Mathematical Analysis 1: Tutorial #10

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indefinite integral $\int f(x)dx$	constraints on the parameter α	constraints on the variable x
$\int 0dx = C$		
$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$	$\alpha \in \mathbb{R} \setminus \{-1\}$	$x > 0$
$\int \frac{dx}{x} = \ln\lvert x\rvert + C$		$x \neq 0$
$\int e^x dx = e^x + C$		
$\int \sin x dx = -\cos x + C$		
$\int \cos x dx = \sin x + C$		
$\int \frac{dx}{\cos^2 x} = \tan x + C$		$x \neq \frac{2k+1}{2}\pi$, $k \in \mathbb{Z}$
$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$		$x \in (-1, 1)$
$\int \frac{dx}{1+x^2} = \arctan x + C$		

The Substitution Rule. *Let $I, J \subseteq \mathbb{R}$ be intervals, let f be a function that has an antiderivative F on the interval I, and let $\varphi : J \to I$ is a continuously differentiable function.*[1] *Then*

$$\int f\big(\varphi(x)\big)\varphi'(x)dx \quad = \quad F\big(\varphi(x)\big) + C$$

on the interval J.

Notation: For $u = \varphi(x)$, where φ is a differentiable function, we write $du(x) = \varphi'(x)dx$, or simply $du = \varphi'(x)dx$. So, under the assumptions of the Substitution Rule, we get the following formula:

$$\int f\big(\varphi(x)\big)\varphi'(x)dx \quad = \quad \int f(u)du, \qquad \begin{array}{l} u = \varphi(x) \\ du = \varphi'(x)dx. \end{array}$$

Integration by parts. *Let function u and v be continuously differentiable on an interval I. Then*

$$\int u(x)dv(x) \quad = \quad u(x)v(x) - \int v(x)du(x)$$

on the interval I.

More succinctly: $\int udv = uv - \int vdu$.

[1] So, we are assuming φ is differentiable at all points $x \in J$, and that $\varphi' : J \to \mathbb{R}$ is continuous.

Exercise 1. *Using substitution, compute the following indefinite integrals:*

(a) $\int x^5 \sin(x^6)dx$;

(b) $\int e^x \cos(e^x)dx$.

(c) $\int e^{\sin x} \cos x dx$;

(d) $\int \cot x dx$;

(e) $\int \frac{dx}{1-x}$;

(f) $\int \frac{7}{6+5x}dx$;

(g) $\int \frac{dx}{1+4x^2}$;

(h) $\int \frac{dx}{2+3x^2}$;

(i) $\int \frac{dx}{\sqrt{1-2x^2}}$.

Exercise 2. *Using integration by parts, compute the following integrals:*

(a) $\int x \cos x dx$;

(b) $\int x e^{-x} dx$;

(c) $\int \ln x dx$.

Exercise 3. *Prove that*

$$\int x^n e^x dx \quad = \quad x^n e^x - n \int x^{n-1} e^x dx \qquad \forall n \in \mathbb{N}.$$

Then, compute $\int x^4 e^x dx$.

Exercise 4. *Compute* $I := \int \frac{x^2}{\sqrt{1-x^2}} dx$.

> ***Hint:*** *Start by integrating by parts with* $u = x$ *and* $dv = \frac{x}{\sqrt{1-x^2}}dx$*, and try to obtain an equation in* I*. Then solve for* I*.*

Exercise 5. *Compute the following integrals (using any method or combination of methods):*

(a) $\int x \sin(x^2)dx$;

(b) $\int \frac{e^{2x}}{1+e^{4x}}dx$;

(c) $\int x^5 \sqrt{x^2-1}dx$;

(d) $\int \arctan x dx$;

(e) $\int x \arctan x dx$;

(f) $\int \sin(\ln x)dx$;

(g) $\int \sqrt{x} e^{\sqrt{x}} dx$;

(h) $\int \frac{\ln x}{\sqrt{x}} dx$.