Two-block paths in oriented graphs of large semidegree

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April 6, 2025

Abstract

We study the existence of oriented paths with two blocks in oriented graphs under semidegree conditions. A *block* of an oriented path is a maximal directed subpath. Given positive integers k and ℓ with $k/2 \leq \ell < k$, we establish a semidegree function that guarantees the containment of every oriented path with two blocks of sizes ℓ and $k - \ell$. As a corollary, we show that every oriented graph with all in- and out-degrees at least $3k/4$ contains every two-block path with k arcs. Our results extend previous work on Stein's conjecture and related problems concerning oriented paths.

1   Introduction

The problem of determining the existence of long paths in a graph with minimum degree constraints is well-studied [3, 15]. In particular, Dirac [3] proved that a minimum degree of at least $n/2$ guarantees the existence of a Hamilton cycle in a graph with n vertices, while a minimum degree of at least $(n-1)/2$ ensures the containment of a Hamilton path. This problem naturally extends to oriented graphs and digraphs. An *oriented graph* is a digraph that contains no directed cycles of length one or two. As a natural analogue to the minimum degree in graphs, the concept of minimum semidegree has been used in the study of oriented Hamilton cycles [6, 7, 8, 11, 12, 13]. The *minimum semidegree* of a digraph D, denoted by $\delta^0(D)$, is defined as the minimum value among the in-degrees and out-degrees of all vertices in D.

A related problem involves finding paths of a given size in a graph. Erdős and Gallai [5] proved that a connected graph with at least $k + 1$ vertices and minimum degree at least $k/2$ contains a k-edge path. In oriented graphs Jackson [10] proved that $\delta^0(G) \geq k/2$ implies G contains a directed path, that is a path where all arcs have the same direction, with k arcs. More recently, Stein [18] conjectured a generalization of Jackson's theorem to all oriented paths.

Conjecture 1.1 (Stein, [18]). *Every oriented graph G with $\delta^0(G) > k/2$ contains every orientation of the k-edge path.*

Note that Conjecture 1.1 becomes straightforward when the minimum semidegree bound is strengthened to $\delta^0(G) \geq k$, as a simple greedy embedding succeeds in all cases. However, the problem becomes significantly more challenging when working with weaker bounds. Thus, in the pursuit of proving Stein's conjecture, the study of certain orientations of the path offers a direction for further exploration. As noted above, Jackson's result [10] confirms Conjecture 1.1 for the case of directed paths. Moreover, Stein and Trujillo-Negrete [19] confirmed Conjecture 1.1, as a consequence of a more general result, for the class of oriented graphs containing no oriented 4-cycle.
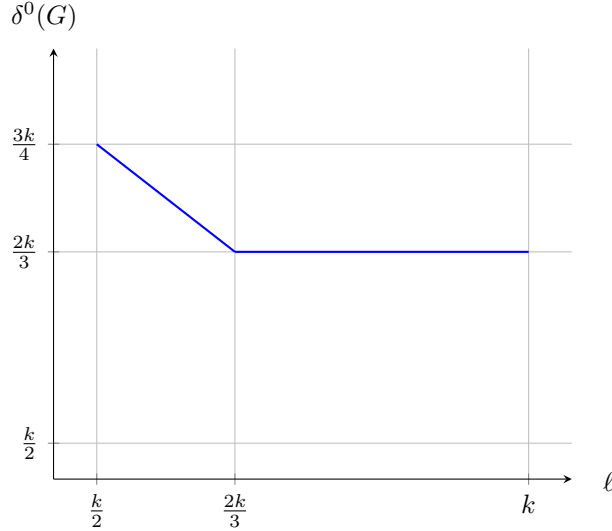
Figure 1: The semidegree function given in Theorem 1.2.

Further progress has been achieved for antidirected paths. An *antidirected path* is an oriented path that alternates the direction of its arcs. Klimošová and Stein [14] showed that every oriented graph G with a minimum semidegree of $3k/4$ contains each antidirected path with k arcs. Their work was later improved by Chen, Hou, and Zhou [2], by reducing the required semidegree to $2k/3$. Most recently, Skokan and Tyomkyn [17] reduced it further to $5k/8$.

The next most natural class of oriented paths to consider are perhaps the paths with two blocks. A *block* of an oriented path P is a maximal directed subpath of P, and its *size* is the number of its arcs. The containment of paths with two blocks in oriented graphs and digraphs has been previously studied under chromatic number constraints [1, 4, 16]. El-Sahili [4] conjectured that a digraph with chromatic number k contains as a subgraph every k-arc oriented path with two blocks. This conjecture was confirmed by Addario-Berry, Havet and Thomassé [1].

In this paper, we study the existence of paths with two blocks in oriented graphs under minimum semidegree conditions. Our main result is the following.

Theorem 1.2. *Let k and ℓ be positive integers with $k/2 \leq \ell < k$. Let G be an oriented graph such that*

$$\delta^0(G) \geq \begin{cases} k - \frac{\ell}{2} & \text{if } \ell \leq \frac{2k}{3}, \\ \frac{2k}{3} & \text{if } \ell > \frac{2k}{3}. \end{cases}$$

Then G contains as a subgraph both possible k-arc oriented paths with two blocks of size ℓ and $k - \ell$.

The semidegree function in Theorem 1.2 is depicted in Figure 1. As a direct consequence, we obtain a general bound on $\delta^0(G)$ for the containment of paths with two blocks, irrespective of the size of the blocks.

Corollary 1.3. *Let $k \geq 2$ be an integer and let G be an oriented graph with $\delta^0(G) \geq 3k/4$. Then G contains as a subgraph every k-arc oriented path with two blocks.*

2   Notation

For an oriented graph G, $V(G)$ denotes the set of vertices and $A(G)$ the set of arcs in G. Given a vertex v of an oriented graph G, let $N^+(v) = \{u \in V(G) : (v, u) \in A(G)\}$ and $N^-(v) = \{u \in V(G) : (u, v) \in A(G)\}$ denote its *out-neighbourhood* and *in-neighbourhood*, respectively. The *out-degree* and *in-degree* of v are $\deg^+(v) = |N^+(v)|$ and $\deg^-(v) = |N^-(v)|$, respectively. The *minimum out-degree* and *minimum in-degree* of G are

$$\delta^+(G) := \min_{v \in V(G)} \{\deg^+(v)\} \quad \text{and} \quad \delta^-(G) := \min_{v \in V(G)} \{\deg^-(v)\},$$
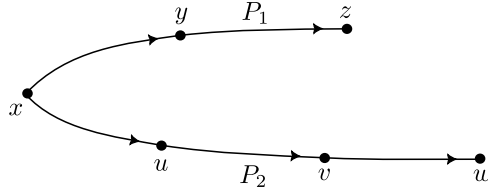
Figure 2: Concatenation $\overleftarrow{P_1}P_2$, where $P_1 = xyz$ and $P_2 = xuvw$.

$$P(\overleftarrow{5}, \overrightarrow{4}) \qquad\qquad P(\overrightarrow{5}, \overleftarrow{5})$$
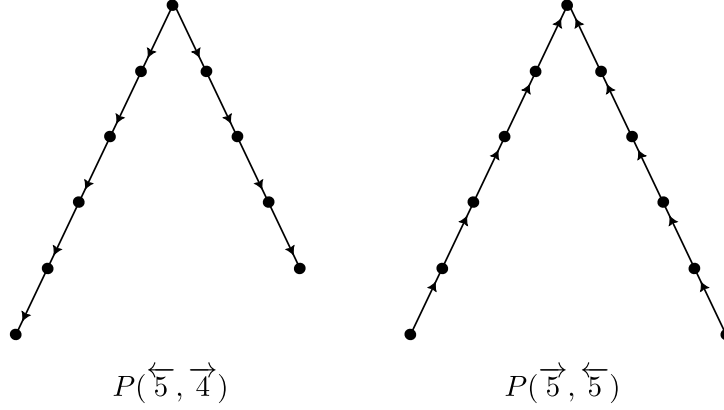
Figure 3: Examples of paths with two blocks. On the left, $P(\overleftarrow{5}, \overrightarrow{4})$ consists of a first block of 5 backward arcs followed by a second block of 4 forward arcs. On the right, $P(\overrightarrow{5}, \overleftarrow{5})$ consists of a first block of 5 forward arcs followed by a second block of 5 backward arcs.

respectively. Additionally, we define the *minimum semidegree* of G as $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. A directed cycle C in G is said to be a *Hamilton cycle* if $V(C) = V(G)$.

The *length* of a path P, denoted by $\mathrm{length}(P)$, is defined as the number of its arcs. Given a directed path $P = u_1 u_2 \ldots u_p$, its *reverse* is the path $\overleftarrow{P} = u_p u_{p-1} \ldots u_1$. (Here we emphasize that as oriented graphs P and $\overleftarrow{P}$ are exactly the same. The only difference is in the order of vertices that we use to represent the path.) For two vertices u_i and u_j in P, with $i < j$, let $u_i \underset{P}{\rightarrow} u_j$ be the subpath of P given by $u_i u_{i+1} \ldots u_j$. Given two directed paths P_1 and P_2 such that their initial vertices coincide, we write $\overleftarrow{P_1}P_2$ to denote the concatenation of $\overleftarrow{P_1}$ and P_2. See an example in Figure 2. We denote by $P(\overleftarrow{r}, \overrightarrow{s})$ the $(r + s)$-arc path with two blocks, where the first block consists of r backward arcs followed by a second block of s forward arcs. Similarly, the notation $P(\overrightarrow{r}, \overleftarrow{s})$ represents the reverse configuration. See Figure 3 for two examples.

Given a path P and an oriented graph G, an *embedding* from P to G is an injective function $f : V(P) \to V(G)$ preserving adjacencies, that is, for each arc (u, v) of P, we have that $(f(u), f(v))$ is an arc of G. If such an embedding exists, we say that P *embeds* in G. In this paper, we interchangeably say that a path P embeds in G or that G contains the path P.

3   Preliminary results

We first state a theorem of Jackson [10] on directed paths which will be a useful tool in the proof of Theorem 1.2.

Theorem 3.1 (Jackson [10])**.** *Every oriented graph G contains a directed path on $2\delta^0(G)$ arcs.*

In the following proposition we gather some properties of G that ensure an embedding of $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$.

Proposition 3.2. *Let k and ℓ be integers with $k/2 \leq \ell < k$. Let G be an oriented graph with $\delta^0(G) \geq k - \ell$. Let $P = v_0 v_1 \ldots v_t$ be a directed path of maximum length in G, and further suppose that $t \geq 2\ell$.*

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Define the sets

$$X = \{v_0, \ldots, v_{k-\ell-1}\}, \quad Y = \{v_{k-\ell}, \ldots, v_{t-k+\ell}\} \quad and \quad Z = \{v_{t-k+\ell+1}, \ldots, v_t\}.$$

Suppose one of the following conditions holds:

(i) $N^-(v_0) \cap Y \neq \emptyset$ or $N^+(v_t) \cap Y \neq \emptyset$.

(ii) $N^+(v_{i-1}) \not\subseteq V(P)$ for some $v_i \in N^+(v_t) \cap (X \setminus \{v_0\})$.

(iii) $G[V(P)]$ contains a Hamilton cycle.

Then $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ embeds in G.

Proof. Observe that X, Y and Z are non-empty and pairwise disjoint. Moreover, $|X| = |Z| = k - \ell$. We proceed by cases according to the proposition.

(i) First, suppose that $N^-(v_0) \cap Y \neq \emptyset$, and let $v_i \in N^-(v_0) \cap Y$. We define the paths

$$P_1 := v_i v_0 \xrightarrow{P} v_{i-1} \quad and \quad P_2 := v_i \xrightarrow{P} v_t.$$

See Figure 4 (left). We have $\mathrm{length}(P_1) = i$ and $\mathrm{length}(P_2) = t - i$. Moreover, since $X \subsetneq V(P_1)$ and $Z \subsetneq V(P_2)$, it follows that $\mathrm{length}(P_1), \mathrm{length}(P_2) \geq k - \ell$.

Next, we show that at least one of P_1 and P_2 has length at least ℓ. Note that this readily implies that $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ is a subpath of $\overleftarrow{P_1} P_2$, and thus embeds in G. Observe that if $i \geq \ell$, then $\mathrm{length}(P_1) = i \geq \ell$, whereas if $i < \ell$, then $\mathrm{length}(P_2) = t - i > t - \ell \geq \ell$ (because $t \geq 2\ell$).

Now, suppose that $N^+(v_t) \cap Y \neq \emptyset$, and let $v_j \in N^+(v_t) \cap Y$. By the maximality of P, we have that $N^-(v_0) \subseteq V(P) \setminus \{v_0\}$, and by the previous case, we may assume that $N^-(v_0) \cap Y = \emptyset$. It follows that $N^-(v_0) \subseteq (X \setminus \{v_0\}) \cup Z$. Since $\delta^0(G) \geq k - \ell = |X| > |X \setminus \{v_0\}|$, we must have $N^-(v_0) \cap Z \neq \emptyset$. Let $v_i \in N^-(v_0) \cap Z$. We define the paths

$$P_1 := v_i v_0 \xrightarrow{P} v_{j-1} \quad and \quad P_2 := v_i \xrightarrow{P} v_t v_j \xrightarrow{P} v_{i-1}.$$

See Figure 4 (right). We have $\mathrm{length}(P_1) = j$ and $\mathrm{length}(P_2) = t - j$. Again, as we have $X \subsetneq V(P_1)$ and $Z \subsetneq V(P_2)$, it follows that $\mathrm{length}(P_1), \mathrm{length}(P_2) \geq k - \ell$. Furthermore, if $j \geq \ell$, then $\mathrm{length}(P_1) \geq \ell$, and if $j < \ell$, then $\mathrm{length}(P_2) > t - \ell \geq \ell$. Thus, $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ is a subpath of $\overleftarrow{P_1} P_2$.
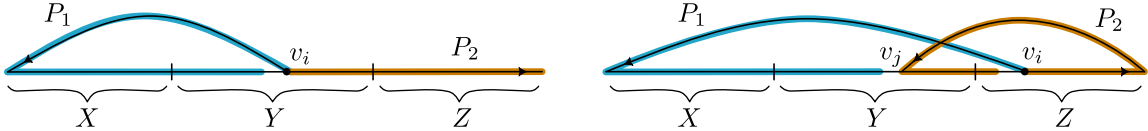

Figure 4: Construction of paths P_1 and P_2 in the proof of Proposition 3.2 (i).

(ii) Fix a vertex $v_i \in N^+(v_t) \cap (X \setminus \{v_0\})$ such that $N^+(v_{i-1}) \not\subseteq V(P)$. Let $w_0 \in N^+(v_{i-1}) \setminus V(P)$, and among all the directed paths in $G[V(G) \setminus V(P)]$ starting at w_0, choose a directed path $P' = w_0 \ldots w_m$ of maximum length. Define the cycle $C := v_i \xrightarrow{P} v_t v_i$. Since $v_i \in X$ and $t \geq 2\ell \geq k$, it follows that

$$|C| > |Y \cup Z| = |V(P)| - |X| = (t+1) - (k-\ell) \geq \ell + 1.$$

Suppose first that $m \geq k - \ell - 1$, and consider the following paths:

$$P_1 := v_{i-1} w_0 \xrightarrow{P'} w_m \quad and \quad P_2 := v_{i-1} \xrightarrow{P} v_t.$$

We have $\mathrm{length}(P_1) \geq m + 1 \geq k - \ell$ and $\mathrm{length}(P_2) = |C| > \ell$. Thus, $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ is a subpath of $\overleftarrow{P_1} P_2$.
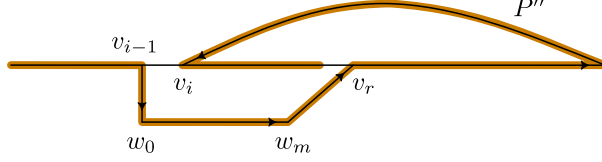
Suppose now that $m \leq k - \ell - 2$. By the maximality of P', we have that

$$N^+(w_m) \subseteq V(P) \cup (V(P') \setminus \{w_m\}).$$

We distinguish two cases:

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- Suppose that $N^+(w_m) \cap V(C) \neq \emptyset$, and let $v_r \in N^+(w_m) \cap V(C)$. Define

$$P'' := \begin{cases} v_0 \underset{P}{\to} v_{i-1}w_0 \underset{P'}{\longrightarrow} w_m v_r \underset{P}{\to} v_t v_i \underset{P}{\to} v_{r-1} & \text{if } r \neq i, \\ v_0 \underset{P}{\to} v_{i-1}w_0 \underset{P'}{\longrightarrow} w_m v_i \underset{P}{\to} v_t & \text{if } r = i. \end{cases}$$

Since $\text{length}(P'') = \text{length}(P) + m + 1 > \text{length}(P)$, this contradicts the maximality of P.

- Suppose that $N^+(w_m) \cap V(C) = \emptyset$, so that $N^+(w_m) \subseteq (V(P) \setminus V(C)) \cup (V(P') \setminus \{w_m\})$. Since

$$|V(P') \setminus \{w_m\}| = m \leq k - \ell - 2 \leq \delta^0(G) - 2,$$

it follows that w_m has at least two out-neighbors in $V(P) \setminus V(C) = \{v_0, \ldots, v_{i-1}\}$. Consequently, we have that $i \geq 2$, and that w_m has at least one out-neighbor in $\{v_0, \ldots, v_{i-2}\}$. Let $j \in \{0, \ldots, i-2\}$ be the smallest index such that $v_j \in N^+(w_m)$. We then define the paths

$$P_1 := v_{i-1}w_0 \underset{P'}{\longrightarrow} w_m v_j \underset{P}{\to} v_{i-2} \quad \text{and} \quad P_2 := v_{i-1} \underset{P}{\to} v_t.$$

Then $\{w_m\} \cup N^+(w_m) \subseteq V(P_1)$, and consequently, $\text{length}(P_1) = |V(P_1)| - 1 \geq |N^+(w_m)| \geq \delta^0(G) \geq k - \ell$. On the other hand, $\text{length}(P_2) = |V(C)| > \ell$. Thus, $P(\overset{\leftarrow}{\ell}, \overrightarrow{k-\ell})$ embeds in G.

(iii) Without loss of generality, suppose the Hamilton cycle is $v_0 v_1 \ldots v_t v_0$; if not, we may relabel the vertices in P. Then, all the in- and out-neighbours of v_0 must belong to $V(P)$, for otherwise we could find a path in G longer than P, contradicting its maximality. Thus, $N^+(v_0) \cup N^-(v_0) \subseteq V(P) \setminus \{v_0\} = (X \setminus \{v_0\}) \cup Y \cup Z$. So, since $|N^+(v_0) \cup N^-(v_0)| \geq 2\delta^0(G) \geq 2(k-\ell) > |X \setminus \{v_0\}| + |Z|$, it follows that $(N^+(v_0) \cup N^-(v_0)) \cap Y \neq \emptyset$. Let $v_i \in (N^+(v_0) \cup N^-(v_0)) \cap Y$. By case (i), we may assume that $v_i \in N^+(v_0)$. Define the paths

$$P_1 := v_0 \underset{P}{\to} v_{i-1} \quad \text{and} \quad P_2 := v_0 v_i \underset{P}{\to} v_t.$$

See Figure 5.

Note that $X \subsetneq V(P_1)$ and $Z \subsetneq V(P_2)$, which implies that $\text{length}(P_1), \text{length}(P_2) \geq k - \ell$. Moreover, at least one of P_1 and P_2 has length at least $t/2 \geq \ell$. Thus, $P(\overset{\leftarrow}{\ell}, \overrightarrow{k-\ell})$ is a subpath of $\overleftarrow{P_1}P_2$, and therefore embeds in G.
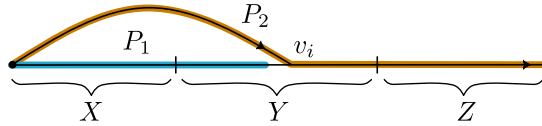

Figure 5: Construction of paths P_1 and P_2 in the proof of Proposition 3.2 (iii).

□

4 Proof of Theorem 1.2

We are now ready to prove Theorem 1.2, restated below for the reader's convenience.

Theorem 1.2. *Let k and ℓ be positive integers with $k/2 \leq \ell < k$. Let G be an oriented graph such that*

$$\delta^0(G) \geq \begin{cases} k - \frac{\ell}{2} & \text{if } \ell \leq \frac{2k}{3}, \\ \frac{2k}{3} & \text{if } \ell > \frac{2k}{3}. \end{cases}$$

Then G contains as a subgraph both possible k-arc oriented paths with two blocks of size ℓ and $k - \ell$.

Proof. Note that it is enough to show that $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ embeds in G, as the embedding of $P(\overrightarrow{\ell}, \overleftarrow{k-\ell})$ within G follows by embedding $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ into the oriented graph obtained by reversing each arc in G.

Consider a directed path $P = v_0 v_1 \ldots v_t$ of maximum length in G. By Theorem 3.1 we have $t \geq 2\delta^0(G)$. We proceed by cases.

- Suppose that $\boldsymbol{k/2 \leq \ell \leq 2k/3}$. Here we have $\delta^0(G) \geq k - \ell/2$, and consequently, $t \geq 2\delta^0(G) \geq 2k - \ell$. It follows that

$$\delta^0(G) > k - \ell \quad \text{and} \quad t - \ell \geq 2k - 2\ell \geq \frac{2k}{3} \geq \ell, \tag{1}$$

  which in particular implies that $t \geq 2\ell$. Define the sets

$$X = \{v_0, \ldots, v_{k-\ell-1}\}, \quad Y = \{v_{k-\ell}, \ldots, v_{t-k+\ell}\} \quad \text{and} \quad Z = \{v_{t-k+\ell+1}, \ldots, v_t\}.$$

  Note that with this set-up, if any one of the conditions (i), (ii) and (iii) from Proposition 3.2 holds, then $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ embeds in G, and we are done.

  We have that

$$|X| = |Z| = k - \ell \quad \text{and} \quad |Y| = t + 1 - 2(k - \ell) \geq (2k - \ell) + 1 - 2(k - \ell) = \ell + 1.$$

  By the maximality of P, we have $N^-(v_0), N^+(v_t) \subseteq V(P)$. Note that if $N^-(v_0) \cap Y \neq \emptyset$ or $N^+(v_t) \cap Y \neq \emptyset$, then Proposition 3.2 (i) guarantees that $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ embeds in G. Moreover, if $v_0 \in N^+(v_t)$ (equivalently: $v_t \in N^-(v_0)$), then $v_0 v_1 \ldots v_t v_0$ is a Hamilton cycle of $G[V(P)]$, and so by Proposition 3.2 (iii), we again obtain that $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ embeds in G. Thus, we may assume that

$$N^-(v_0) \subseteq X \cup (Z \setminus \{v_t\}) \quad \text{and} \quad N^+(v_t) \subseteq (X \setminus \{v_0\}) \cup Z. \tag{2}$$

  Choose $i^* \in \{1, \ldots, k - \lceil 3\ell/2 \rceil\}$ such that $v_{i^*} \in N^+(v_t)$. If no such vertex exists, then by (2), we must have

$$N^+(v_t) \subseteq \{v_{k-\lceil 3\ell/2 \rceil+1}, \ldots, v_{k-\ell-1}\} \cup (Z \setminus \{v_t\}),$$

  which implies that

$$\deg^+(v_t) \leq k - \left\lfloor \frac{\ell}{2} \right\rfloor - 2 < k - \frac{\ell}{2} \leq \delta^0(G),$$

  a contradiction. Note that $v_{i^*} \in X \setminus \{v_0\}$.

  Since $\{v_1, \ldots, v_{k-\lceil 3\ell/2 \rceil}\} \subseteq X$, by Proposition 3.2 (ii), we may assume that $N^+(v_{i^*-1}) \subseteq V(P)$. We now distinguish two cases.

  - Suppose that $N^+(v_{i^*-1}) \cap Y = \emptyset$. Clearly, we have

$$|N^-(v_0) \cap X| \leq k - \ell - 1 \quad \text{and} \quad |N^+(v_{i^*-1}) \cap X| \leq k - \ell - 1.$$

    Then, since $\delta^0(G) \geq k - \ell/2$, and $N^-(v_0) \cup N^+(v_{i^*-1}) \subseteq X \cup Z$ by (2), it follows that

$$|N^+(v_{i^*-1}) \cap Z| \geq \frac{\ell}{2} + 1 > \frac{|Z|}{2} \quad \text{and} \quad |N^-(v_0) \cap Z| \geq \frac{\ell}{2} + 1 > \frac{|Z|}{2}.$$
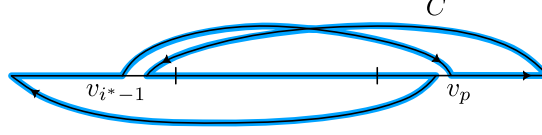
    Define the set

$$S := \{v_j : v_{j-1} \in N^-(v_0) \cap Z\}.$$

    Clearly, $S \subseteq Z$. Moreover, since $v_t \notin N^-(v_0)$ (by (2)), we see that $|S| = |N^-(v_0) \cap Z|$. We now have $|S|, |N^+(v_{i^*-1}) \cap Z| > |Z|/2$, and so by the pigeonhole principle, we must have

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$S \cap (N^+(v_{i^*-1}) \cap Z) \neq \emptyset$. This ensures the existence of a vertex $v_p \in N^+(v_{i^*-1}) \cap Z$ such that $v_{p-1} \in N^-(v_0) \cap Z$. Define

$$C := v_0 \underset{P}{\longrightarrow} v_{i^*-1} v_p \underset{P}{\longrightarrow} v_t v_{i^*} \underset{P}{\longrightarrow} v_{p-1} v_0.$$
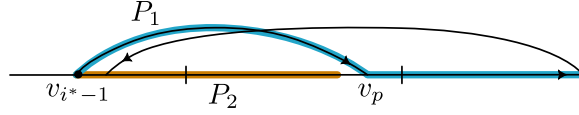

Then, C is a directed cycle with $V(C) = V(P)$, i.e. C is a Hamilton cycle of $G[V(P)]$. By Proposition 3.2 (iii), we conclude that $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ embeds in G.

– Suppose that $N^+(v_{i^*-1}) \cap Y \neq \emptyset$, and let $v_p \in N^+(v_{i^*-1}) \cap Y$.

* Suppose that $p \geq \ell + i^*$. Define the paths

$$P_1 := v_{i^*-1} v_p \underset{P}{\longrightarrow} v_t \quad \text{and} \quad P_2 := v_{i^*-1} \underset{P}{\longrightarrow} v_{p-1}.$$

Since $\ell + i^* \leq p \leq t - k + \ell$, we have

$$\text{length}(P_1) = t - p + 1 > k - \ell \quad \text{and} \quad \text{length}(P_2) = p - i^* \geq \ell.$$

Thus, the path $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ is a subpath of $\overleftarrow{P_1} P_2$.

* Suppose that $p \leq \ell + i^* - 1$. Choose $j^* \in \{t - k + \ell + 1, \ldots, t - \lfloor \ell/2 \rfloor\}$ such that $v_{j^*} \in N^-(v_0)$. The existence of such a vertex v_{j^*} follows by a similar argument to that of v_{i^*}. Indeed, if such a vertex v_{j^*} does not exist, then by (2), we have

$$N^-(v_0) \subseteq (X \setminus \{v_0\}) \cup \{v_{t-\lfloor \ell/2 \rfloor+1}, \ldots, v_t\}.$$
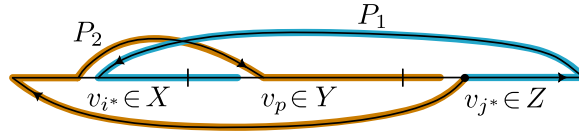
This would imply that

$$\deg^-(v_0) \leq (k - \ell - 1) + \left\lfloor \frac{\ell}{2} \right\rfloor = k - \left\lceil \frac{\ell}{2} \right\rceil - 1 < \delta^0(G),$$

a contradiction.
Define the paths

$$P_1 := v_{j^*} \underset{P}{\longrightarrow} v_t v_{i^*} \underset{P}{\longrightarrow} v_{p-1} \quad \text{and} \quad P_2 := v_{j^*} v_0 \underset{P}{\longrightarrow} v_{i^*-1} v_p \underset{P}{\longrightarrow} v_{j^*-1}.$$

Since $j^* \leq t - \lfloor \ell/2 \rfloor$, $i^* \leq k - \lceil 3\ell/2 \rceil$ and $p \geq k - \ell$ (because $v_p \in Y$), it follows that $t - j^* \geq \lfloor \ell/2 \rfloor$ and $p - i^* \geq \lceil \ell/2 \rceil$. Therefore,

$$\text{length}(P_1) = t - j^* + p - i^* \geq \ell.$$

Moreover, since $j^* \geq t - k + \ell + 1$, $p \leq \ell + i^* - 1$ and $t \geq 2\delta^0(G) \geq 2k - \ell$, we obtain

$$\text{length}(P_2) = i^* + j^* - p \geq i^* + t - k + \ell + 1 - \ell - i^* + 1 = t - k + 2 \geq k - \ell + 2.$$

Thus, the path $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ is a subpath of $\overleftarrow{P_1} P_2$.

- Suppose that $\ell > 2k/3$. Here we have $\delta^0(G) \geq 2k/3$, and consequently, $t \geq 2\delta^0(G) \geq 4k/3$. Define the sets

$$Q := \{v_{k-\ell}, \ldots, v_{t-\ell}\} \quad \text{and} \quad R := \{v_\ell, \ldots, v_{t-k+\ell}\}.$$

Note that $|Q| = |R| = t - k + 1$. Moreover, note that $Q \cap R \neq \emptyset$ if and only if $\ell \leq t - \ell$, and in this case, we have that $Q \cup R = \{v_{k-\ell}, \ldots, v_{t-k+\ell}\}$.

First, suppose that $N^-(v_0) \cap (Q \cup R) = \emptyset$. Since $N^-(v_0) \subseteq V(P)$ by the maximality of P, it follows that $N^-(v_0) \subseteq V(P) \setminus (Q \cup R)$. Then, keeping in mind that $t \geq 2\delta^0(G) \geq 4k/3$ and $\ell > 2k/3$, we obtain

$$\deg^-(v_0) \leq \begin{cases} t - 2(t-k+1) = 2k - t - 2 < 2k/3 & \text{if } Q \cap R = \emptyset, \\ 2(k-\ell) - 1 < 2k/3 & \text{if } Q \cap R \neq \emptyset, \end{cases}$$

a contradiction to $\delta^0(G) \geq 2k/3$. Thus, we may assume that $N^-(v_0) \cap (Q \cup R) \neq \emptyset$.

Let $v_i \in N^-(v_0) \cap (Q \cup R)$. Define the paths

$$P_1 := v_i v_0 \underset{P}{\rightarrow} v_{i-1} \quad \text{and} \quad P_2 := v_i \underset{P}{\rightarrow} v_t.$$

See Figure 6. We have $\text{length}(P_1) = i$ and $\text{length}(P_2) = t - i$. Then,

- if $v_i \in Q$, then $\text{length}(P_1) \geq k - \ell$ and $\text{length}(P_2) \geq \ell$; and,
- if $v_i \in R$, then $\text{length}(P_1) \geq \ell$ and $\text{length}(P_2) \geq k - \ell$.

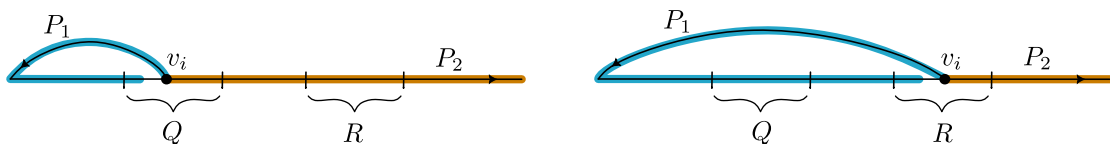Thus, in both cases, $P(\overleftarrow{\ell}, \overrightarrow{k-\ell})$ is a subpath of $\overleftarrow{P_1}P_2$.

Figure 6: Construction of paths P_1 and P_2 when $\ell > 2k/3$.

$\square$

5 Concluding remarks

As mentioned in [14], the tightness of Stein's conjecture (Conjecture 1.1) for the antidirected orientation is witnessed by the $k/2$-blowup of the directed triangle. But more generally, Stein's conjecture is near-tight for every orientation: for an even k take a regular tournament T on $k+1$ vertices. We have $\delta^0(T) = k/2$ and, by a theorem of Havet and Thomassé [9] (proved first for large k by Thomason [20]), T contains every oriented path with k arcs (with three small exceptions all involving the antidirected path), but clearly T does not contain any path with $k+1$ arcs. Thus we wonder if Conjecture 1.1 can be strengthened for all orientations except antidirected.

Question 5.1. *Is it true that every oriented graph G with $\delta^0(G) \geq k/2$ contains every orientation of the k-edge path, save for the antidirected orientations?*

Note that Jackson's theorem (Theorem 3.1) gives an affirmative answer for the directed orientation.

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