Mathematical Analysis 1

Tutorial #9

Sketching graphs of functions

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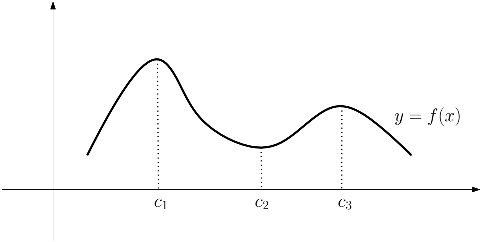
Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A$.

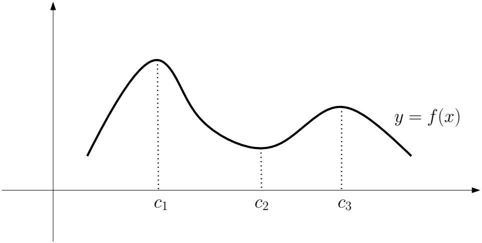
- We say that f reaches a *local maximum* at c if there exists some $\delta > 0$ s.t. for all $x \in A$, if $|x - c| < \delta$, then $f(c) \geq f(x)$.
- We say that f reaches a *local minimum* at c if there exists some $\delta > 0$ s.t. for all $x \in A$, if $|x - c| < \delta$, then $f(c) \leq f(x)$.
- We say that f reaches a *local extremum* at c if f reaches a local maximum or a local minimum at c.

- For instance, in the picture below, the function f reaches local maxima at c_1 and c_3, and it reaches a local minimum at c_2.



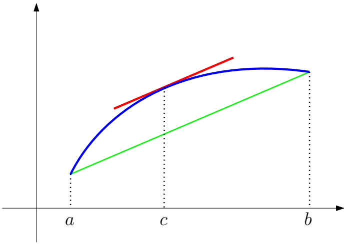
Theorem 4.10.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and assume that f reaches a local extremum at a point $c_0 \in I \subseteq A$, where I is an open interval. If f is differentiable at c_0, then $f'(c_0) = 0$.



Lagrange's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Theorem 4.13.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $I \subseteq A$ be an interval. Assume that f is continuous on I and differentiable on I°. Then all the following hold:

- (a) if $f'(x) = 0$ for all $x \in I^\circ$, then f constant on I;
- (b) if $f'(x) \geq 0$ for all $x \in I^\circ$, then f is non-decreasing on I;
- (c) if $f'(x) > 0$ for all $x \in I^\circ$, then f is strictly increasing on I;
- (d) if $f'(x) \leq 0$ for all $x \in I^\circ$, then f is non-increasing on I;
- (e) if $f'(x) < 0$ for all $x \in I^\circ$, then f is strictly decreasing on I.

The First Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).[a]

ⓐ If both the following hold:

- $f'(x) \leq 0$ for all $x \in (a, c)$,
- $f'(x) \geq 0$ for all $x \in (c, b)$,

then f reaches a local minimum at c.

ⓑ If both the following hold:

- $f'(x) \geq 0$ for all $x \in (a, c)$,
- $f'(x) \leq 0$ for all $x \in (c, b)$,

then f reaches a local maximum at c.

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[a] The function f may or may not be differentiable on the point c itself. However, by hypothesis, f is defined and continuous at c.

- **Remark:** Schematically, the First Derivative Test (and its proof) can be represented by the two tables below.

$a \qquad c \qquad b$

x	(a,c)	(c,b)
$f'(x)$	$-$	$+$
$f(x)$	$\searrow$ min	$\nearrow$

$a \qquad c \qquad b$

x	(a,c)	(c,b)
$f'(x)$	$+$	$-$
$f(x)$	$\nearrow$ max	$\searrow$

  - Technically, we would use this sort of table representation for the case when the inequalities from the First Derivative Test are strict, i.e. when we have $f'(x) > 0$ and $f'(x) < 0$, rather than merely $f'(x) \geq 0$ and $f'(x) \leq 0$.

Proposition 4.13.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in R$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).

1. If $f'(x) > 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly increasing on (a, b), and in particular, f does not reach a local extremum at c.
2. If $f'(x) < 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly decreasing on (a, b), and in particular, f does not reach a local extremum at c.

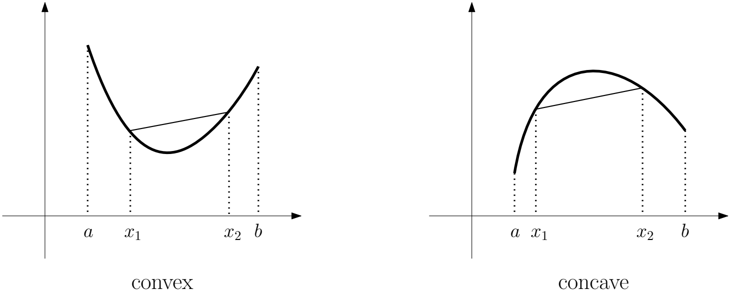
- **Remark:** The proposition applies even if f is not differentiable at c. However, for the proposition to hold, it is indeed necessary for f to be continuous on the whole interval (a, b), and in particular, to be defined at and continuous at the point c.

The Second Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is differentiable on the interval (a, b), that $f'(c) = 0$, and that $f''(c)$ exists.

- (a) If $f''(c) > 0$, then f reaches a local minimum at c.
- (b) If $f''(c) < 0$, then f reaches a local maximum at c.

- The idea behind convexity and concavity is in the picture below.



- Now let's give a proper definition!

Definition

For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and an interval $I \subseteq A$, we say that:

- f is *convex* on the interval I provided that for all $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 \geq 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad \leq \quad \lambda_1 f(x_1) + \lambda_2 f(x_2);$$

- f is *strictly convex* on the interval I provided that for all distinct $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 > 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad < \quad \lambda_1 f(x_1) + \lambda_2 f(x_2);$$

Definition (continued)

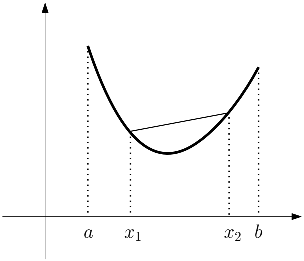
For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and an interval $I \subseteq A$, we say that:

- f is *concave* on the interval I provided that for all $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 \geq 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

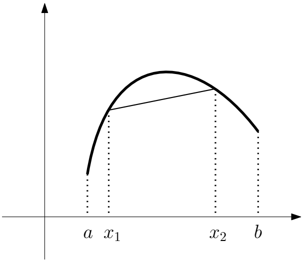
$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad \geq \quad \lambda_1 f(x_1) + \lambda_2 f(x_2);$$

- f is *strictly concave* on the interval I provided that for all distinct $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 > 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad > \quad \lambda_1 f(x_1) + \lambda_2 f(x_2).$$



convex



concave

Theorem 4.14.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on an open interval $I \subseteq A$. Then all the following hold:

(a) f' is non-decreasing on I iff f is convex on I;

(b) f' is strictly increasing on I iff f is strictly convex on I;

(c) f' is non-increasing on I iff f is concave on I;

(d) f' is strictly decreasing on I iff f is strictly concave on I.

Corollary 4.14.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on an open interval $I \subseteq A$. Then all the following hold:

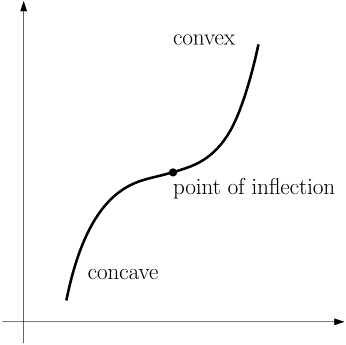
(a) f is convex on I iff $f''(x) \geq 0$ for all $x \in I$;

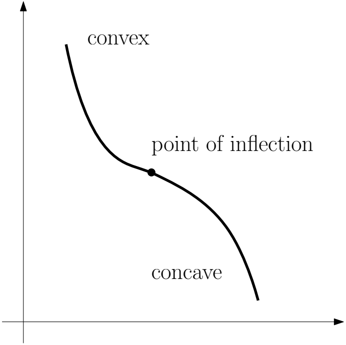
(b) if $f''(x) > 0$ for all $x \in I$, then f is strictly convex on I;

(c) f is concave on I iff $f''(x) \leq 0$ for all $x \in I$.

(d) if $f''(x) < 0$ for all $x \in I$, then f is strictly concave on I.

- A "point of inflection" is a point at which a function switches from being convex to being concave, or vice versa.





- Let us be more precise.

Definition

For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, a point $c \in A$ is said to be a *point of inflection* (or an *inflection point*) of f if f is continuous at c, and there exists some $\delta > 0$ s.t. all the following hold:

- $(c - \delta, c + \delta) \subseteq A$;
- f is convex on one of the open intervals $(c - \delta, c)$ and $(c, c + \delta)$, and is concave on the other;
- for all $\delta' \in (0, \delta)$, f is affine neither on $(c - \delta', c)$ nor on $(c, c + \delta')$.

- In view of Proposition 4.14.1, the third bullet point simply states that f is not simultaneously convex and concave on any small interval on one side (to the left or to the right) of c. So, points of inflection mark a “true switch” from convexity to concavity or vice versa.

Proposition 4.14.4

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $c \in A$, and let $\delta > 0$ be s.t. $(c - \delta, c + \delta) \subseteq A$. Assume that f is continuous at c, and is twice differentiable on $(c - \delta, c)$ and $(c, c + \delta)$. If one of the following holds:

1. $f''(x) > 0$ for all $x \in (c - \delta, c)$, and $f''(x) < 0$ for all $x \in (c, c + \delta)$,
2. $f''(x) < 0$ for all $x \in (c - \delta, c)$, and $f''(x) > 0$ for all $x \in (c, c + \delta)$,

then c is a point of inflection of f.

- To sketch the graph of a function $y = f(x)$, we need to do all the following:
  1. Determine the domain of the function (i.e. the largest subset of $\mathbb{R}$ on which the function is defined).
  2. Examine the behavior of the function near any "gaps" or "edges" of the domain, at and near any points at which f is discontinuous, as well as at $+\infty$ and $-\infty$. In particular, determine whether the function has any vertical, horizontal, or slant asymptotes.
     - A "gap" in the domain would be a point c at which f is undefined, whereas it is defined near c (both to the left and to the right). An "edge" of the domain would be a point c s.t. f defined near c to the left or to the right, but not both.

- To sketch the graph of a function $y = f(x)$, we need to do all the following (continued):
  3. Determine whether the function is periodic (and if so, find its period), and whether it is even and/or odd.
     - A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *periodic* if there exists a real number $p > 0$ s.t. for all $a \in A$ and all $k \in \mathbb{Z}$, we have that $a + kp \in A$ and $f(a + kp) = f(a)$. The smallest such p, if it exists, is called the *period* of the function.
     - A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *even* if for all $x \in A$, we have that $-x \in A$ and $f(-x) = f(x)$. Meanwhile, f is *odd* if for all $x \in A$, we have that $-x \in A$ and $f(-x) = -f(x)$. Graphs of even functions are symmetric about the y-axis, and graphs of odd functions are symmetric about the origin.
  4. Find the x- and y-intercepts of f. Determine intervals at which $f(x) > 0$ and at which $f(x) < 0$.
     - An *x-intercept* of f is any value of x s.t. $f(x) = 0$ (i.e. the value at which f intersects the x-axis). The y-intercept of f is the value $y = f(0)$ (i.e. the value at which f intersects the y-axis), if it exists.

- To sketch the graph of a function $y = f(x)$, we need to do all the following (continued):
    5. Find f', and using this, determine at which intervals the function increases or decreases. Find all local maxima and minima.
        - Sometimes, local maxima and minima can more easily be found using f''.
    6. Find f'', and using this, determine the intervals at which the function is convex or concave, and find all its inflection points.

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution. The domain of the function f is $\mathbb{R} \setminus \{-1\}$, i.e. the function is defined for all real numbers $x \neq -1$. Moreover, it is clear that f is continuous (at all points in its domain). Note that

$$\lim_{x \to -1} f(x) = -\infty,$$

and so $x = -1$ is a vertical asymptote of $f(x)$. Furthermore, the graph of the function approaches the line $x = -1$ "downward" both from the left and from the right. Next, note that

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = -\infty,$$

and so f has no horizontal asymptotes.

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Further, note that

$$\lim_{x\to+\infty} \frac{f(x)}{x} \quad = \quad \lim_{x\to+\infty} \frac{(x-1)^3}{x(x+1)^2} \quad = \quad 1 \quad =: \quad a,$$

and that

$$\lim_{x\to+\infty} \Big(f(x) - ax\Big) \quad = \quad \lim_{x\to+\infty} \frac{-5x^2+2x-1}{x^2+2x+1} \quad = \quad -5 \quad =: \quad b.$$

So, $y = x - 5$ is a slant asymptote of $y = f(x)$ as $x \to +\infty$. A similar calculation shows that $y = x - 5$ is a slant asymptote of $y = f(x)$ as $x \to -\infty$.

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). The function f is neither even nor odd (e.g. because it is defined as 1, but not at -1). The function is also not periodic because, otherwise, there would be some constant $p > 0$ (the period of the function) s.t. f is undefined at $-1 + kp$ for all $k \in \mathbb{Z}$.

Note that $f(0) = -1$ (so, $y = f(x)$ intersects the y axis at $y = -1$), and that $f(x) = 0$ only for $x = 1$ (so, $y = f(x)$ intersects the x-axis at $x = 1$).

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Further, we have the following table, showing where f is positive, negative, or zero.

		-1		1	
x	$(-\infty,-1)$		$(-1,1)$		$(1,+\infty)$
$x-1$	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
$f(x)$	$-$	undef	$-$	0	$+$

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Further, a straightforward calculation shows that

$$f'(x) = \frac{(x-1)^2(x+5)}{(x+1)^3} \quad \text{and} \quad f''(x) = 24\frac{x-1}{(x+1)^4}$$

for all $x \neq -1$. So, for the first derivative, we get the table below.

-5 $\quad$ -1 $\quad$ 1

x	$(-\infty,-5)$	$(-5,-1)$	$(-1,1)$	$(1,+\infty)$
$x-1$	$-$	$-$	$-$ $\quad 0$	$+$
$x+1$	$-$	$-$ $\quad 0$	$+$	$+$
$x+5$	$-$ $\quad 0$	$+$	$+$	$+$
$f'(x)$	$+$ $\quad 0$	$-$ $\quad$ undef	$+$ $\quad 0$	$+$
$f(x)$	$\nearrow$ $\quad$ max	$\searrow$ $\quad$ undef	$\nearrow$	$\nearrow$

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Reminder:

$$f'(x) = \frac{(x-1)^2(x+5)}{(x+1)^3} \quad \text{and} \quad f''(x) = 24\frac{x-1}{(x+1)^4}$$

for all $x \neq -1$.

For the second derivative, we get the table below. (Here, "$\smile$" means "convex," and "$\frown$" means "concave.")

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,+\infty)$
$x-1$	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
$f''(x)$	$-$	undef	$-$	0	$+$
$f(x)$	$\frown$	undef	$\frown$	inflec	$\smile$

Example 4.15.1

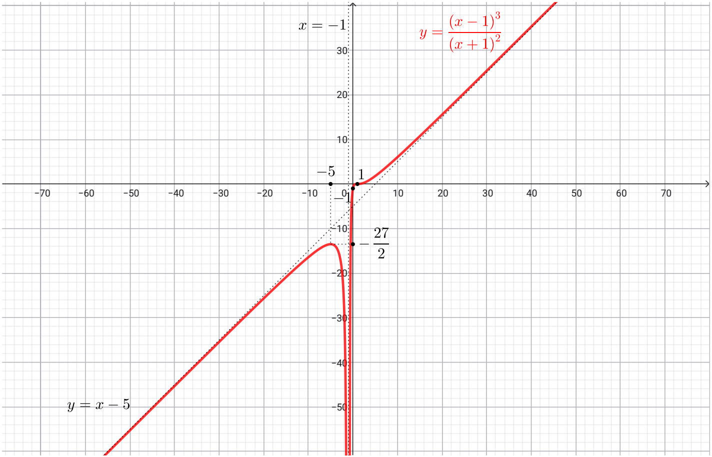
Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). We now combine the three tables above to obtain the table below.

		-5		-1		1	
x	$(-\infty, -5)$		$(-5, -1)$		$(-1, 1)$		$(1, +\infty)$
$f'(x)$	$+$	0	$-$	undef	$+$	0	$+$
$f''(x)$	$-$		$-$	undef	$-$	0	$+$
$f(x)$	$-$ $\nearrow$ $\frown$	max	$-$ $\searrow$ $\frown$	undef	$-$ $\nearrow$ $\frown$	0 inflec	$+$ $\nearrow$ $\smile$

Note that f reaches a local maximum at $x = -5$, and we have that $f(-5) = -\frac{27}{2}$. (There are no other local extrema.) Furthermore, the only point of inflection is $x = 1$, and we have that $f(1) = 0$.

Solution (continued). Putting all this together, we get the following picture.



$\square$

Example 4.15.2

Sketch the graph of the function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$.

Solution. The domain of the function is $\mathbb{R}$, and clearly, f is continuous. Therefore, f has no vertical asymptotes. Note that for all $x \in \mathbb{R} \setminus \{0\}$, we have that

$$f(x) \quad = \quad \frac{x-2}{\sqrt{x^2+1}} \quad = \quad \frac{x(1-\frac{2}{x})}{|x|\sqrt{1+\frac{1}{x^2}}}.$$

So,

$$\lim_{x \to +\infty} f(x) \quad = \quad \lim_{x \to +\infty} \frac{x(1-\frac{2}{x})}{|x|\sqrt{1+\frac{1}{x^2}}} \quad \overset{(*)}{=} \quad \lim_{x \to +\infty} \frac{1-\frac{2}{x}}{\sqrt{1+\frac{1}{x^2}}} \quad = \quad 1,$$

where (*) follows from the fact that $x > 0$. On the other hand,

$$\lim_{x \to -\infty} f(x) \quad = \quad \lim_{x \to -\infty} \frac{x(1-\frac{2}{x})}{|x|\sqrt{1+\frac{1}{x^2}}} \quad \overset{(*)}{=} \quad \lim_{x \to -\infty} \frac{-(1-\frac{2}{x})}{\sqrt{1+\frac{1}{x^2}}} \quad = \quad -1,$$

where (*) follows from the fact that $x < 0$. Thus, $y = 1$ is a horizontal asymptote of f as $x \to +\infty$, whereas $y = -1$ is a horizontal asymptote of f as $x \to -\infty$. Since f has a horizontal asymptote both as $x \to +\infty$ and as $x \to -\infty$, we see that f has no slant asymptotes.

Example 4.15.2

Sketch the graph of the function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$.

Solution (continued). Note that for $x \in \mathbb{R}$, we have that

$$f(-x) \;=\; \frac{-x-2}{\sqrt{x^2+1}},$$

and we see that f is neither even nor odd. Moreover, it is clear that f is not periodic.

Note that $f(0) = -2$ (so, $y = f(x)$ intersects the y-axis at $y = -2$), and that $f(x) = 0$ only for $x = 2$ (so, $y = f(x)$ intersects the x-axis at $x = 2$). Further, we have the following table, showing where f is positive, negative, or zero.

		2	
x	$(-\infty, 2)$		$(2, +\infty)$
$x-2$	$-$	0	$+$
$\sqrt{x^2+1}$	$+$		$+$
$f(x)$	$-$	0	$+$

Example 4.15.2

Sketch the graph of the function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$.

Solution (continued). Next, a straightforward calculation shows that

$$f'(x) = \frac{2x+1}{(x^2+1)^{3/2}} \qquad \text{and} \qquad f''(x) = -\frac{4x^2+3x-2}{(x^2+1)^{5/2}} = -\frac{\left(x-\frac{-3-\sqrt{41}}{8}\right)\left(x-\frac{-3+\sqrt{41}}{8}\right)}{(x^2+1)^{5/2}}$$

for all $x \in \mathbb{R}$. So, for the first derivative, we get the table below.

$-\frac{1}{2}$

x	$(-\infty, -\frac{1}{2})$		$(-\frac{1}{2}, +\infty)$
$2x+1$	$-$	0	$+$
x^2+1	$+$		$+$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	min	$\nearrow$

Example 4.15.2

Sketch the graph of the function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$.

Solution (continued). Reminder:

$$f'(x) = \frac{2x+1}{(x^2+1)^{3/2}} \qquad \text{and} \qquad f''(x) = -\frac{4x^2+3x-2}{(x^2+1)^{5/2}} = -\frac{\left(x-\frac{-3-\sqrt{41}}{8}\right)\left(x-\frac{-3+\sqrt{41}}{8}\right)}{(x^2+1)^{5/2}}$$

for all $x \in \mathbb{R}$.

For the second derivative, we get the table below.

		$\frac{-3-\sqrt{41}}{8}$		$\frac{-3+\sqrt{41}}{8}$	
x	$(-\infty, \frac{-3-\sqrt{41}}{8})$		$(\frac{-3-\sqrt{41}}{8}, \frac{-3+\sqrt{41}}{8})$		$(\frac{-3+\sqrt{41}}{8}, +\infty)$
$x - \frac{-3-\sqrt{41}}{8}$	$-$	0	$+$		$+$
$x - \frac{-3+\sqrt{41}}{8}$	$-$		$-$	0	$+$
x^2+1	$+$		$+$		$+$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	⌢	inflec	⌣	inflec	⌢

Example 4.15.2

Sketch the graph of the function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$.

Solution (continued). We observe that $\frac{-3-\sqrt{41}}{8} < -\frac{1}{2} < 0 < \frac{-3+\sqrt{41}}{8} < 2$, and we combine the three tables above, as follows.

		$\frac{-3-\sqrt{41}}{8}$		$-\frac{1}{2}$		$\frac{-3+\sqrt{41}}{8}$		2	
x	$(-\infty, \frac{-3-\sqrt{41}}{8})$		$(\frac{-3-\sqrt{41}}{8}, -\frac{1}{2})$		$(-\frac{1}{2}, \frac{-3+\sqrt{41}}{8})$		$(\frac{-3+\sqrt{41}}{8}, 2)$		$(2, +\infty)$
$f'(x)$	$-$		$-$	0	$+$		$+$		$+$
$f''(x)$	$-$	0	$+$		$+$	0	$-$		$-$
$f(x)$	$-$ $\searrow$ $\frown$	inflec	$-$ $\searrow$ $\smile$	min	$-$ $\nearrow$ $\smile$	inflec	$-$ $\nearrow$ $\frown$	0	$+$ $\nearrow$ $\frown$

Example 4.15.2

Sketch the graph of the function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$.

Solution (continued). Reminder:

x	$(-\infty, \frac{-3-\sqrt{41}}{8})$	$\frac{-3-\sqrt{41}}{8}$	$(\frac{-3-\sqrt{41}}{8}, -\frac{1}{2})$	$-\frac{1}{2}$	$(-\frac{1}{2}, \frac{-3+\sqrt{41}}{8})$	$\frac{-3+\sqrt{41}}{8}$	$(\frac{-3+\sqrt{41}}{8}, 2)$	2	$(2, +\infty)$
$f'(x)$	$-$		$-$	0	$+$		$+$		$+$
$f''(x)$	$-$	0	$+$		$+$	0	$-$		$-$
$f(x)$	$-$ $\searrow$ $\frown$	inflec	$-$ $\searrow$ $\smile$	min	$-$ $\nearrow$ $\smile$	inflec	$-$ $\nearrow$ $\frown$	0	$+$ $\nearrow$ $\frown$

Note that f reaches a local maximum at $x = -\frac{1}{2}$, and we have that $f(-\frac{1}{2}) = -\sqrt{5}$. (There are no other local extrema.) The inflection points of f are

$$x_1 := \frac{-3-\sqrt{41}}{8} \qquad \text{and} \qquad x_2 := \frac{-3+\sqrt{41}}{8},$$

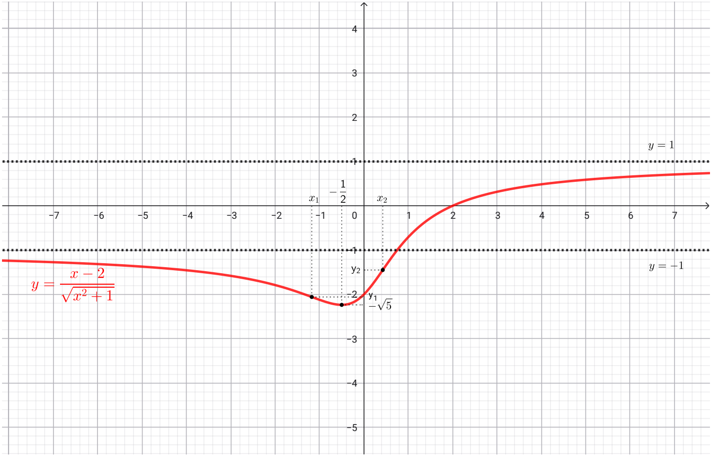
and we have that

$$y_1 := f(x_1) = \frac{-19-\sqrt{41}}{\sqrt{114+6\sqrt{41}}} \qquad \text{and} \qquad y_2 := f(x_2) = \frac{-19+\sqrt{41}}{\sqrt{114-6\sqrt{41}}}.$$

Example 4.15.2

Sketch the graph of the function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$.

Solution (continued). Putting all this together, we get the following picture.



□

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution. The domain of the function is $\mathbb{R}$, and clearly, f is continuous. Therefore, f has no vertical asymptotes. Next, note that for all $x \in \mathbb{R}$, we have that

$$f(-x) \quad = \quad (-x)\arctan(-x) \quad \overset{(*)}{=} \quad (-x)(-\arctan x) \quad = \quad x\arctan x \quad = \quad f(x),$$

where (*) follows from the fact that the arctangent function is odd. So, f is even, which means that its graph is symmetric about the y-axis.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). Now, recall that

$$\lim_{x \to +\infty} \arctan x = \tfrac{\pi}{2} \qquad \text{and} \qquad \lim_{x \to -\infty} \arctan x = -\tfrac{\pi}{2}.$$

It then readily follows that

$$\lim_{x \to +\infty} f(x) \;=\; \lim_{x \to +\infty} \left(x \arctan x\right) \;=\; +\infty$$

and

$$\lim_{x \to -\infty} f(x) \;=\; \lim_{x \to -\infty} \left(x \arctan x\right) \;=\; +\infty$$

Thus, f has no horizontal asymptotes.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). On the other hand, we have that

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \arctan x = \frac{\pi}{2} =: a,$$

and that

$$\begin{aligned}
\lim_{x \to +\infty} \Big(f(x) - ax\Big) &= \lim_{x \to +\infty} \Big(x \arctan x - \tfrac{\pi}{2} x\Big) \\
&= \lim_{x \to +\infty} x(\arctan x - \tfrac{\pi}{2}) \\
&= \lim_{x \to +\infty} \frac{\arctan x - \frac{\pi}{2}}{\frac{1}{x}} \\
&= \lim_{x \to +\infty} \frac{\frac{d}{dx}(\arctan x - \frac{\pi}{2})}{\frac{d}{dx}(\frac{1}{x})} \qquad \text{via L'Hôpital's Rule, since } \lim_{x \to +\infty} (\arctan x - \tfrac{\pi}{2}) = 0 \text{ and } \lim_{x \to +\infty} (\tfrac{1}{x}) = 0 \\
&= \lim_{x \to +\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x \to +\infty} \left(-\frac{x^2}{x^2+1}\right) \\
&= \lim_{x \to +\infty} \left(-\frac{1}{1+\frac{1}{x^2}}\right) \\
&= -1 =: b.
\end{aligned}$$

Thus, $y = \frac{\pi}{2}x - 1$ is a slant asymptote of f as $x \to +\infty$.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). Reminder: $y = \frac{\pi}{2}x - 1$ is a slant asymptote of f as $x \to +\infty$.

Now, the line obtained by reflecting $y = \frac{\pi}{2}x - 1$ about the y-axis is $y = -\frac{\pi}{2}x - 1$. Since f is even, it follows that $y = -\frac{\pi}{2}x - 1$ is a slant asymptote of f as $x \to -\infty$.

Note that $f(0) = 0$, and in fact, $f(x) = 0$ only for $x = 0$. Thus, the graph of f passes through the origin, and it does not intersect the x-axis or the y-axis anywhere else.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). Next, a straightforward calculation shows that

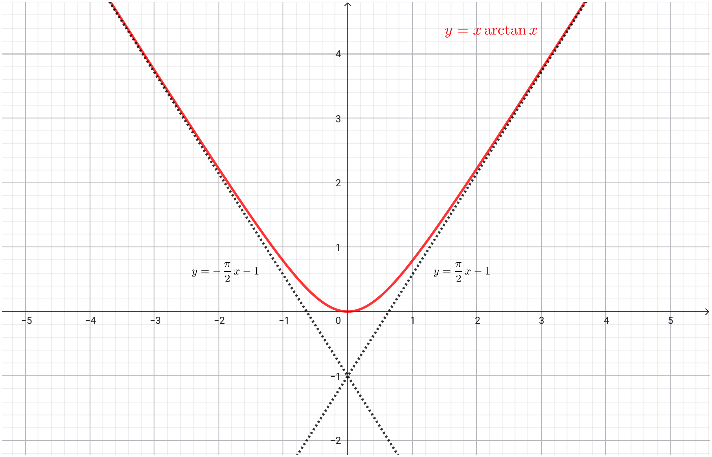
$$f'(x) = \arctan x + \tfrac{x}{1+x^2} \qquad \text{and} \qquad f''(x) = \tfrac{2}{(1+x^2)^2}$$

Note that $f'(0) = 0$, and that $f''(0) > 0$. Therefore, by the Second Derivative Test, f reaches a local minimum at $x = 0$, and clearly, we have that $f(0) = 0$. But in fact, $f''(x) > 0$ for all $x \in \mathbb{R}$, and consequently, f is strictly convex. Thus, the graph of f is as in the picture on the next slide.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued).



Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued).

- **Remark:** Note that the fact that $f''(x) > 0$ for all $x \in \mathbb{R}$ implies that f' is strictly increasing. Since $f'(0) = 0$, it follows that $f'(x) < 0$ for all $x \in (-\infty, 0)$, whereas $f'(x) > 0$ for all $x \in (0, +\infty)$. This yields the table below, which is consistent with the graph that we obtained above. Note, however, that the behavior of f' would have been quite difficult to analyze without using f''.

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	↘	min	↗

□