all $i \in \{1, \dots, k\}$ and $j, \ell \in \{1, \dots, |X_i|\}$ such that $j \leq \ell$ (resp. $j < \ell$), we say that u_i^j is *lower* (resp. *strictly lower*) than u_i^ℓ, and that u_i^ℓ is *higher* (resp. *strictly higher*) than u_i^j; under these circurmstances, we also write $u_i^j \leq u_i^\ell$ (resp. $u_i^j < u_i^\ell$) and $u_i^\ell \geq u_i^j$ (resp. $u_i^\ell > u_i^j$). For each $i \in \{1, \dots, k\}$ let $s_i = u_i^1$ and $t_i = u_i^{|X_i|}$.[12] Further, suppose that c is a proper coloring of $R \setminus t_2$. For all $X \subseteq V(R) \setminus \{t_2\}$, set $c(X) = \{c(x) \mid x \in X\}$. Given distinct colors $a, b \in c(V(R) \setminus \{t_2\})$ and an index $i \in \{1, \dots, k\}$, we say that a is *lower* than b in X_i, and that b is *higher* than a in X_i, provided that either

- $b \notin c(X_i)$, or
- there exist indices $j, \ell \in \{1, \dots, |X_i|\}$ such that $j < \ell$, $c(u_i^j) = a$, and $c(u_i^\ell) = b$.

Let $c_1 = c(s_1)$.[13] We say that c is *unimprovable* if for all colors $a \in c(V(R) \setminus \{t_2\})$ such that $a \neq c_1$, and all components Q of $R^{c_1,a}_{R \setminus t_2, c}$ that do not contain s_1, both the following are satisfied:

- for all odd $i \in \{3, \dots, k\}$ such that Q intersects X_i, c_1 is lower than a in X_i.
- for all even $i \in \{3, \dots, k\}$ such that Q intersects X_i, c_1 is higher than a in X_i;

[11] As usual, subscripts are understood to be modulo k.

[12] Thus, s_i is the lowest and t_i the highest vertex in X_i. Note that this means that s_i is the highest-degree and t_i the lowest-degree vertex in X_i.

[13] Note that this means that $c_1 \notin c(X_2 \setminus \{t_2\})$. This is because $c(s_1) = c_1$, s_1 is complete to X_2 in R, and c is a proper coloring of $R \setminus t_2$.

We remark that if c is an unimprovable coloring of $R \setminus t_2$, then by definition, c is a proper coloring of $R \setminus t_2$, but it need not be an optimal coloring of $R \setminus t_2$, i.e. it may possibly use more than $\chi(R \setminus t_2)$ colors.

Our next lemma shows that any proper coloring of $R \setminus t_2$ (where R and t_2 are as above) can be turned into an unimprovable coloring that uses no more colors than the original coloring of $R \setminus t_2$.[14]

Lemma 3.4. *There exists an algorithm with the following specifications:*

- *Input: An odd ring R with ring partition $(X_1, \dots, X_k)$, for each $i \in \{1, \dots, k\}$, an ordering $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, and a proper coloring c of $R \setminus u_2^{|X_2|}$;*

- *Output: An unimprovable coloring of $R \setminus u_2^{|X_2|}$ that uses no more colors than c does;*

- *Running time: $O(n^4)$.*

Proof. To simplify notation, for all $i \in \{1, \dots, k\}$, we set $s_i = u_i^1$ and $t_i = u_i^{|X_i|}$. (Thus, c is a proper coloring of $R \setminus t_2$.) Let r be the number of colors used by c; by symmetry, we may assume that $c : V(R) \setminus \{t_2\} \to \{1, \dots, r\}$. Set $c_1 = c(s_1)$.

Now, for every proper coloring $\widetilde{c} : V(R) \setminus \{t_2\} \to \{1, \dots, r\}$ of $R \setminus t_2$ such that $\widetilde{c}(s_1) = c_1$,[15] we define the *rank* of $\widetilde{c}$, denoted by $\mathrm{rank}(\widetilde{c})$, as follows.

- For all odd $i \in \{3, \dots, k\}$, if there exists an index $j \in \{1, \dots, |X_i|\}$ such that $\widetilde{c}(u_i^j) = c_1$,[16] then we set $r_i(\widetilde{c}) = j$, and otherwise, we set $r_i(\widetilde{c}) = |X_i| + 1$.

- For all even $i \in \{3, \dots, k\}$, if there exists an index $j \in \{1, \dots, |X_i|\}$ such that $\widetilde{c}(u_i^j) = c_1$,[17] then we set $r_i(\widetilde{c}) = |X_i| - j + 2$, and otherwise, we set $r_i(\widetilde{c}) = 1$.

- We set $\mathrm{rank}(\widetilde{c}) = \sum\limits_{i=3}^{k} r_i(\widetilde{c})$.[18]

[14]In particular, this implies that if $R \setminus t_2$ is r-colorable, then there exists an unimprovable coloring of $R \setminus t_2$ that uses at most r colors.

[15]Note that this implies that $c_1 \notin \widetilde{c}(X_2 \setminus \{t_2\})$. This is because $\widetilde{c}(s_1) = c_1$, s_1 is complete to X_2 in R, and $\widetilde{c}$ is a proper coloring of $R \setminus t_2$.

[16]Note that if j exists, then it is unique. This is because X_i is a clique of $R \setminus t_2$, and $\widetilde{c}$ is a proper coloring of $R \setminus t_2$.

[17]As before, if j exists, then it is unique.

[18]Note that $k - 2 \leq \mathrm{rank}(\widetilde{c}) \leq k - 2 + \sum\limits_{i=3}^{k} |X_i|$.

Note that if $\widetilde{c} : V(R) \setminus \{t_2\} \to \{1, \dots, r\}$ is a proper coloring of $R \setminus t_2$ with $\widetilde{c}(s_1) = c_1$, then $\widetilde{c}$ is unimprovable if and only if for all $a \in \{1, \dots, r\} \setminus \{c_1\}$, and all components Q of $R^{c_1,a}_{R\setminus t_2,c}$ that do not contain s_1, the coloring c' of $R \setminus \{t_2\}$ obtained from c by swapping colors c_1 and a on Q, satisfies $\text{rank}(c') \geq \text{rank}(c)$.

Algorithmically, our goal is to recursively transform c until we obtain an unimprovable coloring of $R \setminus t_2$ that uses only colors from the set $\{1, \dots, r\}$.

First, for all colors $a \in \{1, \dots, r\} \setminus \{c_1\}$, we form the graph $R^{c_1,a}_{R\setminus t_2,c}$, we find all components of this graph, and for each component $Q(c_1, a)$ of $R^{c_1,a}_{R\setminus t_2,c}$ that does not contain s_1, we form the coloring $c_{Q(c_1,a)}$ by starting with c and then swapping colors c_1 and a on $Q(c_1, a)$, and finally, we check whether $\text{rank}(c_{Q(c_1,a)}) < \text{rank}(c)$. Doing this for all possible choices of a and $Q(c_1, a)$ takes $O(n^3)$ time. If for some color $a \in C \setminus \{c_1\}$, and some component $Q(c_1, a)$ of $R^{c_1,a}_{R\setminus t_2,c}$ that does not contain s_1, we found that $\text{rank}(c_{Q(c_1,a)}) < \text{rank}(c)$, then we update c as $c := c_{Q(c_1,a)}$, and we repeat the process. Otherwise, we return c, and we stop.

The algorithm stops because the rank of the coloring decreases after each iteration. Clearly, the algorithm is correct. Since each iteration takes $O(n^3)$ time, and there are $O(n)$ iterations, we see that the total running time of the algorithm is $O(n^4)$. □

We now prove a technical lemma (Lemma 3.5) that is the heart of our proof of Theorem 1.2 for odd rings. We also rely on Lemma 3.5 in our coloring algorithm for rings.[19] We remark that in our proof of Lemma 3.5, we repeatedly rely on Lemma 3.3 without explicitly stating this.[20]

Lemma 3.5. *Let $k \geq 5$ be an odd integer, let R be a k-ring with ring partition $(X_1, \dots, X_k)$, and for each $i \in \{1, \dots, k\}$, let $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ be an ordering of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. For all $i \in \{1, \dots, k\}$, set $s_i = u_i^1$ and $t_i = u_i^{|X_i|}$. Let c be an unimprovable coloring of $R \setminus t_2$, let r be the number of colors used by c.*[21] *Let $c_1 = c(s_1)$, and let $S = \{x \in V(R) \mid x \neq t_2, c(x) = c_1\}$.*[22] *Then both the following hold:*

(a) either $\omega(R \setminus S) \leq r - 1$, or R contains a k-hyperhole of chromatic number $r + 1$;

[19] More precisely, our coloring algorithm for rings relies on Lemma 3.6, which is an easy corollary of Lemma 3.5 and Theorem 1.2.

[20] Essentially, every time we consider a component Q as in Lemma 3.3, we keep in mind the structure of Q, as described in Lemma 3.3.

[21] In particular, the coloring c is proper and $r \geq \chi(R \setminus t_2)$, and this inequality may possibly be strict.

[22] Note that this implies that S is a stable set in $R \setminus t_2$.

(b) if every k-ring R' such that $|V(R')| < |V(R)|$ contains a k-hyperhole of chromatic number $\chi(R')$, then either $\chi(R \setminus S) \leq r-1$, or R contains a k-hyperhole of chromatic number $r+1$.

Proof. By hypotheses, we have that $\chi(R \setminus t_2) \leq r$; it follows that $\omega(R) \leq \chi(R) \leq r+1$. If $\omega(R) = r+1$, then both (a) and (b) follow from Lemma 3.1; thus, we may assume that $\omega(R) \leq r$.

Set $Y_1 = N_R(t_2) \cap X_1$, $X'_2 = X_2 \setminus \{t_2\}$, and $Y_3 = N_R(t_2) \cap X_3$. Note that $N_R(t_2) = Y_1 \cup X'_2 \cup Y_3$, with Y_1, X'_2, Y_3 pairwise disjoint. Note, furthermore, that $Y_1 \cup X_2$ and $X_2 \cup Y_3$ are maximal cliques of R. Let C be the set of colors used by c; then $|C| = r$. To simplify notation, for all distinct colors $a, b \in C$, we write $R^{a,b}$ instead of $R^{a,b}_{R \setminus t_2, c}$.

Claim 1. *Either $\omega(R \setminus S) \leq r-1$, or R contains a k-hyperhole of chromatic number $r+1$. In other words, (a) holds.*

Proof of Claim 1. Recall that $\omega(R \setminus S) \leq r$. Thus, we may assume that $\omega(R \setminus S) = r$, for otherwise we are done. Since c is a proper coloring of $R \setminus t_2$ that uses only r colors, and since S is a color class of the coloring c, we see that S intersects all cliques of size r in R that do not contain t_2. Furthermore, there are exactly two maximal cliques in R that contain t_2, namely $Y_1 \cup X_2$ and $X_2 \cup Y_3$. Since S intersects $Y_1 \cup X_2$ (because $s_1 \in Y_1 \cap S$), we deduce that $X_2 \cup Y_3$ is the unique clique of $R \setminus S$ of size r. (Note that this implies that $X'_2 \cup Y_3$ is a clique of size $r-1$.) In particular, $c_1 \notin c(X'_2 \cup Y_3)$.

Consider any color $a \in c(Y_3)$, and let Q be the component of $R^{c_1,a}$ that contains the vertex of Y_3 colored a. If $s_1 \notin V(Q)$, then by swapping colors c_1 and a on Q, we obtain a coloring that contradicts the fact that c is unimprovable. Thus, $s_1 \in V(Q)$. It follows that the following hold:

- for every odd $i \neq 1$, we have that $c(Y_3) \subseteq c(X_i)$;
- for every even $i \neq 2$, some vertex of X_i is colored c_i, and furthermore, this vertex is adjacent to all vertices of $X_{i-1} \cup X_{i+1}$ that received a color used on Y_3.

For odd $i \geq 5$, let h_i be the highest indexed vertex of X_i that is adjacent both to the vertex of X_{i-1} colored c_1, and to the vertex of X_{i+1} colored c_1. Let $Z_1 = \{s_1\}$, $Z_2 = X_2$, $Z_3 = Y_3$. For all even $i \geq 4$, let Z_i be the set that consists of the vertex of X_i colored c_1, and all the vertices of X_i lower than that vertex. For all odd $i \geq 5$, let Z_i consist of h_i and all the vertices in X_i lower than it. Let $H = R[Z_1 \cup Z_2 \cup \cdots \cup Z_k]$. By construction, H is a k-hyperhole of R; thus, $\chi(H) \leq \chi(R) \leq r+1$. If $\chi(H) = r+1$, then we are done. So assume that $\chi(H) \leq r$. Then $\left\lceil \frac{2|V(H)|}{k-1} \right\rceil = \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil \leq \chi(H) \leq r$. It follows that $|V(H)| \leq \frac{k-1}{2}r$, and consequently, $|V(H) \setminus \{t_2\}| < \frac{k-1}{2}r$. Now, $X_2' \cup Y_3$ is a clique of size $r-1$, and so $|c(X_2' \cup Y_3)| = r-1$. Furthermore, we know that $c_1 \notin c(X_2' \cup Y_3)$, and so since c uses precisely r colors, it follows that $|\{c_1\} \cup c(X_2' \cup Y_3)| = r$. Since $|V(H) \setminus \{t_2\}| < \frac{k-1}{2}r$, we see that some color from $\{c_1\} \cup c(X_2' \cup Y_3)$ appears on fewer than $\frac{k-1}{2}$ vertices of $V(H) \setminus \{t_2\}$; since t_2 is not colored by c, it follows that some color from $\{c_1\} \cup c(X_2' \cup Y_3)$ appears on fewer than $\frac{k-1}{2}$ vertices of H. Now, by construction, every color from $\{c_1\} \cup c(Y_3)$ appears $\frac{k-1}{2}$ times on H. It follows that some color $d \in c(X_2')$ appears fewer than $\frac{k-1}{2}$ times on H. Thus, there exists some even $i \geq 4$ such that either d does not appear on X_i, or d appears higher than c_1 in X_i; let i be the smallest such index. Thus, d appears on each Z_j, for even $j < i$, and there are $\frac{i}{2} - 1$ such j's. On the other hand, let Q be the component of $R^{c_1,d}$ that contains the vertex of X_i colored c_1. If $s_1 \notin V(Q)$, then we swap colors c_1 and d on Q, thus obtaining a coloring of $R \setminus t_2$ that contradicts the fact that c is unimprovable. Thus, $s_1 \in V(Q)$, and it follows that each Z_j, for odd $j > i$, contains a vertex colored d; there are $\lceil \frac{k-i}{2} \rceil = \frac{k-i+1}{2}$ such j's. In total, we get that at least $(\frac{i}{2} - 1) + \frac{k-i+1}{2} = \frac{k-1}{2}$ vertices of H are colored d, contrary to our choice of d. ■

It remains to prove (b). For this, we assume that

- every k-ring R' such that $|V(R')| < |V(R)|$ contains a k-hyperhole of chromatic number $\chi(R')$,
- $\chi(R \setminus S) \geq r$,

and we prove that R contains a k-hyperhole of chromatic number $r+1$.

Since S is a color class of a proper coloring of $R \setminus t_2$ that uses at most r colors, we see that $\chi\Big(R \setminus (S \cup \{t_2\})\Big) \leq r-1$; consequently, $\chi(R \setminus S) \leq r$. Since $\chi(R \setminus S) \geq r$, it follows that $\chi(R \setminus S) = r$. Further, in view of (a), we may assume that $\omega(R \setminus S) \leq r-1$.

Claim 2. *$R \setminus S$ contains a k-hyperhole H such that $\chi(H) = \left\lceil \frac{2|V(H)|}{k-1} \right\rceil = r$.*

Proof of Claim 2. Let $v_1, \dots, v_t$ (with $t \geq 0$) be a maximal sequence of vertices in $R \setminus S$ such that for all $i \in \{1, \dots, t\}$, v_i is simplicial in $R \setminus (S \cup \{v_1, \dots, v_{i-1}\})$. Set $A = \{v_1, \dots, v_t\}$. Suppose first that $R \setminus S = A$. Then $v_1, \dots, v_t$ is a simplicial elimination ordering of $R \setminus S$, and so by coloring $R \setminus S$ greedily using the ordering $v_t, \dots, v_1$, we obtain a proper coloring of $R \setminus S$ that uses only $\omega(R \setminus S)$ colors, contrary to the fact that $\chi(R \setminus S) = r > r-1 \geq \omega(R \setminus S)$. So, $R \setminus S \neq A$. Lemma 2.7 and the maximality of A now imply that $R \setminus (S \cup A)$ is a k-ring. Since $S \neq \emptyset$, the k-ring $R \setminus (S \cup A)$

has fewer vertices than R, and so $R \setminus (S \cup A)$ contains a k-hyperhole H such that $\chi(H) = \chi\Big(R \setminus (S \cup A)\Big)$.

Now, Lemma 2.10 and an easy induction guarantee that

$$\chi(R \setminus S) = \max\Big\{\omega(R \setminus S), \chi\Big(R \setminus (S \cup A)\Big)\Big\}.$$

Since $\chi(R \setminus S) = r$, $\omega(R \setminus S) = r - 1$, and $\chi(H) = \chi\Big(R \setminus (S \cup A)\Big)$, we deduce that $\chi(H) = r$. Since $\omega(H) \leq \omega(R \setminus S) \leq r - 1$, we see that $\omega(H) < \chi(H)$, and so Lemma 1.1 implies that $\chi(H) = \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil = \left\lceil \frac{2|V(H)|}{k-1} \right\rceil$. Thus, $\chi(H) = \left\lceil \frac{2|V(H)|}{k-1} \right\rceil = r$. ■

From now on, let H be as in Claim 2. Our goal is to find a hyperhole in R of size at least $|V(H)| + \frac{k+1}{2}$; this will imply[23] that the chromatic number of that hyperhole is at least $r + 1$,[24] which is what we need.

Recall that $c_1 = c(s_1)$. Let j be the largest odd index such that $c(s_i) = c_1$ for all odd $i \in \{1, \dots, j\}$.

Claim 3. $c_1 \in c(X_i)$ *for every even index* $i \geq j + 3$.

Proof of Claim 3. Suppose otherwise, and fix the smallest even index $i \geq j+3$ such that $c_1 \notin c(X_i)$. If $c(s_{i-1}) = c_1$, then:

- if $i - 1 = j + 2$, then the choice of j is contradicted;
- if $i - 1 \geq j + 4$, then the choice of i is contradicted.[25]

It follows that $c(s_{i-1}) \neq c_1$. Set $c_{i-1} = c(s_{i-1})$, and let Q be the component of $R^{c_1, c_{i-1}}$ that contains s_{i-1}. We know that $c_1, c_{i-1} \notin c(X_i)$, and so $V(Q) \cap X_i = \emptyset$. On the other hand, the parity of i and j implies that $V(Q) \cap X_{j+1} = \emptyset$. Thus, $V(Q) \subseteq X_{j+2} \cup \dots \cup X_{i-1}$. But now by swapping colors c_1 and c_{i-1} on Q, we obtain a coloring of $R \setminus t_2$ that contradicts the fact that c is unimprovable. ■

For all $i \in \{1, \dots, k\} \setminus \{2\}$, if $c_1 \in c(X_i)$, the let $x_i^{c_1}$ be the (unique) vertex of X_i to which c assigns color c_1.[26]

For each $i \in \{1, \dots, k\}$, let h_i be the highest indexed vertex of $X_i \cap V(H)$. Let ℓ be the largest odd index such that for every odd $i \in \{1, \dots, \ell\}$, we have that $c_1 \in c(X_i \cap V(H))$. Clearly, $\ell \geq j$. For $i \in \{1, \dots, \ell + 2\}$, let

[23]The details are given at the end of the proof of the lemma.

[24]Since $\chi(R) \leq r + 1$, we see that any hyperhole in R of chromatic number at least $r + 1$ in fact has chromatic number exactly $r + 1$.

[25]We are using the fact that s_{i-1} is complete to X_{i-2}, and so $c_1 \notin c(X_{i-2})$.

[26]If $c_1 \notin c(X_i)$, then $x_i^{c_1}$ is undefined.

$W_i = \{x \in X_i \mid x \leq h_i\}$. Next, for even $i \geq \ell + 3$, Claim 3 guarantees that $c_1 \in c(X_i)$, and we set $W_i = \{x \in X_i \mid x \leq \max\{h_i, x_i^{c_1}\}\}$. Finally, for odd $i \geq \ell+4$, let $h_i^{c_1}$ be the highest indexed vertex of $X_i \cap V(H)$ that is adjacent to $x_{i-1}^{c_1}$, and let $W_i = \{x \in X_i \mid x \leq h_i^{c_1}\}$. Let $W = R[W_1 \cup W_2 \cup \cdots \cup W_k]$, and for all $i \in \{1, \dots, k\}$, let w_i be the highest indexed vertex of W_i.

Claim 4. *W is a k-hyperhole.*

Proof of Claim 4. Suppose otherwise. Then there exists some even $i \geq \ell+3$ such that $x_i^{c_1}$ is non-adjacent to w_{i-1}. Let $a = c(w_{i-1})$.

Suppose that $i = \ell + 3$. Then by the choice of ℓ, no vertex in $W_{\ell+2}$ is colored c_1. Let Q be the component of $R^{c_1,a}$ that contains w_{i-1}. By construction, $V(Q) \cap X_{\ell+3} = \emptyset$, and by the parity of j and i, we see that $V(Q) \cap X_{j+1} = \emptyset$. Thus, $V(Q) \subseteq X_{j+2} \cup \cdots \cup X_{\ell+2}$. We now swap colors c_1 and a on Q, thus obtaining a coloring of $R \setminus t_2$ that contradicts the fact that c is unimprovable.[27]

Thus, $i \geq \ell + 5$. By construction, $x_{i-2}^{c_1}$ is adjacent to w_{i-1}, and so if $c_1 \in c(X_{i-1})$, then $w_{i-1} < x_{i-1}^{c_1}$. Let Q be the component of $R^{c_1,a}$ that contains w_{i-1}. Then $V(Q) \cap X_{j+1} = V(C) \cap X_i = \emptyset$, and we deduce that $V(Q) \subseteq X_{j+2} \cup \cdots \cup X_{i-1}$. But now by swapping colors c_1 and a on Q, we obtain a coloring of $R \setminus t_2$ that contradicts the fact that c is unimprovable. ■

Claim 5. $|V(W)| \geq |V(H)| + \frac{k-1}{2}$.

Proof of Claim 5. To simplify notation, for all $i \in \{1, \dots, k\}$, we set $H_i = V(H) \cap X_i$. Further, let S_W be the set of all vertices of W to which c assigns color c_1; thus, $S_W \subseteq S$, S_W is a stable set in $R \setminus t_2$, and $V(H) \cap S_W = \emptyset$. By the construction of W, we have that $|S_W| \geq \frac{k-1}{2}$. Thus, it suffices to show that $|V(H)| \leq |V(W) \setminus S_W|$.

By the construction of W, for all indices $i \in \{1, \dots, k\}$ such that either $i \leq \ell+2$ or i is even, we have that $H_i \subseteq W_i \setminus S_W$. We may now assume that for some even index $i \geq \ell+3$, we have that $|W_i \setminus (H_i \cup S_W)| < |H_{i+1} \setminus W_{i+1}|$, for otherwise we are done. Since $W_i \setminus (H_i \cup S_W)$ and $H_{i+1} \setminus W_{i+1}$ are both cliques of $R \setminus t_2$, and since c is a proper coloring of $R \setminus t_2$, we have that $|c(W_i \setminus (H_i \cup S_W))| < |c(H_{i+1} \setminus W_{i+1})|$; fix $a \in c(H_{i+1} \setminus W_{i+1}) \setminus c(W_i \setminus (H_i \cup S_W))$. Then $a \neq c_1$.[28] Furthermore, $a \notin c(W_i)$,[29] and so it follows from the construction of W that a is higher than c_1 in X_i (possibly $a \notin c(X_i)$).

[27] We are using the fact that $V(Q) \subseteq X_{j+2} \cup \cdots \cup X_{\ell+3}$, that $\ell + 2$ is odd, and that $c_1 \notin c(X_{\ell+2})$.

[28] This is because $a \in c(H_{i+1})$, and c does not assign color c_1 to any vertex in H.

[29] By construction, $a \notin c(W_i \setminus (H_i \cup S_W))$, and since $a \neq c_1$, we also have that $a \notin S_W$. Further, $a \in c(H_{i+1})$, and so since H_i is complete to H_{i+1}, we have that $a \notin c(H_i)$. Thus, $a \notin c(W_i)$.

By construction, $a \in c(X_{i+1})$; let x^a_{i+1} be the (unique) vertex of X_{i+1} such that $c(x^a_{i+1}) = a$. As before, let $x^{c_1}_i$ be the unique vertex of X_i to which c assigns color c_1 (such a vertex exists by Claim 3).

Now, we have that $x^a_{i+1} \in X_{i+1} \setminus W_{i+1}$, and that $i+1$ is odd with $i+1 \geq \ell + 4$. We then see from the construction of W that x^a_{i+1} is nonadjacent to $x^{c_1}_i$. Let Q be the component of $R^{c_1,a}$ that contains $x^{c_1}_i$. Then $V(Q) \cap X_{j+1} = V(Q) \cap X_{i+1} = \emptyset$, and it follows that $V(Q) \subseteq X_{j+2} \cup \cdots \cup X_i$. We now swap colors c_1 and a on Q, and we thus obtain a coloring of $R \setminus t_2$ that contradicts the fact that c is unimprovable. ■

By Claim 4, W is a k-hyperhole; since k is odd, we see that $\alpha(W) = \frac{k-1}{2}$. Using Claims 2 and 5, we now get that

$$\chi(W) \geq \left\lceil \frac{|V(W)|}{\alpha(W)} \right\rceil = \left\lceil \frac{2|V(H)|}{k-1} \right\rceil + 1 = r + 1.$$

On the other hand, we have that $\chi(W) \leq \chi(R) \leq r + 1$, and we deduce that $\chi(W) = r + 1$. This proves (b), and we are done. □

We are now ready to prove Theorem 1.2, restated below for the reader's convenience.

Theorem 1.2. *Let $k \geq 4$ be an integer, and let R be a k-ring. Then $\chi(R) = \max\{\chi(H) \mid H$ is a k-hyperhole in $R\}$.*

Proof. If k is even, then the result follows from Lemmas 3.1 and 3.2. So from now on, we assume that k is odd. Clearly, it suffices to show that R contains a k-hyperhole of chromatic number $\chi(R)$. We assume inductively that this holds for smaller k-rings, i.e. we assume that every k-ring R' such that $|V(R')| < |V(R)|$ contains a k-hyperhole of chromatic number $\chi(R')$.

Let $(X_1, \ldots, X_k)$ be a ring partition of R. For each $i \in \{1, \ldots, k\}$ let $X_i = \{u^1_i, \ldots, u^{|X_i|}_i\}$ be an ordering of X_i such that $X_i \subseteq N_R[u^{|X_i|}_i] \subseteq \cdots \subseteq N_R[u^1_i] = X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a ring. For all $i \in \{1, \ldots, k\}$, set $s_i = u^1_i$ and $t_i = u^{|X_i|}_i$. Set $r = \chi(R \setminus t_2)$, and note that this implies that $r \leq \chi(R) \leq r + 1$. Thus, we may assume that R contains no hyperhole of chromatic number $r + 1$, for otherwise we are done.

Let c be an unimprovable coloring of $R \setminus t_2$ that uses exactly r colors (the existence of such a coloring follows from Lemma 3.4). Let C be the set of colors used by c (thus, $|C| = r$), and set $c_1 = c(s_1)$ and $S = \{x \in V(R) \mid x \neq t_2, c(x) = c_1\}$. Lemma 3.5 now implies that $\omega(R \setminus S) \leq r - 1$ and $\chi(R \setminus S) \leq r - 1$. Since S is a stable set in R, we see that $\chi(R) \leq \chi(R \setminus S) + 1 \leq r$, and we deduce that $\chi(R) = r$. Further, since $\omega(R \setminus S) \leq r - 1$, and since S is a stable set, we see that $\omega(R) \leq r$. If $\omega(R) = r$, then $\chi(R) = \omega(R)$, and the result follows from Lemma 3.1. Thus, we may assume that $\omega(R) \leq r - 1$. Clearly, this implies that $\omega(R \setminus t_2) \leq r - 1$. Since $\chi(R \setminus t_2) = r$, we have that $\omega(R \setminus t_2) < \chi(R \setminus t_2)$.

Let $v_1, \dots, v_t$ (with $t \geq 0$) be a maximal sequence of pairwise distinct vertices of $R \setminus t_2$ such that for all $i \in \{1, \dots, t\}$, v_i is simplicial in $R \setminus t_2$. Set $A = \{v_1, \dots, v_t\}$. If $V(R) \setminus \{t_2\} = A$, then $v_1, \dots, v_t$ is a simplicial elimination ordering of $R \setminus t_2$, and so by coloring $R \setminus t_2$ greedily using the ordering $v_t, \dots, v_1$, we obtain a proper coloring of $R \setminus t_2$ that uses only $\omega(R \setminus t_2)$ colors, contrary to the fact that $\omega(R \setminus t_2) < \chi(R \setminus t_2)$. Thus, $V(R) \setminus \{t_2\} \neq A$. The maximality of $v_1, \dots, v_t$ guarantees that $R \setminus (\{t_2\} \cup A)$ has no simplicial vertices, and so by Lemma 2.7, $R \setminus (\{t_2\} \cup A)$ is a k-ring. Further, Lemma 2.10 and an easy induction guarantee that $\chi(R \setminus t_2) = \max\Big\{\omega(R \setminus t_2), \chi\Big(R \setminus (\{t_2\} \cup A)\Big)\Big\}$; since $\omega(R \setminus t_2) < \chi(R \setminus t_2)$ and $\chi(R) = \chi(R \setminus t_2)$, we deduce that $\chi(R) = \chi\Big(R \setminus (\{t_2\} \cup A)\Big)$. The induction hypothesis applied to the k-ring $R \setminus (\{t_2\} \cup A)$ guarantees that $R \setminus (\{t_2\} \cup A)$ contains a k-hyperhole H such that $\chi(H) = \chi(R \setminus (\{t_2\} \cup A))$. But then $\chi(H) = \chi(R)$. This completes the argument. $\square$

We complete this section by stating an easy corollary (Lemma 3.6) of Lemma 3.5 and Theorem 1.2. We will rely on Lemma 3.6 to construct a coloring algorithm for rings in section 5.

Lemma 3.6. *Let $k \geq 5$ be an odd integer, let R be a k-ring with ring partition $(X_1, \dots, X_k)$, and for each $i \in \{1, \dots, k\}$, let $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ be an ordering of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. For all $i \in \{1, \dots, k\}$, set $s_i = u_i^1$ and $t_i = u_i^{|X_i|}$. Let c be an unimprovable coloring of $R \setminus t_2$, let r be the number of colors used by c.*[30] *Let $c_1 = c(s_1)$, and let $S = \{x \in V(R) \mid x \neq t_2, c(x) = c_1\}$.*[31] *Then either $\chi(R \setminus S) \leq r - 1$ or $\chi(R) = r + 1$.*

Proof. By hypothesis, $\chi(R \setminus t_2) \leq r$; consequently, $\chi(R) \leq r + 1$. The result now readily follows from Theorem 1.2 and Lemma 3.5. $\square$

4 Computing the chromatic number of a ring

In this section, we use Corollary 1.3 and Lemma 2.4 to show that the chromatic number of a ring can be computed in polynomial time (see Theorem 4.2). We also give a polynomial-time coloring algorithm that computes the chromatic number of graphs in $\mathcal{G}_{\mathrm{T}}$ (see Theorem 4.3).

First, we give an algorithm that computes a maximum hyperhole of a ring.[32] The reader may have noticed that the proof of Theorem 1.2 is not

[30] In particular, the coloring c is proper and $r \geq \chi(R \setminus t_2)$, and this inequality may possibly be strict.

[31] Note that this implies that S is a stable set in $R \setminus t_2$.

[32] We remind the reader that, by Lemma 2.6(b), every hyperhole in a ring is of the same length as that ring.

constructive, since there are steps in that proof where we need to compare the (unknown) value of $\chi(R)$ with some other value. Moreover R may contain exponentially many hyperholes. So, our first goal is to construct a polynomial-time algorithm that finds a hyperhole of maximum size in an input ring.

We begin with some terminology and notation. Let $k \geq 4$ be an integer, let R be a k-ring with ring partition $(X_1, \dots, X_k)$, and for all $i \in \{1, \dots, k\}$, let $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ be an ordering of X_i such that $X_i \subseteq N_G[u_i^{|X_i|}] \subseteq \cdots \subseteq N_G[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a ring. Let H be a hyperhole in R. For all $i \in \{1, \dots, k\}$, let $t_i = \max\{t \in \{1, \dots, |X_i|\} \mid u_i^t \in V(H)\}$ and $Y_i = \{u_i^1, \dots, u_i^{t_i}\}$. Finally, let $\widetilde{H} = R[Y_1 \cup \cdots \cup Y_k]$ and $C_H = \{u_1^{t_1}, \dots, u_k^{t_k}\}$. Clearly, $\widetilde{H}$ is a hyperhole, with $V(H) \subseteq V(\widetilde{H})$. Furthermore, C_H induces a hole in R, and it uniquely determines $\widetilde{H}$. We say that H is *normal* in R if $H = \widetilde{H}$. Clearly, any maximal hyperhole (and therefore, any hyperhole of maximum size) in R is normal. Thus, to find a maximum hyperhole in an input ring, we need only consider normal hyperholes in that ring.

Lemma 4.1. *There exists an algorithm with the following specifications:*

- *Input: A graph R;*
- *Output: Either a maximum hyperhole H in R, or the true statement that R is not a ring;*
- *Running time: $O(n^3)$.*

Proof. We first run the algorithm from Lemma 2.3 input R; this takes $O(n^2)$ time. If the algorithm returns the answer that R is not a ring, then we return that answer as well and stop. So assume the algorithm returned the statement that R is a ring, along with the length k and ring partition $(X_1, \dots, X_k)$ of R, and for each $i \in \{1, \dots, k\}$ an ordering $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \cdots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$.

For each $j \in \{1, \dots, |X_1|\}$, we will find a normal hyperhole H_j of R such that $V(H_j) \cap X_1 = \{u_1^1, \dots, u_1^j\}$, and subject to that, $|V(H_j)|$ is maximum. We will then compare the sizes of all the H_j's (with $1 \leq j \leq |X_1|$), and clearly the one with maximum size will be a maximum hyperhole in R. Let us show how to find such a hyperhole H_j for fixed j.

First, define a set of $|X_1|$ new vertices, $X_{k+1} = \{u_{k+1}^1, \dots, u_{k+1}^{|X_1|}\}$, with $X_{k+1} \cap V(R) = \emptyset$ (so $|X_{k+1}| = |X_1|$, and X_{k+1} is a copy of X_1). Let D be the directed graph with vertex set $V(D) = V(R) \cup X_{k+1}$ and arc set:

$$\begin{aligned} A(D) \quad = \quad & \bigcup_{i=1}^{k-1} \left(\left\{ \overrightarrow{xy} \mid x \in X_i,\ y \in X_{i+1},\ xy \in E(R) \right\} \right. \\ & \left. \cup \left\{ \overrightarrow{xu_{k+1}^{\ell}} \mid x \in X_k,\ xu_1^{\ell} \in E(R),\ 1 \leq \ell \leq |X_1| \right\} \right). \end{aligned}$$

We define a weight function $w : A(D) \to \mathbb{N}$ as follows. For every arc $\overrightarrow{u_i^h u_{i+1}^\ell}$ in $A(D)$, with $i \in \{1, \dots, k\}$, $h \in \{1, \dots, |X_i|\}$, and $\ell \in \{1, \dots, |X_{i+1}|\}$, we set $w(\overrightarrow{u_i^h u_{i+1}^\ell}) = (|X_i| - h) + (|X_{i+1}| - \ell)$.

Let P_j be a minimum weight directed path between u_1^j and u_{k+1}^j in the weighted digraph (D, w). Such a path can be found in time $O(n^2)$ using Dijkstra's algorithm [4, 12]. For each $i \in \{1, \dots, k\}$, let $s_{i,j} \in \{1, \dots, |X_i|\}$ be the (unique) index such that $u_i^{s_{i,j}} \in V(P_j)$, and let $Y_{i,j} = \{u_i^\ell \mid 1 \leq \ell \leq s_{i,j}\}$. Then let $H_j = R[Y_{1,j} \cup \dots \cup Y_{k,j}]$. Clearly, H_j is a normal hyperhole of R. Moreover we have $|V(H_j)| = \sum_{i=1}^k |Y_{i,j}| = \sum_{i=1}^k s_{i,j} = |V(R)| - \sum_{i=1}^k (|X_i| - s_{i,j}) = |V(R)| - \frac{1}{2} w(P_j)$, and so the fact that P_j has minimum weight implies that H_j has maximum size among all hyperholes H that satisfy $V(H) \cap X_1 = \{u_1^1, \dots, u_1^j\}$. So, H_j is the desired hyperhole for a given j.

We now compare the sizes of the hyperholes $H_1, \dots, H_{|X_1|}$ (this takes $O(n^2)$ time), and we return the one of maximum size.

The total running time is $O(n^3)$ since we construct H_j for $O(n)$ values of j. This completes the proof. □

We are now ready to give a polynomial-time coloring algorithm for rings. In fact, we give a slightly more general algorithm. We remind the reader that $\mathcal{R}_{\geq 4}$ is the class of all graphs G that have the property that every induced subgraph of G either is a ring or has a simplicial vertex. By Lemma 2.8, $\mathcal{R}_{\geq 4}$ is hereditary and contains all rings. We now give a polynomial-time algorithm that computes the chromatic number of graphs in $\mathcal{R}_{\geq 4}$.

Theorem 4.2. *There exists an algorithm with the following specifications:*

- *Input: A graph G;*
- *Output: Either $\chi(G)$, or the true statement that $G \notin \mathcal{R}_{\geq 4}$;*
- *Running time: $O(n^3)$.*

Proof. First, we form a maximal sequence $v_1, \dots, v_t$ ($t \geq 0$) of vertices such that, for all $i \in \{1, \dots, t\}$, v_i is simplicial in $G \setminus \{v_1, \dots, v_{i-1}\}$; this can be done in $O(n^3)$ time by calling the algorithm from Lemma 2.5 with input G. If $V(G) = \{v_1, \dots, v_t\}$, then we greedily color G using the ordering $v_t, \dots, v_1$, we return the number of colors that we used, and we stop; this takes $O(n^2)$ time. From now on, we assume that $V(G) \neq \{v_1, \dots, v_t\}$, and we form the graph $R := G \setminus \{v_1, \dots, v_t\}$ in $O(n^2)$ time. The maximality of $v_1, \dots, v_t$ guarantees that R contains no simplicial vertices, and so by the definition of $\mathcal{R}_{\geq 4}$, we have that either R is a ring, or $R \notin \mathcal{R}_{\geq 4}$.

We now run the algorithm from Lemma 2.2 input R; this takes $O(n^2)$ time. If the algorithm returns the answer that R is not a ring, then we return the answer that $G \notin \mathcal{R}_{\geq 4}$,[33] and we stop. So assume the algorithm returned the statement that R is a ring, along with the length k and ring partition $(X_1, \dots, X_k)$ of R. Next, we call the algorithm from Lemma 2.4; this takes $O(n^3)$ time. Since R is a ring, we know that the algorithm returns a maximum clique C of R. Next, run the algorithm from Lemma 4.1 with input R; this takes $O(n^3)$ time. Since R is a ring, we know that the algorithm returns a hyperhole H of R of maximum size; since R is a k-ring, Lemma 2.6(b) guarantees that H is a k-hyperhole. Set $c := \max\{|C|, \left\lceil \frac{|V(H)|}{\lfloor k/2 \rfloor} \right\rceil\}$; by Corollary 1.3, we have that $\chi(R) = c$, and so the algorithm is correct.

If $t = 0$ (so that $G = R$), then we return c, and we stop. So assume that $t \geq 1$. For each $i \in \{1, \dots, t\}$, set $c_i = |N_G[v_i] \setminus \{v_1, \dots, v_{i-1}\}|$; computing the constants $c_1, \dots, c_t$ takes $O(n^2)$ time. An easy induction using Lemma 2.10 now establishes that $\chi(G) = \max\{c_1, \dots, c_t, c\}$. So, we return $\max\{c_1, \dots, c_t, c\}$, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. □

We complete this section by showing how to compute the chromatic number of graphs in $\mathcal{G}_{\mathrm{T}}$ in polynomial time.

Theorem 4.3. *There exists an algorithm with the following specifications:*

- *Input: A graph G;*
- *Output: Either $\chi(G)$, or the true statement that $G \notin \mathcal{G}_T$;*
- *Running time: $O(n^5)$.*

Proof. We first check whether G has a clique-cutset, and if so, we obtain a clique-cut-partition (A, B, C) of G such that $G[A \cup C]$ does not admit a clique-cutset; this can be done by running the algorithm from [13] with input G, and it takes $O(n^3)$ time. If we obtained the answer that G does not admit a clique-cutset, then we set $A = V(G)$, $B = \emptyset$, and $C = \emptyset$, and we set $c = 0$. On the other hand, if we obtained (A, B, C), then we make a recursive call to the algorithm with input $G[B \cup C]$; if we obtained the answer that $G[B \cup C] \notin \mathcal{G}_{\mathrm{T}}$, then we return the answer that $G \notin \mathcal{G}_{\mathrm{T}}$ and stop, and otherwise (i.e. if we obtained the chromatic number of $G[B \cup C]$) we set $c = \chi(G[B \cup C])$.

We may now assume that we have obtained the number c (for otherwise, we terminated the algorithm). Clearly, $\chi(G) = \max\{\chi(G[A \cup C]), c\}$. Next, we run the algorithm from Theorem 4.2 with input $G[A \cup C]$; this takes $O(n^3)$ time. If the algorithm returned $\chi(G[A \cup C])$, then we return the number $\max\{\chi(G[A \cup C]), c\}$, and we stop. So assume the algorithm returned the answer that $G[A \cup C]$ is not a ring.

[33] This is correct because, as explained above, if R is not a ring, then $R \notin \mathcal{R}_{\geq 4}$, and (since $\mathcal{R}_{\geq 4}$ is hereditary) this implies that $G \notin \mathcal{R}_{\geq 4}$.

So far, we know that $G[A \cup C]$ does not admit a clique-cutset and is not a ring. Theorem 2.11 now guarantees that either $G[A \cup C]$ is a complete graph or a 7-hyperantihole, or $G[A \cup C] \notin \mathcal{G}_{\mathrm{T}}$ (in which case, $G \notin \mathcal{G}_{\mathrm{T}}$, since $\mathcal{G}_{\mathrm{T}}$ is hereditary). Clearly, complete graphs have stability number one, and hyperantiholes have stability number two. Thus, either $\alpha(G[A \cup C]) \leq 2$ or $G \notin \mathcal{G}_{\mathrm{T}}$. Now, we determine whether $\alpha(G[A \cup C]) \leq 2$ by examining all triples of vertices in $G[A \cup C]$; this takes $O(n^3)$ time. If $\alpha(G[A \cup C]) \geq 3$, then we return the answer that $G \notin \mathcal{G}_{\mathrm{T}}$ and stop. Assume now that $\alpha(G[A \cup C]) \leq 2$. Now, we form the graph $\overline{G}[A \cup C]$ (the complement of $G[A \cup C]$) in $O(n^2)$ time, and we find a maximum matching M in $\overline{G}[A \cup C]$ by running the algorithm from [5]; this takes $O(n^4)$ time. Since $\alpha(G[A \cup C]) \leq 2$, we see that $\chi(G) = |V(G)| - |M|$; we now return the number $\max\{|V(G)| - |M|, c\}$, and we stop.

Clearly, the algorithm is correct. The slowest step takes $O(n^4)$ time, and we make $O(n)$ recursive calls. Thus, the total running time of the algorithm is $O(n^5)$. □

5 Coloring rings

We remind the reader that $\mathcal{R}_{\geq 4}$ is the class of all graphs G that have the property that every induced subgraph of G either is a ring or has a simplicial vertex. By Lemma 2.8, $\mathcal{R}_{\geq 4}$ is hereditary and contains all rings. Our goal in this section is to construct a polynomial-time coloring algorithm for graphs in $\mathcal{R}_{\geq 4}$ (see Theorem 5.2), and more generally, for graphs in $\mathcal{G}_{\mathrm{T}}$ (see Theorem 5.3). We already know how to color even rings (see Lemma 3.2). In the remainder of the section, we focus primarily on odd rings.

Lemma 5.1. *There exists an algorithm with the following specifications:*

- *Input: A positive integer r, an r-colorable odd ring R with ring partition $(X_1, \dots, X_k)$, for each $i \in \{1, \dots, k\}$, an ordering $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, and a proper coloring c of $R \setminus u_2^{|X_2|}$ that uses at most r colors;*
- *Output: A proper coloring of R that uses at most r colors;*
- *Running time: $O(n^5)$.*

Proof. To simplify notation, for all $i \in \{1, \dots, k\}$, we set $s_i = u_i^1$ and $t_i = u_i^{|X_i|}$. Thus, c is a proper coloring of $R \setminus t_2$ that uses at most r colors. We may assume that the set of colors used by c is included in $\{1, \dots, r\}$.

First, we update c by running the algorithm from Lemma 3.4 and transforming it into an unimprovable coloring of $R \setminus t_2$; this takes $O(n^4)$ time. We may assume that $c(s_1) = r$. Let $S = \{x \in V(R) \setminus \{t_2\} \mid c(x) = r\}$. Since $\chi(R) \leq r$, Lemma 3.6 guarantees that $\chi(R \setminus S) \leq r - 1$. (Note that this implies that $\omega(R \setminus S) \leq r-1$.) Next, we run the algorithm from Lemma 2.5, and we obtain a (possibly empty) sequence $v_1, \dots, v_t$ of vertices of $R \setminus S$ such that for all $i \in \{1, \dots, t\}$, v_i is simplicial in $R \setminus (S \cup \{v_1, \dots, v_{i-1}\})$.

Suppose first that $V(R) \setminus S = \{v_1, \dots, v_t\}$; then $v_1, \dots, v_t$ is a simplicial elimination ordering of $R \setminus S$. We now greedily color $R \setminus S$ using the ordering $v_t, \dots, v_1$; since $\omega(R \setminus S) \leq r-1$, and since $v_1, \dots, v_t$ is a simplicial elimination ordering of R, we see that at most $r-1$ colors are used to color $R \setminus S$. We now extend this coloring of $R \setminus S$ to a proper coloring of R that uses at most r colors by assigning the same new color to all vertices in S. We return this coloring of R, and we stop.

Suppose now that $V(G) \setminus S \neq \{v_1, \dots, v_t\}$. Set $R' := R \setminus (S \cup \{v_1, \dots, v_t\})$. The maximality of $v_1, \dots, v_t$ guarantees that R' has no simplicial vertices, and so it follows from Lemma 2.7 that R' is a k-ring with ring partition $\Big(X_1 \cap V(R'), \dots, X_k \cap V(R')\Big)$. Furthermore, $\chi(R') \leq \chi(R \setminus S) \leq r-1$, i.e. R' is $(r-1)$-colorable. Let $c' = c \restriction (V(R') \setminus \{t_2\})$. If $t_2 \notin V(R')$, then $c' : V(R') \to \{1, \dots, r-1\}$ is a proper coloring of R', and otherwise, $c' : V(R') \setminus \{t_2\} \to \{1, \dots, r-1\}$ is a proper coloring of $R' \setminus t_2$. In the former case, we set $c'' = c'$; in the latter case, we make a recursive call to the algorithm, and we obtain a proper coloring $c'' : V(R') \to \{1, \dots, r-1\}$ of R'. In either case, we extend c'' to a proper coloring of $R \setminus S$ by assigning colors greedily to the vertices $v_t, \dots, v_1$ (in that order); since $\omega(R \setminus S) \leq r-1$, this gives us a proper coloring of $R \setminus S$ that uses only colors $1, \dots, r-1$. Finally, we assign color r to all the vertices in S. This produces a proper coloring of R that uses at most r colors. We return this coloring of R, and we stop.

Clearly, the algorithm is correct. We make $O(n)$ recursive calls, and otherwise, the slowest step of the algorithm takes $O(n^4)$ time. Thus, the total running time of the algorithm is $O(n^5)$. □

Theorem 5.2. *There exists an algorithm with the following specifications:*

- *Input: A graph G;*
- *Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{R}_{\geq 4}$;*
- *Running time: $O(n^6)$.*

Proof. First, we form a maximal sequence $v_1, \dots, v_t$ ($t \geq 0$) of vertices such that, for all $i \in \{1, \dots, t\}$, v_i is simplicial in $G \setminus \{v_1, \dots, v_{i-1}\}$; this can be done in $O(n^3)$ time by running the algorithm from Lemma 2.5.

Suppose first that $t \geq 1$. If $V(G) = \{v_1, \dots, v_t\}$, so that $v_1, \dots, v_t$ is a simplicial elimination ordering of G, then we color G greedily in $O(n^2)$ time using the ordering $v_t, \dots, v_1$; we return this coloring of G, and we stop. So assume that $V(G) \setminus \{v_1, \dots, v_t\} \neq \emptyset$. We then make a recursive call to the algorithm with input $G \setminus \{v_1, \ldots, v_t\}$. If we obtain an optimal coloring of $G \setminus \{v_1, \ldots, v_t\}$, then we greedily extend this coloring to an optimal coloring of G using the ordering $v_t, \ldots, v_1$, we return this coloring of G, and we stop. On the other hand, if the algorithm returns the statement that $G \setminus \{v_1, \ldots, v_t\} \notin \mathcal{R}_{\geq 4}$, then we return the answer that $G \notin \mathcal{R}_{\geq 4}$ (this is correct because $R_{\geq 4}$ is hereditary), and we stop.

From now on, we assume that $t = 0$. Thus, G contains no simplicial vertices, and so by Lemma 2.7, either G is a ring, or $G \notin \mathcal{R}_{\geq 4}$. We now run the algorithm from Lemma 2.3 input G; this takes $O(n^2)$ time. If the algorithm returns the answer that G is not a ring, then we return that answer that $G \notin \mathcal{R}_{\geq 4}$. So assume the algorithm returned the statement that G is a ring, along with the length k and ring partition $(X_1, \ldots, X_k)$ of G, and for each $i \in \{1, \ldots, k\}$ an ordering $X_i = \{u_i^1, \ldots, u_i^{|X_i|}\}$ of X_i such that $X_i \subseteq N_G[u_i^{|X_i|}] \subseteq \cdots \subseteq N_G[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. If k is even, then we obtain an optimal coloring of G in $O(n^3)$ time by running the algorithm from Lemma 3.2, we return this coloring, and we stop. So from now on, we assume that k is odd, so that G is an odd ring. We now compute the chromatic number r of G in $O(n^3)$ time by running the algorithm from Theorem 4.2. Set $t_2 = u_2^{|X_2|}$. We now make a recursive call to the algorithm and obtain an optimal coloring of $G \setminus t_2$, and then, in $O(n^5)$ time, we obtain a coloring of G that uses at most r colors by running the algorithm from Lemma 5.1. Since $\chi(G) = r$, our coloring of G is optimal; we return this coloring, and we stop.

Clearly, the algorithm is correct. We make $O(n)$ recursive calls to the algorithm, and otherwise, the slowest step of the algorithm takes $O(n^5)$ time. Thus, the total running time of the algorithm is $O(n^6)$. □

We complete this section by giving a polynomial-time coloring algorithm for graphs in $\mathcal{G}_{\mathrm{T}}$. We remark that the algorithm is very similar to the one from Theorem 4.3, except that we use Theorem 5.2 instead of Theorem 4.2. Nevertheless, for the sake of completeness, we give all the details.

Theorem 5.3. *There exists an algorithm with the following specifications:*

- *Input: A graph G;*
- *Output: Either an optimal coloring of G, or the true statement that $G \notin \mathcal{G}_T$;*
- *Running time: $O(n^7)$.*

Proof. We first check whether G has a clique-cutset, and if so, we obtain a clique-cut-partition (A, B, C) of G such that $G[A \cup C]$ does not admit a clique-cutset; this can be done in $O(n^3)$ time by running the algorithm from [13] with input G. If we obtained the answer that G does not admit a clique-cutset, then we set $A = V(G)$, $B = \emptyset$, and $C = \emptyset$. On the other hand, if we obtained (A, B, C), then we make a recursive call to the algorithm with input $G[B \cup C]$; if we obtained the answer that $G[B \cup C] \notin \mathcal{G}_{\text{T}}$, then we return the answer that $G \notin \mathcal{G}_{\text{T}}$ (this is correct because $\mathcal{G}_{\text{T}}$ is hereditary), and we stop. So from now on, we assume that one of the following holds:

- $B = C = \emptyset$;
- (A, B, C) is a clique-cut-partition, and we recursively obtained an optimal coloring c_B of $G[B \cup C]$.

In either case, we also have that $G[A \cup C]$ does not admit a clique-cutset.

We now run the algorithm from Theorem 5.2 with input $G[A \cup C]$; this takes $O(n^6)$ time. The algorithm either returns an optimal coloring c_A of $G[A \cup C]$, or it returns the answer that $G[A \cup C] \notin \mathcal{R}_{\geq 4}$. If the algorithm returned the answer that $G[A \cup C] \notin \mathcal{R}_{\geq 4}$, then our goal is to either produce an optimal coloring c_A of $G[A \cup C]$ in another way, or to determine that $G \notin \mathcal{G}_{\text{T}}$. In this case (i.e. if the algorithm returned the answer that $G[A \cup C] \notin \mathcal{R}_{\geq 4}$), we proceed as follows. Since $\mathcal{R}_{\geq 4}$ contains all rings (by Lemma 2.8), we have that $G[A \cup C]$ is not a ring. Recall that $G[A \cup C]$ does not admit a clique-cutset. Thus, Theorem 2.11 implies that either $G[A \cup C]$ is a complete graph, or $G[A \cup C]$ is a 7-hyperantihole, or $G[A \cup C] \notin \mathcal{G}_{\text{T}}$ (in which case, $G \notin \mathcal{G}_{\text{T}}$, since $\mathcal{G}_{\text{T}}$ is hereditary). Clearly, complete graphs have stability number one, and hyperantiholes have stability number two. Thus, either $\alpha(G[A \cup C]) \leq 2$ or $G \notin \mathcal{G}_{\text{T}}$. We determine whether $\alpha(G) \leq 2$ by examining all triples of vertices in G; this takes $O(n^3)$ time. If $\alpha(G) \geq 3$, then we return the answer that $G \notin \mathcal{G}_{\text{T}}$, and we stop. So suppose that $\alpha(G[A \cup C]) \leq 2$. This means that each color class of a proper coloring of $G[A \cup C]$ is of size at most two, and that, taken together, color classes of size exactly two correspond to a matching of $\overline{G}[A \cup C]$ (the complement of $G[A \cup C]$). So, we form the graph $\overline{G}[A \cup C]$ in $O(n^2)$ time, and we find a maximum matching M in $\overline{G}[A \cup C]$ in $O(n^4)$ time by running the algorithm from [5]. We assign a different color to each member of the matching M, plus a new color to each vertex of $G[A \cup C]$ that is not an endpoint of an edge of M. This produces an optimal coloring c_A of $G[A \cup C]$.

So from now on, we may assume that we have obtained an optimal coloring c_A of $G[A \cup C]$. If $B = C = \emptyset$, then c_A is in fact an optimal coloring of G; in this case, we return c_A, and we stop. So assume that $B \cup C \neq \emptyset$. Then we have already obtained an optimal coloring c_B of $G[B \cup C]$. After possibly renaming colors, we may assume that the color set used by one of c_A, c_B is included in the color set used by the other one. Now, C is a clique in G, and so c_A assigns a different color to each vertex of C, and the same is true for c_B. So, after possibly permuting colors, we may assume that $c_A \restriction C = c_B \restriction C$. Now $c := c_A \cup c_B$ is an optimal coloring of G. We return c, and we stop.

Clearly, the algorithm is correct. The slowest step takes $O(n^6)$ time, and we make $O(n)$ recursive calls. Thus, the total running time of the algorithm is $O(n^7)$. $\square$

6 Optimal χ-bounding functions

For all integers $k \geq 4$, let $\mathcal{H}_k$ be the class of all induced subgraphs of k-hyperholes, and let $\mathcal{A}_k$ be the class of all induced subgraphs of k-hyperantiholes; clearly, classes $\mathcal{H}_k$ and $\mathcal{A}_k$ are both hereditary, and they contain all complete graphs.[34] Recall that for all integers $k \geq 4$, $\mathcal{R}_k$ is the class of all graphs G that have the property that every induced subgraph of G either is a k-ring or has a simplicial vertex; clearly, $\mathcal{R}_k$ is hereditary and contains all complete graphs, and by Lemma 2.8, all k-rings belong to $\mathcal{R}_k$ (in particular, $\mathcal{H}_k \subseteq \mathcal{R}_k$). In this section, for all integers $k \geq 4$, we find the optimal χ-bounding functions for the classes $\mathcal{H}_k$ (see Theorem 6.5), $\mathcal{A}_k$ (see Theorem 6.12), and $\mathcal{R}_k$ (see Theorem 6.8). Further, for all integers $k \geq 4$, we set $\mathcal{H}_{\geq k} = \bigcup_{i=k}^{\infty} \mathcal{H}_i$ and $\mathcal{A}_{\geq k} = \bigcup_{i=k}^{\infty} \mathcal{A}_i$, and we remind the reader that $\mathcal{R}_{\geq k} = \bigcup_{i=k}^{\infty} \mathcal{R}_i$.[35] For all integers $k \geq 4$, we find the optimal χ-bounding functions for the classes $\mathcal{H}_{\geq k}$ (see Corollary 6.5), $\mathcal{A}_{\geq k}$ (see Corollary 6.13), and $\mathcal{R}_{\geq k}$ (see Corollary 6.9); see also Theorem 6.14. Finally, we find the optimal χ-bounding function for the class $\mathcal{G}_{\mathrm{T}}$ (see Theorem 6.15).

Recall that $\mathbb{N}$ is the set of all positive integers, and let $i_{\mathbb{N}}$ be the identity function on $\mathbb{N}$, i.e. let $i_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ be given by $i_{\mathbb{N}}(n) = n$ for all $n \in \mathbb{N}$.

We define the function $f_{\mathrm{T}} : \mathbb{N} \to \mathbb{N}$ by setting

$$f_{\mathrm{T}}(n) \quad = \quad \begin{cases} \lfloor 5n/4 \rfloor & \text{if} \quad n \equiv 0, 1 \pmod 4 \\ \\ \lceil 5n/4 \rceil & \text{if} \quad n \equiv 2, 3 \pmod 4 \end{cases}$$

for all $n \in \mathbb{N}$.

For all odd integers $k \geq 5$, we define the function $f_k : \mathbb{N} \to \mathbb{N}$ by setting

$$f_k(n) \quad = \quad \begin{cases} \left\lfloor \frac{kn}{k-1} \right\rfloor & \text{if} \quad n \equiv 0, 1 \pmod{k-1} \\ \\ \left\lceil \frac{kn}{k-1} \right\rceil & \text{if} \quad n \equiv 2, \dots, k-2 \pmod{k-1} \end{cases}$$

for all $n \in \mathbb{N}$.

[34] The reason we emphasize that these classes contain all complete graphs is that we defined optimal χ-bounding functions only for hereditary, χ-bounded classes that contain all complete graphs.

[35] Clearly, for all integers $k \geq 4$ we have that: $\mathcal{H}_{\geq k}$ is the class of all induced subgraphs of hyperholes of length at least k; $\mathcal{A}_{\geq k}$ is the class of all induced subgraphs of hyperantiholes of length at least k; and $\mathcal{R}_{\geq k}$ contains all induced subgraphs of rings of length at least k. In particular, $\mathcal{H}_{\geq k} \subseteq \mathcal{R}_{\geq k}$. It is clear that $\mathcal{H}_{\geq k}$, $\mathcal{A}_{\geq k}$, and $\mathcal{R}_{\geq k}$ are hereditary and contain all complete graphs.

For all odd integers $k \geq 5$, we define the function $g_k : \mathbb{N} \to \mathbb{N}$ by setting

$$g_k(n) \quad = \quad \begin{cases} n & \text{if} \quad n \leq \frac{k-3}{2} \\ \\ \left\lfloor \frac{kn}{k-1} \right\rfloor & \text{if} \quad n \geq \frac{k+1}{2} \text{ and } n \equiv 0, \dots, \frac{k-3}{2} \pmod{k-1} \\ \\ \left\lceil \frac{kn}{k-1} \right\rceil & \text{if} \quad n \equiv \frac{k-1}{2}, \dots, k-2 \pmod{k-1} \end{cases}$$

for all $n \in \mathbb{N}$.

Note that $f_{\mathrm{T}} = f_5 = g_5$. Before turning to the classes mentioned at the beginning of this section, we prove a few technical lemmas concerning functions f_{T}, f_k, and g_k.

Lemma 6.1. *Let $k \geq 5$ be an odd integer, and let $n \in \mathbb{N}$. Then $f_k(n) = n + \left\lceil \frac{2\lfloor n/2 \rfloor}{k-1} \right\rceil$.*

Proof. Set $m = \lfloor \frac{n}{k-1} \rfloor$ and $\ell = n - (k-1)m$. Clearly, m is a nonnegative integer, $\ell \in \{0, \dots, k-2\}$, $n = (k-1)m + \ell$, and $n \equiv \ell \pmod{k-1}$.

Since k is odd, we have that $k-1$ is even, and so

$$\left\lceil \frac{2\lfloor n/2 \rfloor}{k-1} \right\rceil \quad = \quad \left\lceil \frac{2\left\lfloor \frac{(k-1)m+\ell}{2} \right\rfloor}{k-1} \right\rceil \quad = \quad m + \left\lceil \frac{2\lfloor \ell/2 \rfloor}{k-1} \right\rceil.$$

If $0 \leq \ell \leq 1$, then $f_k(n) = \lfloor \frac{n}{k-1} \rfloor$, and we have that

$$\begin{aligned} n + \left\lceil \frac{2\lfloor n/2 \rfloor}{k-1} \right\rceil \quad &= \quad n + m + \left\lceil \frac{2\lfloor \ell/2 \rfloor}{k-1} \right\rceil \\ &= \quad n + m \\ &= \quad \lfloor \tfrac{kn}{k-1} \rfloor \\ &= \quad f_k(n), \end{aligned}$$

and we are done.

Suppose now that $2 \leq \ell \leq k-2$; then $f_k(n) = \lceil \frac{kn}{k-1} \rceil$. First, we have that

$$n + \left\lceil \frac{2\lfloor n/2 \rfloor}{k-1} \right\rceil \quad = \quad n + m + \left\lceil \frac{2\lfloor \ell/2 \rfloor}{k-1} \right\rceil \quad = \quad n + m + 1 \quad = \quad \lfloor \tfrac{kn}{k-1} \rfloor + 1.$$

Since $\ell \neq 0$, we see that $\frac{kn}{k-1}$ is not an integer, and so $\lfloor \frac{kn}{k-1} \rfloor + 1 = \lceil \frac{kn}{k-1} \rceil$. It now follows that

$$n + \left\lceil \frac{2\lfloor n/2 \rfloor}{k-1} \right\rceil \quad = \quad \lfloor \tfrac{kn}{k-1} \rfloor + 1 \quad = \quad \lceil \tfrac{kn}{k-1} \rceil \quad = \quad f_k(n),$$

which is what we needed. This completes the argument. $\square$

Lemma 6.2. *Let $k \geq 5$ be an odd integer, and let $n \in \mathbb{N}$. Then $g_k(n) = n + \left\lceil \lfloor \frac{2n}{k-1} \rfloor / 2 \right\rceil$.*

Proof. If $n \leq \frac{k-3}{2}$, then $n + \left\lceil \lfloor \frac{2n}{k-1} \rfloor / 2 \right\rceil = n = g_k(n)$, and we are done. So from now on, we assume that $n \geq \frac{k-1}{2}$.

Set $m = \lfloor \frac{n}{k-1} \rfloor$ and $\ell = n-(k-1)m$. Clearly, m is a nonnegative integer, $\ell \in \{0, \dots, k-2\}$, $n = (k-1)m + \ell$, and $n \equiv \ell \pmod{k-1}$.

First, we have that

$$\left\lceil \lfloor \tfrac{2n}{k-1} \rfloor / 2 \right\rceil \quad = \quad \left\lceil \left\lfloor \tfrac{2((k-1)m+\ell)}{k-1} \right\rfloor / 2 \right\rceil \quad = \quad m + \left\lceil \lfloor \tfrac{2\ell}{k-1} \rfloor / 2 \right\rceil.$$

Suppose first that $0 \leq \ell \leq \frac{k-3}{2}$; then $g_k(n) = \lfloor \frac{kn}{k-1} \rfloor$. We now have that

$$\begin{aligned} n + \left\lceil \lfloor \tfrac{2n}{k-1} \rfloor / 2 \right\rceil &= n + m + \left\lceil \lfloor \tfrac{2\ell}{k-1} \rfloor / 2 \right\rceil \\ &= n + m \\ &= \lfloor \tfrac{kn}{k-1} \rfloor \\ &= g_k(n), \end{aligned}$$

which is what we needed.

Suppose now that $\frac{k-1}{2} \leq \ell \leq k-2$; then $g_k(n) = \lceil \frac{kn}{k-1} \rceil$. Now, note that

$$\begin{aligned} n + \left\lceil \lfloor \tfrac{2n}{k-1} \rfloor / 2 \right\rceil &= n + m + \left\lceil \lfloor \tfrac{2\ell}{k-1} \rfloor / 2 \right\rceil \\ &= n + m + 1 \\ &= \lfloor \tfrac{kn}{k-1} \rfloor + 1. \end{aligned}$$

Since $\ell \neq 0$, we see that $\frac{kn}{k-1}$ is not an integer, and so $\lfloor \frac{kn}{k-1} \rfloor + 1 = \lceil \frac{kn}{k-1} \rceil$. We now have that

$$n + \left\lceil \lfloor \tfrac{2n}{k-1} \rfloor / 2 \right\rceil \quad = \quad \lfloor \tfrac{kn}{k-1} \rfloor + 1 \quad = \quad \lceil \tfrac{kn}{k-1} \rceil \quad = \quad g_k(n),$$

which is what we needed. This completes the argument. $\square$

Given functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \leq g$ and $g \geq f$, if for all $n \in \mathbb{N}$, we have that $f(n) \leq g(n)$. As usual, a function $f : \mathbb{N} \to \mathbb{N}$ is said to be *nondecreasing* if for all $m, n \in \mathbb{N}$ such that $m \leq n$, we have that $f(m) \leq f(n)$.

Lemma 6.3. *Function f_T is nondecreasing, and $f_T = f_5 = g_5$. Furthermore, for all odd integers $k \geq 5$, all the following hold:*

(a) $f_T \geq f_k \geq g_k$;

(b) functions f_k and g_k are nondecreasing;

(c) $f_k \geq f_{k+2}$ and $g_k \geq g_{k+2}$.

Proof. The fact that f_{T} is nondecreasing, and that $f_{\mathrm{T}} = f_5 = g_5$, follows from the definitions of f_{T}, f_5, and g_5. Further, it follows from construction that for all odd integers $k \geq 5$, we have that $f_k \geq g_k$. The rest readily follows from Lemmas 6.1 and 6.2. $\square$

Lemma 6.4. *Let $k \geq 5$ be an odd integer. Then all k-hyperholes H satisfy $\chi(H) \leq f_k(\omega(H))$. Furthermore, there exists a sequence $\{H_n^k\}_{n=2}^{\infty}$ of k-hyperholes such that for all integers $n \geq 2$, we have that $\omega(H_n^k) = n$ and $\chi(H_n^k) = f_k(n)$.*

Proof. We begin by proving the first statement of the lemma. Let H be a k-hyperhole, and let $(X_1, \dots, X_k)$ be a partition of $V(H)$ into nonempty cliques such that for all $i \in \{1, \dots, k\}$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$, as in the definition of a k-hyperhole. Since H is a k-hyperhole, and since k is odd, we have that $\alpha(H) = \lfloor k/2 \rfloor = \frac{k-1}{2}$. Then by Lemma 1.1, we have that

$$\chi(H) \;=\; \max\{\omega(H), \left\lceil \tfrac{|V(H)|}{\alpha(H)} \right\rceil\} \;=\; \max\{\omega(H), \left\lceil \tfrac{2|V(H)|}{k-1} \right\rceil\}.$$

It is clear that $\omega(H) \leq f_k(\omega(H))$, and so it suffices to show that $\left\lceil \frac{2|V(H)|}{k-1} \right\rceil \leq f_k(\omega(H))$. Clearly, for all $i \in \{1, \dots, k\}$, $X_i \cup X_{i+1}$ is a clique, and so $|X_i| + |X_{i+1}| \leq \omega(H)$. In particular, $|X_k| + |X_1| \leq \omega(H)$, and so either $|X_k| \leq \lfloor \omega(H)/2 \rfloor$ or $|X_1| \leq \lfloor \omega(H)/2 \rfloor$; by symmetry, we may assume that $|X_k| \leq \lfloor \omega(H)/2 \rfloor$. Now, using the fact that k is odd, we get that

$$\begin{aligned} |V(H)| &= \sum_{i=1}^{k} |X_i| \\ &= \Big(\sum_{i=1}^{(k-1)/2} (|X_{2i-1}| + |X_{2i}|) \Big) + |X_k| \\ &\leq \tfrac{k-1}{2}\omega(H) + \lfloor \omega(H)/2 \rfloor. \end{aligned}$$

But now by Lemma 6.1, we have that

$$\begin{aligned} \left\lceil \tfrac{2|V(H)|}{k-1} \right\rceil &\leq \left\lceil \frac{2\Big(\frac{k-1}{2}\omega(H) + \lfloor \omega(H)/2 \rfloor\Big)}{k-1} \right\rceil \\ &= \omega(H) + \left\lceil \tfrac{2\lfloor \omega(H)/2 \rfloor}{k-1} \right\rceil \\ &= f_k(\omega(H)), \end{aligned}$$

which is what we needed. This proves the first statement of the lemma.

It remains to prove the second statement of the lemma. We fix an integer $n \geq 2$, and we construct H_n^k as follows. Let $X_1, \dots, X_k$ be pairwise disjoint sets such that for all $i \in \{1, \dots, k\}$,

- if i is odd, then $|X_i| = \lfloor n/2 \rfloor$, and
- if i is even, then $|X_i| = \lceil n/2 \rceil$.

Since $n \geq 2$, sets $X_1, \dots, X_n$ are all nonempty. Now, let H_n^k be the graph whose vertex set is $V(H_n^k) = X_1 \cup \dots \cup X_k$, and with adjacency as follows:

- $X_1, \dots, X_k$ are all cliques;
- for all $i \in \{1, \dots, k\}$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H_n^k) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$.

Clearly, H_n^k is a k-hyperhole, and $\omega(H_n^k) = \lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. It remains to show that $\chi(H_n^k) = f_k(n)$. But by the first statement of the lemma, we have that $\chi(H_n^k) \leq f_k(n)$, and so in fact, it suffices to show that $\chi(H_n^k) \geq f_k(n)$.

It is clear that $\chi(H_n^k) \geq \left\lceil \frac{|V(H_n^k)|}{\alpha(H_n^k)} \right\rceil$. Further, by construction, and by the fact that k is odd, we have that

- $\alpha(H_n^k) = \lfloor k/2 \rfloor = \frac{k-1}{2}$, and
- $|V(H_n^k)| = \lceil k/2 \rceil \lfloor n/2 \rfloor + \lfloor k/2 \rfloor \lceil n/2 \rceil = \frac{k-1}{2} n + \lfloor n/2 \rfloor$.

Thus,

$$\chi(H_n^k) \quad \geq \quad \left\lceil \frac{|V(H_n^k)|}{\alpha(H_n^k)} \right\rceil \quad = \quad \left\lceil \frac{2\left(\frac{k-1}{2}n + \lfloor n/2 \rfloor\right)}{k-1} \right\rceil \quad = \quad n + \left\lceil \frac{2\lfloor n/2 \rfloor}{k-1} \right\rceil.$$

Lemma 6.1 now implies that

$$\chi(H_n^k) \quad \geq \quad n + \left\lceil \frac{2\lfloor n/2 \rfloor}{k-1} \right\rceil \quad = \quad f_k(n),$$

which is what we needed. This proves the second statement of the lemma. □

Theorem 6.5. *Let $k \geq 4$ be an integer. Then $\mathcal{H}_k$ is χ-bounded. Furthermore, if k is even, then $i_{\mathbb{N}}$ is the optimal χ-bounding function for $\mathcal{H}_k$, and if k is odd, then f_k is the optimal χ-bounding function for $\mathcal{H}_k$.*

Proof. Note that every induced subgraph of a k-hyperhole is either a k-hyperhole or a chordal graph.[36] Since chordal graphs are perfect (by [3]), it follows that all graphs in $\mathcal{H}_k$ are either k-hyperholes or perfect graphs.

[36]This is easy to see by inspection, but it also follows from Lemma 2.6(c).

Furthermore, by construction, $\mathcal{H}_k$ contains all k-hyperholes. Thus, if k is odd, then Lemma 6.4 implies that f_k is the optimal χ-bounding function for $\mathcal{H}_k$. Suppose now that k is even. By Lemma 3.2, all even hyperholes are perfect, and we deduce that all graphs in $\mathcal{H}_k$ are perfect. So, $i_{\mathbb{N}}$ is the optimal χ-bounding function for $\mathcal{H}_k$. $\square$

Corollary 6.6. *Let $k \geq 4$ be an integer. Then $\mathcal{H}_{\geq k}$ is χ-bounded. Furthermore, if k is even, then f_{k+1} is the optimal χ-bounding function for $\mathcal{H}_{\geq k}$, and if k is odd, then f_k is the optimal χ-bounding function for $\mathcal{H}_{\geq k}$.*

Proof. This follows immediately from Lemma 6.3(c) and Theorem 6.5. $\square$

Lemma 6.7. *Let $k \geq 5$ be an odd integer. Then all k-rings R satisfy $\chi(R) \leq f_k(\omega(R))$. Furthermore, there exists a sequence $\{R_n^k\}_{n=2}^{\infty}$ of k-rings such that for all integers $n \geq 2$, we have that $\omega(R_n^k) = n$ and $\chi(R_n^k) = f_k(n)$.*

Proof. Since every k-hyperhole is a k-ring, the second statement of the lemma follows immediately from the second statement of Lemma 6.4. It remains to prove the first statement. Let R be a k-ring. Then by Theorem 1.2, there exists a k-hyperhole H in R such that $\chi(R) = \chi(H)$. Thus, by Lemma 6.4, $\chi(H) \leq f_k(\omega(H))$. Clearly, $\omega(H) \leq \omega(R)$, and by Lemma 6.3(b), f_k is a nondecreasing function. We now have that

$$\chi(R) \;=\; \chi(H) \;\leq\; f_k(\omega(H)) \;\leq\; f_k(\omega(R)),$$

which is what we needed. This completes the argument. $\square$

Theorem 6.8. *Let $k \geq 4$ be an integer. Then $\mathcal{R}_k$ is χ-bounded. Furthermore, if k is even, then $i_{\mathbb{N}}$ is the optimal χ-bounding function for $\mathcal{R}_k$, and if k is odd, then f_k is the optimal χ-bounding function for $\mathcal{R}_k$.*

Proof. Suppose first that k is even. By Lemma 3.2, every k-ring R satisfies $\chi(R) = \omega(R)$. Lemma 2.10 and an easy induction now imply that $\mathcal{R}_k$ is χ-bounded by $i_{\mathbb{N}}$, and it is obvious that this χ-bounding function is optimal.

Suppose now that k is odd. By Lemma 2.8, all k-rings belong to $\mathcal{R}_k$. Thus, it suffices to show that $\mathcal{R}_k$ is χ-bounded by f_k, for optimality will then follow immediately from Lemma 6.7.

So, fix $G \in \mathcal{R}_k$, and assume inductively that all graphs $G' \in \mathcal{R}_k$ with $|V(G')| < |V(G)|$ satisfy $\chi(G') \leq f_k(\omega(G'))$. We must show that $\chi(G) \leq f_k(\omega(G))$. If G is a complete graph, then $\chi(G) = \omega(G) \leq f_k(\omega(G))$, and we are done. So assume that G is not complete, and in particular, G has at least two vertices.

Suppose that G has a simplicial vertex v. Then by Lemma 2.10, $\chi(G) = \max\{\omega(G), \chi(G \setminus v)\}$. Clearly, $\omega(G) \leq f_k(\omega(G))$. On the other hand, using the induction hypothesis and the fact that f_k is nondecreasing (by Lemma 6.3(b)), we get that $\chi(G \setminus v) \leq f_k(\omega(G \setminus v)) \leq f(\omega(G))$. It now follows that $\chi(G) = \max\{\omega(G), \chi(G \setminus v)\} \leq f_k(\omega(G))$, which is what we needed.

Suppose now that G does not contain a simplicial vertex. Then by the definition of $\mathcal{R}_k$, G is a k-ring, and so Lemma 6.7 implies that $\chi(G) \leq f_k(\omega(G))$. This completes the argument. $\square$

Corollary 6.9. *Let $k \geq 4$ be an integer. Then $\mathcal{R}_{\geq k}$ is χ-bounded. Furthermore, if k is even, then f_{k+1} is the optimal χ-bounding function for $\mathcal{R}_{\geq k}$, and if k is odd, then f_k is the optimal χ-bounding function for $\mathcal{R}_{\geq k}$.*

Proof. This follows immediately from Lemma 6.3(c) and Theorem 6.8. $\square$

A *cobipartite graph* is a graph whose complement is bipartite. Equivalently, a graph is *cobipartite* if its vertex set can be partitioned into two (possibly empty) cliques.

Lemma 6.10. *Let $k \geq 4$ be an integer, let A be a k-hyperantihole, and let $(X_1, \dots, X_k)$ be a partition of $V(A)$ into cliques such that for all $i \in \{1, \dots, k\}$, X_i is complete to $V(A) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ and anticomplete to $X_{i-1} \cup X_{i+1}$. Then for all $i \in \{1, \dots, k\}$, $A \setminus X_i$ is perfect. Furthermore, if k is even, then A is perfect.*

Proof. The Perfect Graph Theorem [9] states that a graph is perfect if and only if its complement is perfect; bipartite graphs are obviously perfect, and it follows that cobipartite graphs are also perfect. Clearly, for all $i \in \{1, \dots, k\}$, $A \setminus X_i$ is cobipartite and consequently perfect. Furthermore, if k is even, then A is cobipartite and consequently perfect. $\square$

Lemma 6.11. *Let $k \geq 5$ be an odd integer. Then all k-hyperantiholes A satisfy $\omega(A) \geq \frac{k-1}{2}$ and $\chi(A) \leq g_k(\omega(A))$. Furthermore, there exists a sequence $\{A_n^k\}_{n=\frac{k-1}{2}}^{\infty}$ of k-hyperantiholes such that for all integers $n \geq \frac{k-1}{2}$, we have that $\omega(A_n^k) = n$ and $\chi(A_n^k) = g_k(n)$.*

Proof. We begin by proving the first statement of the lemma. Let A be a k-hyperantihole, and let $(X_1, \dots, X_k)$ be a partition of $V(A)$ into nonempty cliques such that for all $i \in \{1, \dots, k\}$, X_i is complete to $V(A) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ and anticomplete to $X_{i-1} \cup X_{i+1}$, as in the definition of a k-hyperantihole. Since A is a k-hyperantihole, and since k is odd, we see that $\omega(A) \geq \lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$.

By symmetry, we may assume that $|X_2| = \min\{|X_1|, \dots, |X_k|\}$. Since $\bigcup_{i=1}^{(k-1)/2} X_{2i}$ is a clique, we see that $\sum_{i=1}^{(k-1)/2} |X_{2i}| \leq \omega(A)$, and so by the minimality of $|X_2|$, we have that $|X_2| \leq \left\lfloor \frac{2\omega(A)}{k-1} \right\rfloor$.

By construction, X_2 is anticomplete to $X_1 \cup X_3$ in A, and $|X_2| \leq |X_1|, |X_3|$. Fix any $X_1^2 \subseteq X_1$ and $X_3^2 \subseteq X_3$ such that either $|X_1^2| = \left\lfloor |X_2|/2 \right\rfloor$

and $|X_3^2| = \left\lceil |X_2|/2 \right\rceil$, or $|X_1^2| = \left\lceil |X_2|/2 \right\rceil$ and $|X_3^2| = \left\lfloor |X_2|/2 \right\rfloor$.[37] Let $X_2^* = X_1^2 \cup X_2 \cup X_3^2$. Note that X_2 and $X_2^* \setminus X_2 = X_1^2 \cup X_3^2$ are cliques in A, they are anticomplete to each other in A, and they are both of size $|X_2|$. Thus, $\chi(A[X_2^*]) = |X_2|$.

By Lemma 6.10, $A \setminus X_2$ is perfect. Since $A \setminus X_2^*$ is an induced subgraph of $A \setminus X_2$, it follows that $\chi(A \setminus X_2^*) = \omega(A \setminus X_2^*)$. Let K be a maximum clique of $A \setminus X_2^*$. (In particular, $K \cap X_2 = \emptyset$.) Then

$$\begin{aligned}
\chi(A) &\leq \chi(A \setminus X_2^*) + \chi(A[X_2^*]) \\
&= \omega(A \setminus X_2^*) + |X_2| \\
&= |K| + |X_2| \\
&= |K \cup X_2|.
\end{aligned}$$

Suppose first that K intersects neither $X_1 \setminus X_1^2$ nor $X_3 \setminus X_3^2$. Since $K \subseteq V(A) \setminus X_2^*$, it follows that $K \cap (X_1 \cup X_3) = \emptyset$. Then X_2 is complete to K. Thus, $K \cup X_2$ is a clique of A, and it follows that $|K \cup X_2| \leq \omega(A)$; consequently,

$$\chi(A) \leq |K \cup X_2| \leq \omega(A) \leq g_k(\omega(A)),$$

and we are done.

Suppose now that K intersects at least one of $X_1 \setminus X_1^2$ and $X_3 \setminus X_3^2$; by symmetry, we may assume that $K \cap (X_1 \setminus X_1^2) \neq \emptyset$. Then $K \cup X_1^2$ is a clique of A, and it follows that $|K \cup X_1^2| \leq \omega(A)$; consequently,

$$|K| \leq \omega(A) - |X_1^2| \leq \omega(A) - \left\lfloor |X_2|/2 \right\rfloor,$$

and so

$$\begin{aligned}
\chi(A) &\leq |K| + |X_2| \\
&\leq (\omega(A) - \left\lfloor |X_2|/2 \right\rfloor) + |X_2| \\
&= \omega(A) + \left\lceil |X_2|/2 \right\rceil \\
&\leq \omega(A) + \left\lceil \left\lfloor \tfrac{2\omega(A)}{k-1} \right\rfloor /2 \right\rceil.
\end{aligned}$$

By Lemma 6.2, we now have that

$$\chi(A) \leq \omega(A) + \left\lceil \left\lfloor \tfrac{2\omega(A)}{k-1} \right\rfloor /2 \right\rceil = g_k(\omega(A)),$$

[37] This way, we maintain full symmetry between X_1 and X_3.

and again we are done. This proves the first statement of the lemma.

It remains to prove the second statement of the lemma. We fix an integer $n \geq \frac{k-1}{2}$, and we construct A_n^k as follows. Set $m = \lfloor \frac{n}{k-1} \rfloor$ and $\ell = n - (k-1)m$. Clearly, m is a nonnegative integer, $\ell \in \{0, \dots, k-2\}$, $n = (k-1)m + \ell$, and $n \equiv \ell \pmod{k-1}$. Now, let $X_1, \dots, X_k$ be pairwise disjoint sets such that for all $i \in \{1, \dots, k\}$,

- if $0 \leq \ell \leq \frac{k-3}{2}$, then $|X_1| = \dots = |X_{2\ell}| = 2m+1$ and $|X_{2\ell+1}| = \dots = |X_k| = 2m$;
- if $\frac{k-1}{2} \leq \ell \leq k-2$, then $|X_1| = \dots = |X_{2\ell-k+1}| = 2m+2$ and $|X_{2\ell-k+2}| = \dots = |X_k| = 2m+1$.

Since $n \geq \frac{k-1}{2}$, sets $X_1, \dots, X_k$ are all nonempty. Let A_n^k be the graph with vertex set $V(A_n^k) = X_1 \cup \dots \cup X_k$, and with adjacency as follows:

- $X_1, \dots, X_k$ are all cliques;
- for all $i \in \{1, \dots, k\}$, X_i is complete to $V(A_n^k) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ and anticomplete to $X_{i-1} \cup X_{i+1}$.

Clearly, A_n^k is a k-hyperantihole. We must show that $\omega(A_n^k) = n$ and $\chi(A_n^k) = g_k(n)$.

We first show that $\omega(A_n^k) = n$. Suppose first that $0 \leq \ell \leq \frac{k-3}{2}$. Now 2ℓ consecutive X_i's are of size $2m+1$ (since they are consecutive, at most ℓ of them can be included in a clique of A_n^k), and all the other X_i's are of size $2m$. So, a maximum clique of A_n^k is the union of ℓ sets X_i of size $2m+1$, and of $\frac{k-1}{2} - \ell$ sets X_i of size $2m$. It follows that

$$\omega(A_n^k) \quad = \quad \ell(2m+1) + \Big(\tfrac{k-1}{2} - \ell\Big)2m \quad = \quad (k-1)m + \ell \quad = \quad n,$$

which is what we needed.

Suppose now that $\frac{k-1}{2} \leq \ell \leq k-2$. Then $2\ell - k + 1$ consecutive X_i's are of size $2m+2$ (since they are consecutive, at most $\lceil \frac{2\ell-k+1}{2} \rceil = \ell - \frac{k-1}{2}$ of them can be included in a clique of A_n^k), and all the other X_i's are of size $2m+1$. So, a maximum clique of A_n^k is the union of $\ell - \frac{k-1}{2}$ sets X_i of size $2m+2$, and of $\frac{k-1}{2} - (\ell - \frac{k-1}{2}) = k - \ell - 1$ sets X_i of size $2m+1$. It follows that

$$\begin{aligned} \omega(A_n^k) &= \Big(\ell - \tfrac{k-1}{2}\Big)(2m+2) + (k-\ell-1)(2m+1) \\ &= (k-1)m + \ell \\ &= n, \end{aligned}$$

which is what we needed.

We have now shown that $\omega(A_n^k) = n$. It remains to show that $\chi(A_n^k) = g_k(n)$. But by the first statement of the lemma, we have that $\chi(A_n^k) \leq$

$g_k(n)$, and so in fact, it suffices to show that $\chi(A_n^k) \geq g_k(n)$. Clearly, $\chi(A_n^k) \geq \left\lceil \frac{|V(A_n^k)|}{\alpha(A_n^k)} \right\rceil$, and since A_n^k is a hyperantihole, we see that $\alpha(A_n^k) = 2$. Thus, $\chi(A_n^k) \geq \left\lceil \frac{1}{2}|V(A_n^k)| \right\rceil$.

Suppose first that $0 \leq \ell \leq \frac{k-3}{2}$. Recall that $n \geq \frac{k-1}{2}$, and so $g_k(n) = \lfloor \frac{kn}{k-1} \rfloor$. We now have that

$$\begin{aligned}
\chi(A_k^n) &\geq \left\lceil \tfrac{1}{2}|V(A_n^k)| \right\rceil \\
&= \left\lceil \tfrac{1}{2}\Big(2\ell(2m+1) + (k-2\ell)2m\Big) \right\rceil \\
&= km + \ell \\
&= n + m \\
&= \lfloor \tfrac{kn}{k-1} \rfloor \\
&= g_k(n),
\end{aligned}$$

which is what we needed.

Suppose now that $\frac{k-1}{2} \leq \ell \leq k-2$. Since $\ell \neq 0$, we see that $\frac{kn}{k-1}$ is not an integer, and so $\lfloor \frac{kn}{k-1} \rfloor + 1 = \lceil \frac{kn}{k-1} \rceil$. Further, since $\frac{k-1}{2} \leq \ell \leq k-2$, we have that $g_k(n) = \lceil \frac{kn}{k-1} \rceil$. We then see that

$$\begin{aligned}
\chi(A_k^n) &\geq \left\lceil \tfrac{1}{2}|V(A_n^k)| \right\rceil \\
&= \left\lceil \tfrac{1}{2}\Big((2\ell-k+1)(2m+2) + (2k-2\ell-1)(2m+1)\Big) \right\rceil \\
&= \left\lceil km + \ell + \tfrac{1}{2} \right\rceil \\
&= km + \ell + 1 \\
&= n + m + 1 \\
&= \lfloor \tfrac{kn}{k-1} \rfloor + 1 \\
&= \lceil \tfrac{kn}{k-1} \rceil \\
&= g_k(n),
\end{aligned}$$

which is what we needed. This proves the second statement of the lemma.

□

Theorem 6.12. *Let $k \geq 4$ be an integer. Then $\mathcal{A}_k$ is χ-bounded. Furthermore, if k is even, then $i_{\mathbb{N}}$ is the optimal χ-bounding function for $\mathcal{A}_k$, and if k is odd, then g_k is the optimal χ-bounding function for $\mathcal{A}_k$.*

Proof. If k is even, then by Lemma 6.10, all graphs in $\mathcal{A}_k$ are perfect, and it follows that $i_{\mathbb{N}}$ is the optimal χ-bounding function for $\mathcal{A}_k$. Suppose now that k is odd. Clearly, all k-hyperantiholes belong to $\mathcal{A}_k$; on the other hand, it follows from Lemma 6.10 that all graphs in $\mathcal{A}_k$ are either k-hyperantiholes or perfect graphs. The fact that g_k is the optimal χ-bounding function for $\mathcal{A}_k$ now follows from Lemma 6.11, and from the definition of g_k. □

Corollary 6.13. *Let $k \geq 4$ be an integer. Then $\mathcal{A}_{\geq k}$ is χ-bounded. If k is even, then g_{k+1} is the optimal χ-bounding function for $\mathcal{A}_{\geq k}$, and if k is odd, then g_k is the optimal χ-bounding function for $\mathcal{A}_{\geq k}$.*

Proof. This follows immediately from Lemma 6.3(c) and Theorem 6.12. □

We remind the reader that the function $f_{\mathrm{T}} : \mathbb{N} \to \mathbb{N}$ is given by

$$f_{\mathrm{T}}(n) \quad = \quad \begin{cases} \lfloor 5n/4 \rfloor & \text{if} \quad n \equiv 0, 1 \pmod 4 \\ \\ \lceil 5n/4 \rceil & \text{if} \quad n \equiv 2, 3 \pmod 4 \end{cases}$$

for all $n \in \mathbb{N}$.

Note that $\mathcal{H}_{\geq 4}$ is the class of all induced subgraphs of hyperholes, and that $\mathcal{A}_{\geq 4}$ is the class of all induced subgraphs of hyperantiholes. Furthermore, by Lemma 2.8, $\mathcal{R}_{\geq 4}$ contains all induced subgraphs of rings. In particular, $\mathcal{H}_{\geq 4} \subseteq \mathcal{R}_{\geq 4}$.

Theorem 6.14. *Classes $\mathcal{H}_{\geq 4}$, $\mathcal{A}_{\geq 4}$, and $\mathcal{R}_{\geq 4}$ are χ-bounded. Furthermore, f_T is the optimal χ-bounding function for all three classes.*

Proof. By Lemma 6.3, we have that $f_{\mathrm{T}} = f_5 = g_5$. The result now follows immediately from Corollaries 6.6, 6.9, and 6.13. □

Theorem 6.15. *$\mathcal{G}_T$ is χ-bounded. Furthermore, f_T is the optimal χ-bounding function for $\mathcal{G}_T$.*

Proof. We begin by showing f_{T} is a χ-bounding function for $\mathcal{G}_{\mathrm{T}}$. First, by Lemma 6.3, we have that f_{T} is nondecreasing, and that $f_{\mathrm{T}} = f_5 = g_5$. Now, fix $G \in \mathcal{G}_{\mathrm{T}}$, and assume inductively that for all $G' \in \mathcal{G}_{\mathrm{T}}$ such that $|V(G')| < |V(G)|$, we have that $\chi(G') \leq f_{\mathrm{T}}(\omega(G'))$.

By Theorem 2.11, we know that either G is a complete graph, a ring, or a 7-hyperantihole, or G admits a clique-cutset. If G is a complete graph, a ring, or a 7-hyperantihole, then $G \in \mathcal{R}_{\geq 4} \cup \mathcal{A}_{\geq 4}$, and Theorem 6.14 guarantees that $\chi(G) \leq f_{\mathrm{T}}(\omega(G))$. It remains to consider the case when G admits a clique-cutset. Let (A, B, C) be a clique-cut-partition of G, and set $G_A = G[A \cup C]$

and $G_B = G[B \cup C]$. Clearly, $\chi(G) = \max\{\chi(G_A), \chi(G_B)\}$. Using the induction hypothesis and the fact that f_{T} is nondecreasing, we now get that

$$\begin{array}{rcl} \chi(G) & = & \max\{\chi(G_A), \chi(G_B)\} \\ & \leq & \max\{f_{\mathrm{T}}(\omega(G_A)), f_{\mathrm{T}}(\omega(G_B))\} \\ & \leq & f_{\mathrm{T}}(\omega(G)), \end{array}$$

which is what we needed. This proves that f_{T} is indeed a χ-bounding function for $\mathcal{G}_{\mathrm{T}}$.

It remains to establish the optimality of f_{T}. Let $n \in \mathbb{N}$; we must exhibit a graph $G \in \mathcal{G}_{\mathrm{T}}$ such that $\omega(G) = n$ and $\chi(G) = f_{\mathrm{T}}(n)$. If $n = 1$, then we observe that $K_1 \in \mathcal{G}_{\mathrm{T}}$, $\omega(K_1) = 1$, and $\chi(K_1) = 1 = f_{\mathrm{T}}(1)$. So assume that $n \geq 2$. Let H_n^5 be as in the statement of Lemma 6.4. Then H_n^5 is a 5-hyperhole, and it is easy to see that all hyperholes belong to $\mathcal{G}_{\mathrm{T}}$; thus, $H_n^5 \in \mathcal{G}_{\mathrm{T}}$. Further, since $f_{\mathrm{T}} = f_5$, Lemma 6.4 guarantees that $\omega(H_n^5) = n$ and $\chi(H_n^5) = f_5(n) = f_{\mathrm{T}}(n)$. Thus, f_{T} is indeed the optimal χ-bounding function for $\mathcal{G}_{\mathrm{T}}$. □

7 Class $\mathcal{G}_{\mathrm{T}}$ and Hadwiger's conjecture

In this section, we prove Hawdiger's conjecture for the class $\mathcal{G}_{\mathrm{T}}$ (see Theorem 7.4).

Lemma 7.1. *Every hyperhole H contains $K_{\chi(H)}$ as a minor.*

Proof. Let H be a hyperhole, and let k be the its length. Let $(X_1, \dots, X_k)$ be a partition of $V(H)$ into nonempty cliques such that for all $i \in \{1, \dots, k\}$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$. By symmetry, we may assume that $|X_1| = \min\{|X_1|, |X_2|, \dots, |X_k|\}$. Clearly, $\chi(H \setminus X_1) = \omega(H \setminus X_1)$,[38] and furthermore, there exists some index $j \in \{2, \dots, k-1\}$ such that $\omega(H \setminus X_1) = |X_j \cup X_{j+1}|$. By the choice of X_1, we see that there are $|X_1|$ vertex-disjoint paths between X_{j-1} and X_{j+2}, none of them passing through $X_j \cup X_{j+1}$. We then take our $|X_1|$ paths and the vertices of $X_j \cup X_{j+1}$ as branch sets, and we obtain a $K_{|X_1| + \omega(H \setminus X_1)}$ minor in G. Since $\chi(H) \leq |X_1| + \chi(H \setminus X_1) = |X_1| + \omega(H \setminus X_1)$, we conclude that H contains $K_{\chi(H)}$ as a minor. □

Lemma 7.2. *Every ring R contains $K_{\chi(R)}$ as a minor.*

Proof. This follows immediately from Theorem 1.2 and Lemma 7.1. □

Lemma 7.3. *Every hyperantihole A contains $K_{\chi(A)}$ as a minor.*

[38] By Lemma 2.6(c), $H \setminus X_1$ is chordal, and by [3], chordal graphs are perfect. So, $H \setminus X_1$ is perfect and therefore satisfies $\chi(H \setminus X_1) = \omega(H \setminus X_1)$.

Proof. Let A be a hyperantihole, and let k be its length. Let $(X_1, \dots, X_k)$, with $k \geq 4$, be a partition of $V(A)$ into nonempty cliques, such that for all $i \in \{1, \dots, k\}$, X_i is complete to $A \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ and anticomplete the $X_{i-1} \cup X_{i+1}$, as in the definition of a hyperantihole. If $k = 4$, then $V(K)$ can be partitioned into two cliques (namely $X_1 \cup X_3$ and $X_2 \cup X_4$), anticomplete to each other, and the result is immediate. From now on, we assume that $k \geq 5$.

By symmetry, we may assume that $|X_1| = \min\{|X_1|, |X_2|, \dots, |X_k|\}$. Clearly, $\chi(A) \leq \chi(A \setminus X_1) + |X_1|$. On the other hand, by Lemma 6.10, $A \setminus X_1$ is perfect, and in particular, $\chi(A \setminus X_1) = \omega(A \setminus X_1)$. Let K be a clique of size $\omega(A \setminus X_1)$ in $A \setminus X_1$. Then, $\chi(A) \leq |K| + |X_1|$, and so it suffices to show that A contains $K_{|K|+|X_1|}$ as a minor.

If $K \cap (X_{k-1} \cup X_2) = \emptyset$, then X_1 is complete to K in A, $K \cup X_1$ is a clique of size $|K| + |X_1|$ in A, and we are done.

Next, suppose that $K \cap X_2 \neq \emptyset$ and $K \cap X_k = \emptyset$. Since K is a clique that intersects X_2, and since X_2 is anticomplete to X_3, we see that $K \cap X_3 = \emptyset$. By construction, $|X_1| \leq |X_3|, |X_k|$; since $k \geq 5$, we see that X_3 is complete to $X_1 \cup X_k$, and we deduce that there exist $|X_1|$ vertex-disjoint two-edge paths, each of the form a, b, c, where $a \in X_1$, $b \in X_3$, and $c \in X_k$. We now take our $|X_1|$ two-edge paths and all the vertices of K as branch sets, and we obtain a $K_{|K|+|X_1|}$ minor in A. The case when $K \cap X_2 = \emptyset$ and $K \cap X_k \neq \emptyset$ is analogous.

By symmetry, it remains to consider the case when $K \cap X_2$ and $K \cap X_k$ are both nonempty. Since K is a clique that intersects X_2, and since X_2 is anticomplete to X_3, we see that $K \cap X_3 = \emptyset$, and similarly, $K \cap X_{k-1} = \emptyset$. By construction, X_1 is complete to $X_3 \cup X_{k-1}$ (note that $k - 1 \neq 3$, since $k \geq 5$), and so there exist $|X_1|$ vertex-disjoint two-edge paths in A, each of the form a, b, c, where $a \in X_3$, $b \in X_1$, and $c \in X_{k-1}$. We now take our $|X_1|$ two-edge paths, and all the vertices of K as branch sets, and we obtain a $K_{|K|+|X_1|}$ minor in A. This completes the argument. □

Theorem 7.4. *Every graph $G \in \mathcal{G}_T$ contains $K_{\chi(G)}$ as a minor.*

Proof. Fix $G \in \mathcal{G}_{\mathrm{T}}$, and assume inductively that every graph $G' \in \mathcal{G}_{\mathrm{T}}$ with $|V(G')| < |V(G)|$ contains $K_{\chi(G')}$ as a minor. We must show that G contains $K_{\chi(G)}$ as a minor. We apply Theorem 2.11. Suppose first that G admits a clique-cutset, and let (A, B, C) be a clique-cut-partition of G. Clearly, $\chi(G) = \max\{\chi(G[A \cup C]), \chi(G[B \cup C])\}$, and the result follows from the induction hypothesis. So assume that G does not admit a clique-cutset. Then Theorem 2.11 implies that G is a complete graph, a ring, or a 7-hyperantihole; in the first case, the result is immediate, in the second, it follows from Lemma 7.2, and in the third, it follows from Lemma 7.3. □

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